9.1 Exponential Functions

Exponents and Properties ▪ Exponential Functions ▪ Exponential Equations ▪ Compound Interest ▪ The Number *e* and Continuous Compounding ▪ Exponential Models and Curve Fitting

TEACHING TIP Have students review Appendix C on Functions and Appendix D on Graphing Techniques before beginning this section.

Exponents and Properties Recall from algebra the definition of a^r, where r is a rational number: if $r = \frac{m}{n}$, then for appropriate values of m and n,

$$a^{m/n} = \left(\sqrt[n]{a}\right)^m.$$

For example,

$$16^{3/4} = \left(\sqrt[4]{16}\right)^3 = 2^3 = 8,$$

$$27^{-1/3} = \frac{1}{27^{1/3}} = \frac{1}{\sqrt[3]{27}} = \frac{1}{3}, \quad \text{and} \quad 64^{-1/2} = \frac{1}{64^{1/2}} = \frac{1}{\sqrt{64}} = \frac{1}{8}.$$

In this section we extend the definition of a^r to include all real (not just rational) values of the exponent r. For example, $2^{\sqrt{3}}$ might be evaluated by approximating the exponent $\sqrt{3}$ with the rational numbers 1.7, 1.73, 1.732, and so on. Since these decimals approach the value of $\sqrt{3}$ more and more closely, it seems reasonable that $2^{\sqrt{3}}$ should be approximated more and more closely by the numbers $2^{1.7}$, $2^{1.73}$, $2^{1.732}$, and so on. (Recall, for example, that $2^{1.7} = 2^{17/10} = \left(\sqrt[10]{2}\right)^{17}$. In fact, this is exactly how $2^{\sqrt{3}}$ is defined (in a more advanced course). To show that this assumption is reasonable, Figure 1 gives graphs of the function $f(x) = 2^x$ with three different domains.

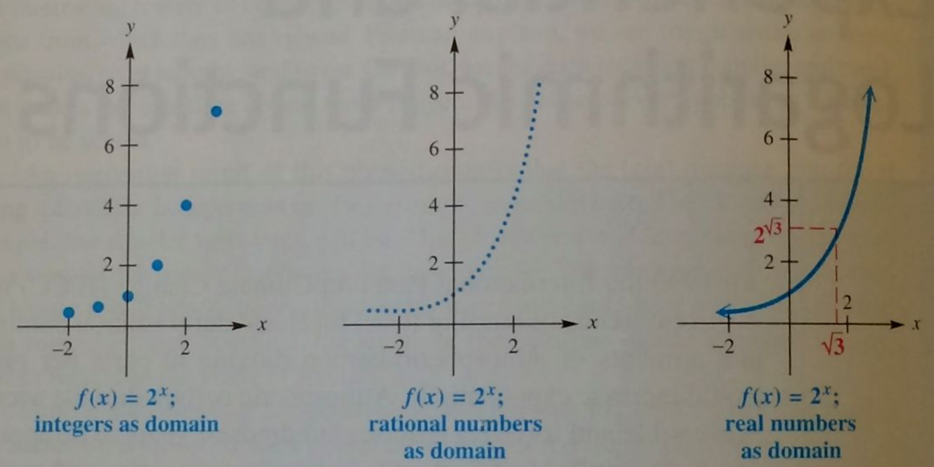


Figure 1

Using this interpretation of real exponents, all rules and theorems for exponents are valid for all real number exponents, not just rational ones. In addition to the rules for exponents, we use several new properties in this chapter. For example, if $y = 2^x$, then each real value of x leads to exactly one value of y, and therefore, $y = 2^x$ defines a function. Furthermore,

$$\text{if} \quad 2^x = 2^4, \quad \text{then} \quad x = 4,$$

$$\text{and} \quad \text{if} \quad x = 4, \quad \text{then} \quad 2^x = 2^4.$$

Also,

$$4^2 < 4^3 \quad \text{but} \quad \left(\frac{1}{2}\right)^2 > \left(\frac{1}{2}\right)^3.$$

In general, when $a > 1$, increasing the exponent on a leads to a *larger* number, but when $0 < a < 1$, increasing the exponent on a leads to a *smaller* number.

These properties are generalized below. Proofs of the properties are not given here, as they require more advanced mathematics.

Additional Properties of Exponents

For any real number $a > 0$, $a \neq 1$, the following statements are true.

(a) a^x **is a unique real number for all real numbers** x.

(b) $a^b = a^c$ **if and only if** $b = c$.

(c) **If** $a > 1$ **and** $m < n$, **then** $a^m < a^n$.

(d) **If** $0 < a < 1$ **and** $m < n$, **then** $a^m > a^n$.

TEACHING TIP Give interpretations of the properties. That is, property (a) says that $f(x) = a^x$ is a function with domain $(-\infty, \infty)$, property (b) says that f is one-to-one, property (c) says that f is increasing if $a > 1$, and property (d) says that f is decreasing if $0 < a < 1$.

Properties (a) and (b) require $a > 0$ so that a^x is always defined. For example, $(-6)^x$ is not a real number if $x = \frac{1}{2}$. This means that a^x will always be positive, since a must be positive. In property (a), a cannot equal 1 because $1^x = 1$ for every real number value of x, so each value of x leads to the same real number, 1. For property (b) to hold, a must not equal 1 since, for example, $1^4 = 1^5$, even though $4 \neq 5$.

EXAMPLE 1 Evaluating an Exponential Expression

If $f(x) = 2^x$, find each of the following.

(a) $f(-1)$ (b) $f(3)$ (c) $f\left(\frac{5}{2}\right)$ (d) $f(4.92)$

Solution

(a) $f(-1) = 2^{-1} = \frac{1}{2}$ Replace x with -1. (Appendix C)

(b) $f(3) = 2^3 = 8$

(c) $f\left(\frac{5}{2}\right) = 2^{5/2} = (2^5)^{1/2} = 32^{1/2} = \sqrt{32} = \sqrt{16} \cdot \sqrt{2} = 4\sqrt{2}$

(d) $f(4.92) = 2^{4.92} \approx 30.2738447$ Use a calculator.

Now try Exercises 3, 9, and 11.

Exponential Functions We can now define a function $f(x) = a^x$ whose domain is the set of all real numbers.

Exponential Function

If $a > 0$ and $a \neq 1$, then

$$f(x) = a^x$$

defines the **exponential function** with base a.

NOTE If $a = 1$, the function becomes the constant function defined by $f(x) = 1$, not an exponential function.

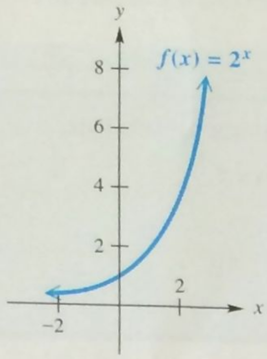


Figure 1 showed the graph of $f(x) = 2^x$ with three different domains. We repeat the final graph (with real numbers as domain) in the margin. The y-intercept is $y = 2^0 = 1$. Since $2^x > 0$ for all x and $2^x \to 0$ as $x \to -\infty$, the x-axis is a horizontal asymptote. As the graph suggests, the domain of the function is $(-\infty, \infty)$ and the range is $(0, \infty)$. The function is increasing on its entire domain, and therefore is one-to-one. Our observations from Figure 1 lead to the following generalizations about the graphs of exponential functions.

EXPONENTIAL FUNCTION $f(x) = a^x$

Domain: $(-\infty, \infty)$ Range: $(0, \infty)$

For $f(x) = 2^x$:

x	$f(x)$
-2	$\frac{1}{4}$
-1	$\frac{1}{2}$
0	1
1	2
2	4
3	8

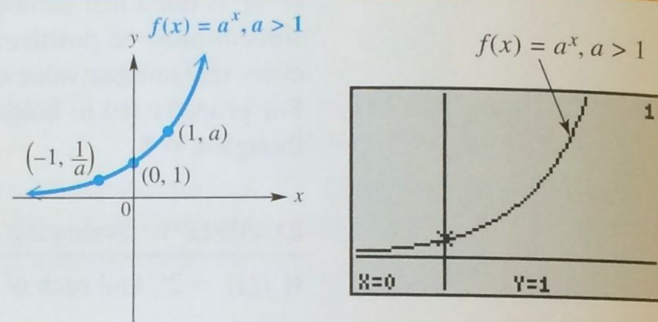


Figure 2

- $f(x) = a^x$, $a > 1$, is increasing and continuous on its entire domain, $(-\infty, \infty)$.
- The x-axis is a horizontal asymptote as $x \to -\infty$.
- The graph goes through the points $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, and $(1, a)$.

For $f(x) = \left(\frac{1}{2}\right)^x$:

x	$f(x)$
-3	8
-2	4
-1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$

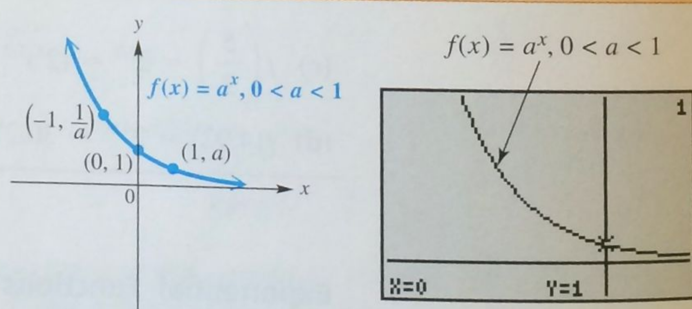
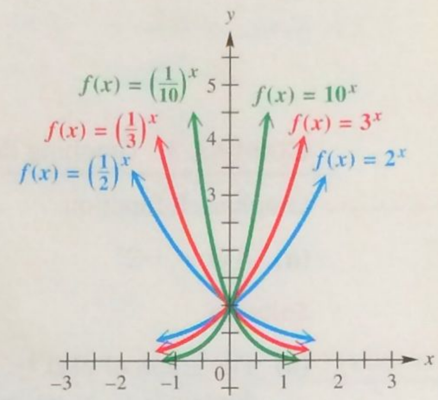


Figure 3

- $f(x) = a^x$, $0 < a < 1$, is decreasing and continuous on its entire domain, $(-\infty, \infty)$.
- The x-axis is a horizontal asymptote as $x \to \infty$.
- The graph goes through the points $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, and $(1, a)$.

Starting with $f(x) = 2^x$ and replacing x with $-x$ gives $f(-x) = 2^{-x} = (2^{-1})^x = \left(\frac{1}{2}\right)^x$. For this reason, the graphs of $f(x) = 2^x$ and $f(x) = \left(\frac{1}{2}\right)^x$ are reflections of each other across the y-axis. This is supported by the graphs in Figures 2 and 3.

The graph of $f(x) = 2^x$ is typical of graphs of $f(x) = a^x$ where $a > 1$. For larger values of a, the graphs rise more steeply, but the general shape is similar to the graph in Figure 2. When $0 < a < 1$, the graph decreases in a manner similar to the graph of $f(x) = \left(\frac{1}{2}\right)^x$. In Figure 4, the graphs of several typical exponential functions illustrate these facts.



$f(x) = a^x$
Domain: $(-\infty, \infty)$; Range: $(0, \infty)$
When $a > 1$, the function is increasing.
When $0 < a < 1$, the function is decreasing.

Figure 4

In summary, the graph of a function of the form $f(x) = a^x$ has the following features.

TEACHING TIP Encourage students to memorize the characteristics of the graph of $f(x) = a^x$. Ask them how the coordinates $(0, 1)$ and $(1, a)$ are affected by different variations of the graph of $f(x) = -a^{x-h} + k$.

Characteristics of the Graph of $f(x) = a^x$

1. The points $\left(-1, \frac{1}{a}\right)$, $(0, 1)$, and $(1, a)$ are on the graph.
2. If $a > 1$, then f is an increasing function; if $0 < a < 1$, then f is a decreasing function.
3. The x-axis is a horizontal asymptote.
4. The domain is $(-\infty, \infty)$, and the range is $(0, \infty)$.

EXAMPLE 2 Graphing an Exponential Function

Graph $f(x) = 5^x$.

Solution The y-intercept is 1, and the x-axis is a horizontal asymptote. Plot a few ordered pairs, and draw a smooth curve through them as shown in Figure 5 on the next page. Like the function $f(x) = 2^x$, this function also has domain $(-\infty, \infty)$ and range $(0, \infty)$ and is one-to-one. The graph is increasing on its entire domain.

x	$f(x)$
-1	.2
0	1
.5	2.2
1	5
1.5	11.2
2	25

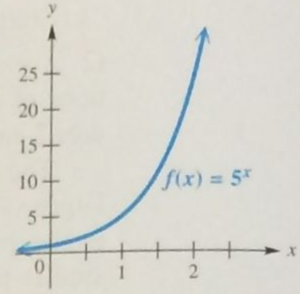


Figure 5

Now try Exercise 13.

EXAMPLE 3 Graphing Reflections and Translations

Graph each function.

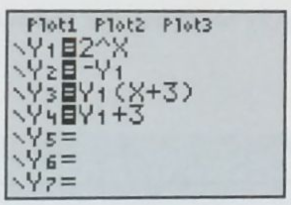
(a) $f(x) = -2^x$ **(b)** $f(x) = 2^{x+3}$ **(c)** $f(x) = 2^x + 3$

Solution

(a) The graph of $f(x) = -2^x$ is that of $f(x) = 2^x$ reflected across the x-axis. The domain is $(-\infty, \infty)$, and the range is $(-\infty, 0)$. See Figure 6.

(b) The graph of $f(x) = 2^{x+3}$ is the graph of $f(x) = 2^x$ translated 3 units to the left, as shown in Figure 7.

(c) The graph of $f(x) = 2^x + 3$ is that of $f(x) = 2^x$ translated 3 units up. See Figure 8.



The screen shows how a graphing calculator can be directed to graph the three functions in Example 3. Y_1 is defined as 2^X, and Y_2, Y_3, and Y_4 are defined as reflections and/or translations of Y_1.

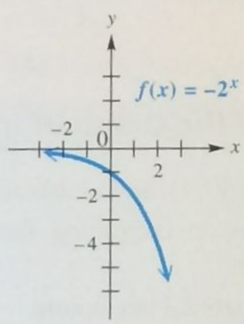


Figure 6

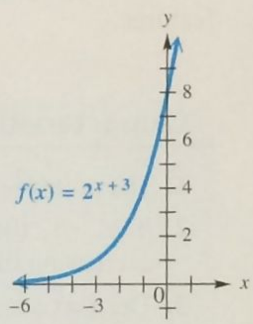


Figure 7

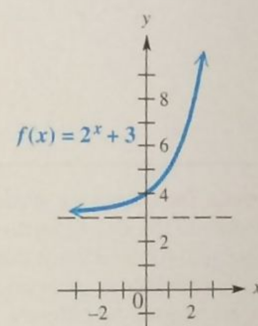


Figure 8

Now try Exercises 25 and 27.

TEACHING TIP Review the properties of exponents.

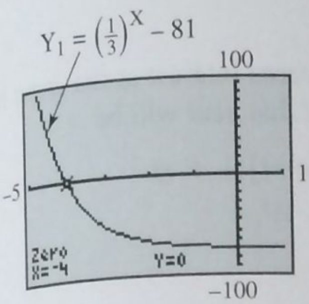
Exponential Equations

Exponential Equations Property (b) given at the beginning of this section is used to solve **exponential equations,** equations with variables as exponents.

$$2^x = 32, \quad 4^{3x-1} = 16^{x+2}, \quad \left(\frac{1}{27}\right)^{x-1} = 9^{2x} \quad \text{Exponential equations}$$

EXAMPLE 4 Using a Property of Exponents to Solve an Equation

Solve $\left(\frac{1}{3}\right)^x = 81.$



The screen supports the algebraic result in Example 4, using the x-intercept method of solution.

Solution Write each side of the equation using a common base.

$$\left(\frac{1}{3}\right)^x = 81$$

$$(3^{-1})^x = 81 \quad \text{Definition of negative exponent; } \frac{1}{a^n} = a^{-n}$$

$$3^{-x} = 81 \quad (a^m)^n = a^{mn}$$

$$3^{-x} = 3^4 \quad \text{Write 81 as a power of 3.}$$

$$-x = 4 \quad \text{Property (b)}$$

$$x = -4 \quad \text{Multiply by } -1.$$

The solution set of the original equation is $\{-4\}$.

Now try Exercise 45.

EXAMPLE 5 Using a Property of Exponents to Solve an Equation

Solve $2^{x+4} = 8^{x-6}$.

Solution Write each side of the equation using a common base.

$$2^{x+4} = 8^{x-6}$$

$$2^{x+4} = (2^3)^{x-6} \quad \text{Write 8 as a power of 2.}$$

$$2^{x+4} = 2^{3x-18} \quad (a^m)^n = a^{mn}$$

$$x + 4 = 3x - 18 \quad \text{Property (b)}$$

$$-2x = -22 \quad \text{Subtract } 3x \text{ and 4. (Appendix A)}$$

$$x = 11 \quad \text{Divide by } -2.$$

Check by substituting 11 for x in the original equation. The solution set is $\{11\}$.

Now try Exercise 53.

TEACHING TIP Give an example such as $16^{x-2} = 9^{1-x}$ that cannot be solved using properties of exponents. Show how a graphing calculator can be used to find the approximate solution $x = 1.558$. (Mention that Section 9.3 presents algebraic techniques for finding the solution.)

EXAMPLE 6 Using a Property of Exponents to Solve an Equation

Solve $81 = b^{4/3}$.

Solution

$$81 = b^{4/3}$$

$$81 = \left(\sqrt[3]{b}\right)^4 \quad a^{m/n} = \left(\sqrt[n]{a}\right)^m$$

$$\pm 3 = \sqrt[3]{b} \quad \text{Take fourth roots on both sides.}$$

$$\pm 27 = b \quad \text{Cube both sides.}$$

Check *both* solutions in the original equation. Both check, so the solution set is $\{-27, 27\}$.

Now try Exercise 61.

TEACHING TIP Point out that it is more convenient to write $b^{4/3}$ as $\left(\sqrt[3]{b}\right)^4$ instead of $\sqrt[3]{b^4}$.

Compound Interest Recall the formula for simple interest, $I = Prt$, where P is principal (amount deposited), r is annual rate of interest expressed as a decimal, and t is time in years that the principal earns interest. Suppose $t = 1$ yr. Then at the end of the year the amount has grown to

$$P + Pr = P(1 + r),$$

the original principal plus interest. If this balance earns interest at the same interest rate for another year, the balance at the end of that year will be

$$\begin{aligned}[P(1 + r)] + [P(1 + r)]r &= [P(1 + r)](1 + r)\\ &= P(1 + r)^2.\end{aligned}$$

After the third year, this will grow to

$$\begin{aligned}[P(1 + r)^2] + [P(1 + r)^2]r &= [P(1 + r)^2](1 + r)\\ &= P(1 + r)^3.\end{aligned}$$

Continuing in this way produces the formula

$$A = P(1 + r)^t$$

for interest compounded annually. The following general formula for *compound interest* can be derived in the same way as the formula given above.

Compound Interest

If P dollars are deposited in an account paying an annual rate of interest r compounded (paid) m times per year, then after t years the account will contain A dollars, where

$$A = P\left(1 + \frac{r}{m}\right)^{tm}.$$

In the formula for compound interest, A is sometimes called the **future value** and P the **present value.** A is also called the **compound amount** and is the balance *after* interest has been earned.

EXAMPLE 7 Using the Compound Interest Formula

Suppose \$1000 is deposited in an account paying 4% interest per year compounded quarterly (four times per year).

(a) Find the amount in the account after 10 yr with no withdrawals.

(b) How much interest is earned over the 10-yr period?

Solution

(a) $A = P\left(1 + \frac{r}{m}\right)^{tm}$ Compound interest formula

$A = 1000\left(1 + \frac{.04}{4}\right)^{10(4)}$ Let $P = 1000$, $r = .04$, $m = 4$, and $t = 10$.

$A = 1000(1 + .01)^{40} = 1488.86$ Round to the nearest cent.

Thus, \$1488.86 is in the account after 10 yr.

(b) The interest earned for that period is

$$\$1488.86 - \$1000 = \$488.86.$$

Now try Exercise 63(a).

EXAMPLE 8 Finding Present Value

Becky Anderson must pay a lump sum of \$6000 in 5 yr.

(a) What amount deposited today at 3.1% compounded annually will grow to \$6000 in 5 yr?

(b) If only \$5000 is available to deposit now, what annual interest rate is necessary for the money to increase to \$6000 in 5 yr?

Solution

(a)

$$A = P\left(1 + \frac{r}{m}\right)^{tm} \quad \text{Compound interest formula}$$

$$6000 = P\left(1 + \frac{.031}{1}\right)^{5(1)} \quad \text{Let } A = 6000,\ r = .031,\ m = 1, \text{ and } t = 5.$$

$$6000 = P(1.031)^5$$

$$P = \frac{6000}{(1.031)^5} \quad \text{Solve for } P\text{; divide by } (1.031)^5.$$

$$P \approx 5150.60 \quad \text{Use a calculator.}$$

If Becky leaves \$5150.60 for 5 yr in an account paying 3.1% compounded annually, she will have \$6000 when she needs it. We say that \$5150.60 is the present value of \$6000 if interest of 3.1% is compounded annually for 5 yr.

(b)

$$A = P\left(1 + \frac{r}{m}\right)^{tm}$$

$$6000 = 5000(1 + r)^5 \quad \text{Let } A = 6000,\ P = 5000,\ m = 1, \text{ and } t = 5.$$

$$\frac{6}{5} = (1 + r)^5 \quad \text{Divide by 5000.}$$

$$\left(\frac{6}{5}\right)^{1/5} = 1 + r \quad \text{Take the fifth root.}$$

$$\left(\frac{6}{5}\right)^{1/5} - 1 = r \quad \text{Subtract 1.}$$

$$r \approx .0371 \quad \text{Use a calculator.}$$

An interest rate of 3.71% will produce enough interest to increase the \$5000 to \$6000 by the end of 5 yr.

Now try Exercises 65 and 69.

m	$\left(1+\frac{1}{m}\right)^m$ (rounded)
1	2
2	2.25
5	2.48832
10	2.59374
25	2.66584
50	2.69159
100	2.70481
500	2.71557
1000	2.71692
10,000	2.71815
1,000,000	2.71828

The Number e and Continuous Compounding The more often interest is compounded within a given time period, the more interest will be earned. Surprisingly, however, there is a limit on the amount of interest, no matter how often it is compounded. To see this, suppose that \$1 is invested at 100% interest per year, compounded m times per year. Then the interest rate (in decimal form) is 1.00 and the interest rate per period is $\frac{1}{m}$. According to the formula (with $P = 1$), the compound amount at the end of 1 yr will be

$$A = \left(1 + \frac{1}{m}\right)^m.$$

A calculator gives the results in the margin for various values of m. The table suggests that as m increases, the value of $\left(1 + \frac{1}{m}\right)^m$ gets closer and closer to some fixed number. This is indeed the case. This fixed number is called e.

Value of e

To nine decimal places, $e \approx 2.718281828.$

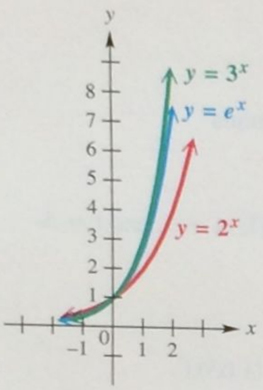


Figure 9

Figure 9 shows the functions defined by $y = 2^x$, $y = 3^x$, and $y = e^x$. Notice that because $2 < e < 3$, the graph of $y = e^x$ lies "between" the other two graphs.

As mentioned above, the amount of interest earned increases with the frequency of compounding, but the value of the expression $\left(1 + \frac{1}{m}\right)^m$ approaches e as m gets larger. Consequently, the formula for compound interest approaches a limit as well, called the compound amount from *continuous compounding*.

TEACHING TIP Tell students that e is defined as $\lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m$.

Continuous Compounding

If P dollars are deposited at a rate of interest r compounded continuously for t years, the compound amount in dollars on deposit is

$$A = Pe^{rt}.$$

EXAMPLE 9 Solving a Continuous Compounding Problem

Suppose \$5000 is deposited in an account paying 3% interest compounded continuously for 5 yr. Find the total amount on deposit at the end of 5 yr.

Solution

$A = Pe^{rt}$ — Continuous compounding formula

$= 5000e^{.03(5)}$ — Let $P = 5000$, $t = 5$, and $r = .03$.

$= 5000e^{.15}$

$\approx 5000(1.161834)$ — Use a calculator.

$= 5809.17$

or \$5809.17. Check that daily compounding would have produced a compound amount about 3¢ less.

Now try Exercise 63(b).

Looking Ahead to Calculus

In calculus, the derivative allows us to determine the slope of a tangent line to the graph of a function. For the function $f(x) = e^x$, the derivative is the function f itself: $f'(x) = e^x$. Therefore, in calculus the exponential function with base e is much easier to work with than exponential functions having other bases.

EXAMPLE 10 Comparing Interest Earned as Compounding Is More Frequent

In Example 7, we found that \$1000 invested at 4% compounded quarterly for 10 yr grew to \$1488.86. Compare this same investment compounded annually, semiannually, monthly, daily, and continuously.

Solution Substituting into the compound interest formula and the formula for continuous compounding gives the following results for amounts of \$1 and \$1000.

Compounded	\$1	\$1000
annually	$(1 + .04)^{10} \approx 1.48024$	\$1480.24
semiannually	$\left(1 + \frac{.04}{2}\right)^{10(2)} \approx 1.48595$	\$1485.95
quarterly	$\left(1 + \frac{.04}{4}\right)^{10(4)} \approx 1.48886$	\$1488.86
monthly	$\left(1 + \frac{.04}{12}\right)^{10(12)} \approx 1.49083$	\$1490.83
daily	$\left(1 + \frac{.04}{365}\right)^{10(365)} \approx 1.49179$	\$1491.79
continuously	$e^{10(.04)} \approx 1.49182$	\$1491.82

Compounding semiannually rather than annually increases the value of the account after 10 yr by \$5.71. Quarterly compounding grows to \$2.91 more than semiannual compounding after 10 yr. Each increase in the frequency of compounding earns less and less additional interest, until going from daily to continuous compounding increases the value by only \$.03.

Now try Exercise 71.

Exponential Models and Curve Fitting The number e is important as the base of an exponential function in many practical applications. For example, in situations involving growth or decay of a quantity, the amount or number present at time t often can be closely modeled by a function defined by

$$y = y_0 e^{kt},$$

where y_0 is the amount or number present at time $t = 0$ and k is a constant.

The next example, which refers to the problem stated at the beginning of this chapter, illustrates exponential growth.

EXAMPLE 11 Using Data to Model Exponential Growth

If current trends of burning fossil fuels and deforestation continue, then future amounts of atmospheric carbon dioxide in parts per million (ppm) will increase as shown in the table on the next page.

Year	Carbon Dioxide (ppm)
1990	353
2000	375
2075	590
2175	1090
2275	2000

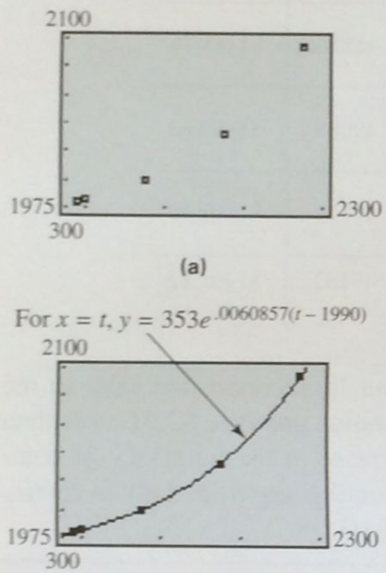
Source: International Panel on Climate Change (IPCC), 1990.

(a) Make a scatter diagram of the data. Do the carbon dioxide levels appear to grow exponentially?

(b) The function defined by

$$y = 353e^{.0060857(t-1990)}$$

is a good model for the data. Use a graph of this model to estimate when future levels of carbon dioxide will double and triple over the preindustrial level of 280 ppm.



(a)

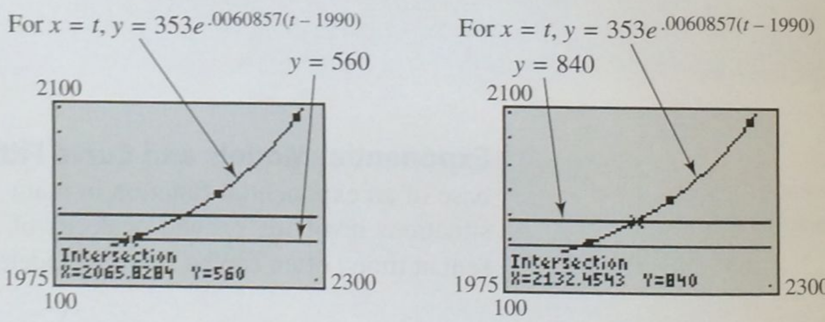
(b)

Figure 10

Solution

(a) We show a calculator graph for the data in Figure 10(a). The data appear to have the shape of the graph of an increasing exponential function.

(b) A graph of $y = 353e^{.0060857(t-1990)}$ in Figure 10(b) shows that it is very close to the data points. We graph $y = 2 \cdot 280 = 560$ in Figure 11(a) and $y = 3 \cdot 280 = 840$ in Figure 11(b) on the same coordinate axes as the given function, and use the calculator to find the intersection points.



(a)

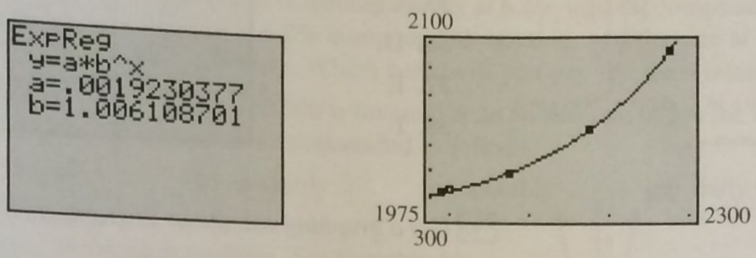
(b)

Figure 11

The graph of the function intersects the horizontal lines at approximately 2065.8 and 2132.5. According to this model, carbon dioxide levels will double by 2065 and triple by 2132.

Now try Exercise 73.

Graphing calculators are capable of fitting exponential curves to scatter diagrams like the one found in Example 11. Figure 12(a) shows how the TI-83 Plus displays another (different) equation for the atmospheric carbon dioxide example: $y = .0019 \cdot 1.0061^x$. (Coefficients are rounded here.) Notice that this calculator form differs from the model in Example 11. Figure 12(b) shows the data points and the graph of this exponential regression equation. ■



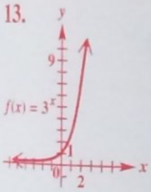
(a) (b)

Figure 12

9.1 Exercises

1. 9 2. 27 3. $\frac{1}{9}$ 4. $\frac{1}{27}$ 5. $\frac{1}{16}$ 6. $\frac{1}{64}$ 7. 16 8. 64 9. 5.196 10. $\frac{1}{8}$ 11. .039 12. 6.332

13.



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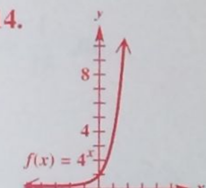

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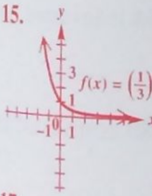

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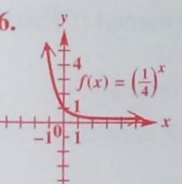

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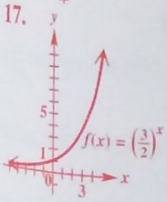

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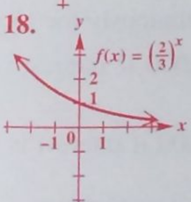


If $f(x) = 3^x$ and $g(x) = (\frac{1}{4})^x$, find each of the following. If a result is irrational, round the answer to the nearest thousandth. See Example 1.

1. $f(2)$ **2.** $f(3)$ **3.** $f(-2)$ **4.** $f(-3)$

5. $g(2)$ **6.** $g(3)$ **7.** $g(-2)$ **8.** $g(-3)$

9. $f\left(\frac{3}{2}\right)$ **10.** $g\left(\frac{3}{2}\right)$ **11.** $g(2.34)$ **12.** $f(1.68)$

Graph each function. See Example 2.

13. $f(x) = 3^x$ **14.** $f(x) = 4^x$ **15.** $f(x) = \left(\frac{1}{3}\right)^x$ **16.** $f(x) = \left(\frac{1}{4}\right)^x$

17. $f(x) = \left(\frac{3}{2}\right)^x$ **18.** $f(x) = \left(\frac{2}{3}\right)^x$ **19.** $f(x) = 10^x$ **20.** $f(x) = 10^{-x}$

21. $f(x) = 4^{-x}$ **22.** $f(x) = 6^{-x}$ **23.** $f(x) = 2^{|x|}$ **24.** $f(x) = 2^{-|x|}$

Sketch the graph of $f(x) = 2^x$. Then refer to it and use graphing techniques to graph each function as defined. See Example 3.

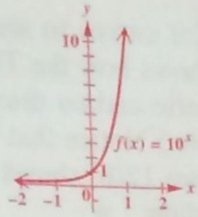
25. $f(x) = 2^x + 1$ **26.** $f(x) = 2^x - 4$ **27.** $f(x) = 2^{x+1}$ **28.** $f(x) = 2^{x-4}$

Sketch the graph of $f(x) = (\frac{1}{3})^x$. Then refer to it and use graphing techniques to graph each function as defined. See Example 3.

29. $f(x) = \left(\frac{1}{3}\right)^x - 2$ **30.** $f(x) = \left(\frac{1}{3}\right)^x + 4$

31. $f(x) = \left(\frac{1}{3}\right)^{x+2}$ **32.** $f(x) = \left(\frac{1}{3}\right)^{x-4}$

19.



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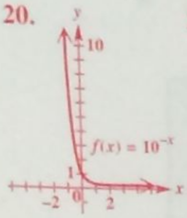

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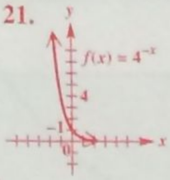

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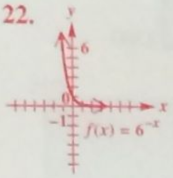

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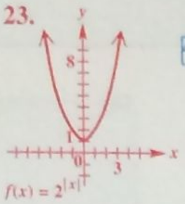

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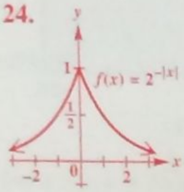

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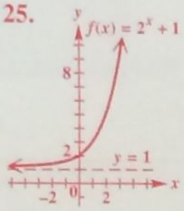

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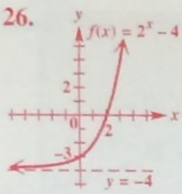

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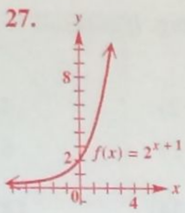

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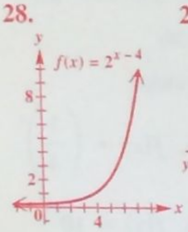

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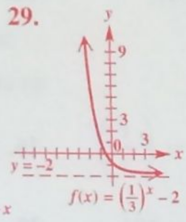

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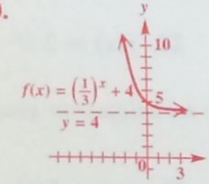

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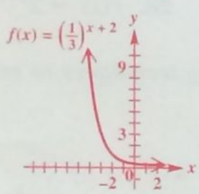
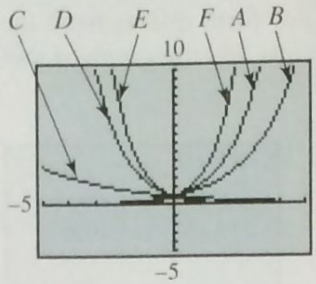


Concept Check *The graphs of $y = a^x$ for $a = 1.8, 2.3, 3.2, .4, .75,$ and $.31$ are given in the figure. They are identified by letter, but not necessarily in the same order as the values just given. Use your knowledge of how the exponential function behaves for various values of the base to identify each lettered graph.*

33. A
34. B
35. C
36. D
37. E
38. F



Use a graphing calculator to graph each function as defined.

39. $f(x) = \dfrac{e^x - e^{-x}}{2}$

40. $f(x) = \dfrac{e^x + e^{-x}}{2}$

41. $f(x) = x \cdot 2^x$

42. $f(x) = x^2 \cdot 2^{-x}$

Solve each equation. See Examples 4–6.

43. $4^x = 2$

44. $125^r = 5$

45. $\left(\frac{1}{2}\right)^k = 4$

46. $\left(\frac{2}{3}\right)^x = \frac{9}{4}$

47. $2^{3-y} = 8$

48. $5^{2p+1} = 25$

49. $e^{4x-1} = (e^2)^x$

50. $e^{3-x} = (e^3)^{-x}$

51. $27^{4z} = 9^{z+1}$

52. $32^t = 16^{1-t}$

53. $4^{x-2} = 2^{3x+3}$

54. $\left(\frac{1}{2}\right)^{3x-6} = 8^{x+1}$

55. $\left(\frac{1}{e}\right)^{-x} = \left(\frac{1}{e^2}\right)^{x+1}$

56. $e^{k-1} = \left(\frac{1}{e^4}\right)^{k+1}$

57. $(\sqrt{2})^{x+4} = 4^x$

58. $(\sqrt[3]{5})^{-x} = \left(\frac{1}{5}\right)^{x+2}$

59. $\frac{1}{27} = b^{-3}$

60. $\frac{1}{81} = k^{-4}$

61. $4 = r^{2/3}$

62. $z^{5/2} = 32$

Solve each problem involving compound interest. See Examples 7–9.

63. **Future Value** Find the future value and interest earned if \$8906.54 is invested for 9 yr at 5% compounded
(a) semiannually (b) continuously.

64. **Future Value** Find the future value and interest earned if \$56,780 is invested at 5.3% compounded
(a) quarterly for 23 quarters (b) continuously for 15 yr.

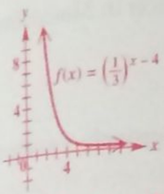
65. **Present Value** Find the present value of \$25,000 if interest is 6% compounded quarterly for 11 quarters.

66. **Present Value** Find the present value of \$45,000 if interest is 3.6% compounded monthly for 1 yr.

67. **Present Value** Find the present value of \$5000 if interest is 3.5% compounded quarterly for 10 yr.

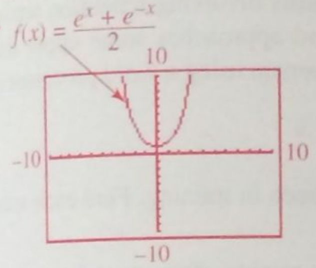
68. **Interest Rate** Find the required annual interest rate to the nearest tenth of a percent for \$65,000 to grow to \$65,325 if interest is compounded monthly for 6 months.

32.



33. 2.3 **34.** 1.8 **35.** .75
36. .4 **37.** .31 **38.** 3.2
Answer graphs for Exercises 39 and 41 are given on page A-26 of the answer section at the back of the text.

40.



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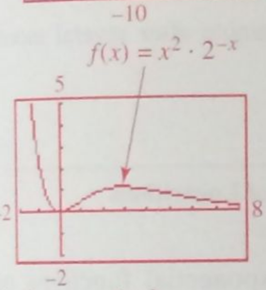


43. $\left\{\frac{1}{2}\right\}$ **44.** $\left\{\frac{1}{3}\right\}$ **45.** $\{-2\}$
46. $\{-2\}$ **47.** $\{0\}$ **48.** $\left\{\frac{1}{2}\right\}$
49. $\left\{\frac{1}{2}\right\}$ **50.** $\left\{-\frac{3}{2}\right\}$
51. $\left\{\frac{1}{5}\right\}$ **52.** $\left\{\frac{4}{9}\right\}$ **53.** $\{-7\}$
54. $\left\{\frac{1}{2}\right\}$ **55.** $\left\{-\frac{2}{3}\right\}$
56. $\left\{-\frac{3}{5}\right\}$ **57.** $\left\{\frac{4}{3}\right\}$
58. $\{-3\}$ **59.** $\{3\}$ **60.** $\{-3, 3\}$
61. $\{-8, 8\}$ **62.** $\{4\}$
63. (a) \$13,891.16; \$4984.62 (b) \$13,968.24; \$5061.70
64. (a) \$76,855.95; \$20,075.95 (b) \$125,735.96; \$68,955.96
65. \$21,223.33 **66.** \$43,411.15
67. \$3528.81 **68.** 1.0%
69. 4.5% **70.** 6.5%
71. Bank A (even though it has the highest stated rate)
72. (a) \$16,288.95 (b) \$16,436.19 (c) \$16,470.09 (d) \$16,486.65

69. ***Interest Rate*** Find the required annual interest rate to the nearest tenth of a percent for \$1200 to grow to \$1500 if interest is compounded quarterly for 5 yr.

70. ***Interest Rate*** Find the required annual interest rate to the nearest tenth of a percent for \$5000 to grow to \$8400 if interest is compounded quarterly for 8 yr.

Solve each problem. See Example 10.

71. ***Comparing Loans*** Bank A is lending money at 6.4% interest compounded annually. The rate at Bank B is 6.3% compounded monthly, and the rate at Bank C is 6.35% compounded quarterly. Which bank will you pay the *least* interest?

72. ***Future Value*** Suppose \$10,000 is invested at an annual rate of 5% for 10 yr. Find the future value if interest is compounded as follows.

(a) annually **(b)** quarterly **(c)** monthly **(d)** daily (365 days)

(Modeling) Solve each problem. See Example 11.

73. ***Atmospheric Pressure*** The atmospheric pressure (in millibars) at a given altitude (in meters) is shown in the table.

Altitude	Pressure	Altitude	Pressure
0	1013	6000	472
1000	899	7000	411
2000	795	8000	357
3000	701	9000	308
4000	617	10,000	265
5000	541		

Source: Miller, A. and J. Thompson, *Elements of Meteorology*, Fourth Edition, Charles E. Merrill Publishing Company, Columbus, Ohio, 1993.

(a) Use a graphing calculator to make a scatter diagram of the data for atmospheric pressure P at altitude x.

(b) Would a linear or exponential function fit the data better?

(c) The function defined by

$$P(x) = 1013e^{-.0001341x}$$

approximates the data. Use a graphing calculator to graph P and the data on the same coordinate axes.

(d) Use P to predict the pressures at 1500 m and 11,000 m, and compare them to the actual values of 846 millibars and 227 millibars, respectively.

74. ***World Population Growth*** Since 1990, world population in millions closely fits the exponential function defined by

$$y = 6073e^{.0137x},$$

where x is the number of years since 1990.

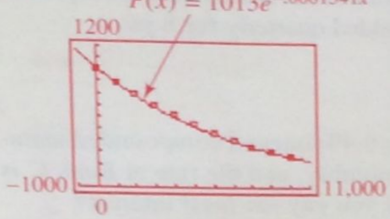
(a) The world population was about 6079 million in 2000. How closely does the function approximate this value?

(b) Use this model to approximate the population in 2005.

(c) Use this model to predict the population in 2015.

(d) Explain why this model may not be accurate for 2015.

73. **(a)** See the answer graph for part (c). **(b)** exponential
(c)



(d) $P(1500) \approx 828$ mb; $P(11{,}000) \approx 232$ mb **74.** **(a)** The function gives approximately 6965 million, which differs by 886 million from the actual value. **(b)** 7458 million **(c)** 8554 million **75.** **(a)** about 63,000 **(b)** about 42,000 **(c)** about 21,000 **76.** **(a)** about 207 **(b)** about 235 **(c)** about 249 **(d)** The number of symbols approaches 250.

75. ***Deer Population*** The exponential growth of the deer population in Massachusetts can be calculated using the model

$$T = 50{,}000(1 + .06)^n,$$

where 50,000 is the initial deer population and .06 is the rate of growth. T is the total population after n years have passed.

(a) Predict the total population after 4 yr.

(b) If the initial population was 30,000 and the growth rate was .12, approximately how many deer would be present after 3 yr?

(c) How many additional deer can we expect in 5 yr if the initial population is 45,000 and the current growth rate is .08?

76. ***Employee Training*** A person learning certain skills involving repetition tends to learn quickly at first. Then learning tapers off and approaches some upper limit. Suppose the number of symbols per minute that a person using a word processor can type is given by

$$p(t) = 250 - 120(2.8)^{-.5t},$$

where t is the number of months the operator has been in training. Find each value.

(a) $p(2)$ **(b)** $p(4)$ **(c)** $p(10)$

(d) What happens to the number of symbols per minute after several months of training?

9.2 Logarithmic Functions

Logarithms ▪ **Logarithmic Equations** ▪ **Logarithmic Functions** ▪ **Properties of Logarithms**

Logarithms The previous section dealt with exponential functions of the form $y = a^x$ for all positive values of a, where $a \neq 1$. The horizontal line test shows that exponential functions are one-to-one, and thus have inverse functions. The equation defining the inverse of a function is found by interchanging x and y in the equation that defines the function. Doing so with

$$y = a^x \quad \text{gives} \quad x = a^y \quad \text{(Section 6.1)}$$

as the equation of the inverse function of the exponential function defined by $y = a^x$. This equation can be solved for y by using the following definition.

Logarithm

For all real numbers y, and all positive numbers a and x, where $a \neq 1$:

$$y = \log_a x \quad \text{if and only if} \quad x = a^y.$$

Exponent
↓
Logarithmic form: $y = \log_a x$
↑
Base

Exponent
↓
Exponential form: $a^y = x$
↑
Base

The "log" in the definition above is an abbreviation for **logarithm.** Read $\log_a x$ as "the logarithm to the base a of x."

By the definition of logarithm, if $y = \log_a x$, then the power to which a must be raised to obtain x is y, or $x = a^y$. To remember the location of the base and the exponent in each form, refer to the diagrams in the margin.

Meaning of $\log_a x$

A logarithm is an exponent. The expression $\log_a x$ is the exponent to which the base a must be raised to obtain x.