9.3 Evaluating Logarithms; Equations and Applications

Common Logarithms ▪ Natural Logarithms ▪ Logarithms to Other Bases ▪ Exponential and Logarithmic Equations

Common Logarithms The two most important bases for logarithms are 10 and e. Base 10 logarithms are called **common logarithms.** The common logarithm of x is written $\log x$, where the base is understood to be 10.

Common Logarithm

For all positive numbers x, $\quad \log x = \log_{10} x$.

A calculator with a log key can be used to find the base 10 logarithm of any positive number. Figure 20 shows how a graphing calculator displays common logarithms. A common logarithm of a power of 10, such as 1000, is an integer (in this case, 3). Most common logarithms used in applications, such as log 142 and log .005832, are irrational numbers.

Figure 21 reinforces the concept presented in the previous section: $\log x$ is the exponent to which 10 must be raised in order to obtain x.

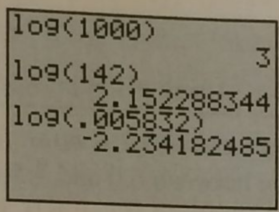


Figure 20

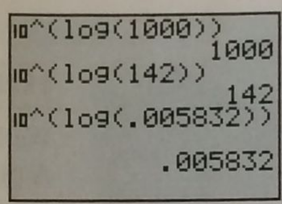


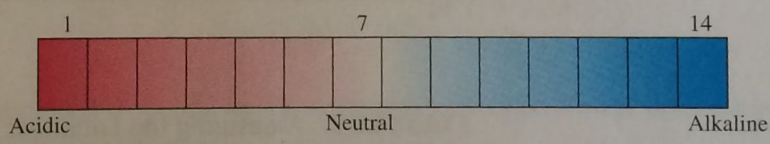
Figure 21

NOTE Base a, $a > 1$, logarithms of numbers less than 1 are always negative, as suggested by the graphs in Section 9.2.

In chemistry, the **pH** of a solution is defined using logarithms as

$$\text{pH} = -\log[H_3O^+],$$

where $[H_3O^+]$ is the hydronium ion concentration in moles* per liter. The pH value is a measure of the acidity or alkalinity of a solution. Pure water has pH 7.0, substances with pH values greater than 7.0 are alkaline, and substances with pH values less than 7.0 are acidic. It is customary to round pH values to the nearest tenth.



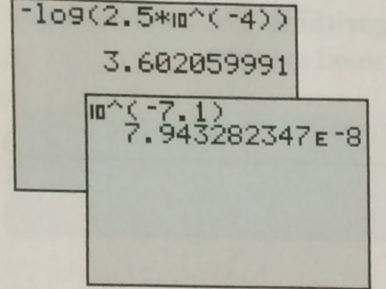
*A *mole* is the amount of a substance that contains the same number of molecules as the number of atoms in exactly 12 grams of carbon 12.

EXAMPLE 1 Finding pH

(a) Find the pH of a solution with $[H_3O^+] = 2.5 \times 10^{-4}$.

(b) Find the hydronium ion concentration of a solution with pH = 7.1.

Solution



The screens show how a graphing calculator evaluates the pH and $[H_3O^+]$ in Example 1.

(a)

$$\begin{aligned} pH &= -\log[H_3O^+] \\ &= -\log(2.5 \times 10^{-4}) && \text{Substitute.} \\ &= -(\log 2.5 + \log 10^{-4}) && \text{Product property (Section 9.2)} \\ &= -(.3979 - 4) && \log 10^{-4} = -4 \text{ (Section 9.2)} \\ &= -.3979 + 4 && \text{Distributive property} \\ pH &\approx 3.6 \end{aligned}$$

(b)

$$\begin{aligned} pH &= -\log[H_3O^+] \\ 7.1 &= -\log[H_3O^+] && \text{Substitute.} \\ -7.1 &= \log[H_3O^+] && \text{Multiply by } -1. \\ [H_3O^+] &= 10^{-7.1} && \text{Write in exponential form. (Section 9.2)} \\ [H_3O^+] &\approx 7.9 \times 10^{-8} && \text{Evaluate } 10^{-7.1} \text{ with a calculator.} \end{aligned}$$

Now try Exercises 23 and 27.

EXAMPLE 2 Using pH in an Application

Wetlands are classified as *bogs*, *fens*, *marshes*, and *swamps* based on pH values. A pH value between 6.0 and 7.5 indicates that the wetland is a "rich fen." When the pH is between 4.0 and 6.0, it is a "poor fen," and if the pH falls to 3.0 or less, the wetland is a "bog." (*Source:* R. Mohlenbrock, "Summerby Swamp, Michigan," *Natural History,* March 1994.)

Suppose that the hydronium ion concentration of a sample of water from a wetland is 6.3×10^{-5}. How would this wetland be classified?

Solution

$$\begin{aligned} pH &= -\log[H_3O^+] && \text{Definition of pH} \\ &= -\log(6.3 \times 10^{-5}) && \text{Substitute.} \\ &= -(\log 6.3 + \log 10^{-5}) && \text{Product property} \\ &= -\log 6.3 - (-5) && \text{Distributive property; } \log 10^n = n \\ &= -\log 6.3 + 5 \\ pH &\approx 4.2 && \text{Use a calculator.} \end{aligned}$$

Since the pH is between 4.0 and 6.0, the wetland is a poor fen.

Now try Exercise 31.

EXAMPLE 3 Measuring the Loudness of Sound

The loudness of sounds is measured in a unit called a *decibel.* To measure with this unit, we assign intensity I_0 to a very faint sound, called the *threshold sound.* If a particular sound has intensity I, then the decibel rating of this louder sound is $d = 10 \log \frac{I}{I_0}$. Find the decibel rating of a sound with intensity $10{,}000I_0$.

Solution

$$d = 10 \log \frac{10{,}000 I_0}{I_0} \qquad \text{Let } I = 10{,}000 I_0.$$
$$= 10 \log 10{,}000$$
$$= 10(4) \qquad \log 10{,}000 = \log 10^4 = 4 \text{ (Section 9.2)}$$
$$= 40$$

The sound has a decibel rating of 40.

Now try Exercise 47.

Looking Ahead to Calculus

The natural logarithmic function defined by $f(x) = \ln x$ and the reciprocal function defined by $g(x) = \frac{1}{x}$ have an important relationship in calculus. The derivative of the natural logarithmic function is the reciprocal function. Using *Leibniz notation* (named after one of the co-inventors of calculus), this fact is written $\frac{d}{dx}(\ln x) = \frac{1}{x}$.

Natural Logarithms In Section 9.1 we introduced the irrational number e. In most practical applications of logarithms, e is used as base. Logarithms to base e are called **natural logarithms,** since they occur in the life sciences and economics in natural situations that involve growth and decay. The base e logarithm of x is written $\ln x$ (read "el-en x").

Natural Logarithm

For all positive numbers x, $\quad \ln x = \log_e x.$

A graph of the natural logarithmic function defined by $f(x) = \ln x$ is given in Figure 22.

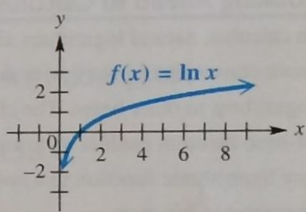


Figure 22

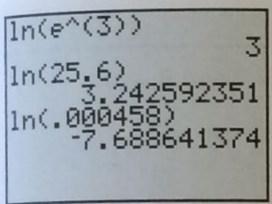


Figure 23

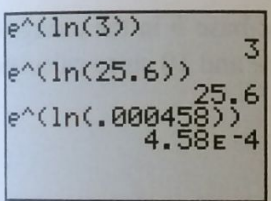


Figure 24

Natural logarithms can be found using a calculator. (Consult your owner's manual.) As in the case of common logarithms, when used in applications natural logarithms are usually irrational numbers. Figure 23 shows how three natural logarithms are evaluated with a graphing calculator. Figure 24 reinforces the fact that $\ln x$ is the exponent to which e must be raised in order to obtain x. ■

EXAMPLE 4 Measuring the Age of Rocks

Geologists sometimes measure the age of rocks by using "atomic clocks." By measuring the amounts of potassium 40 and argon 40 in a rock, the age t of the specimen in years is found with the formula

$$t = (1.26 \times 10^9)\frac{\ln\left[1 + 8.33\left(\frac{A}{K}\right)\right]}{\ln 2},$$

where A and K are the numbers of atoms of argon 40 and potassium 40, respectively, in the specimen.

(a) How old is a rock in which $A = 0$ and $K > 0$?

(b) The ratio $\frac{A}{K}$ for a sample of granite from New Hampshire is .212. How old is the sample?

Solution

(a) If $A = 0$, $\frac{A}{K} = 0$ and the equation becomes

$$t = (1.26 \times 10^9)\frac{\ln 1}{\ln 2} = (1.26 \times 10^9)(0) = 0.$$

The rock is new (0 yr old).

(b) Since $\frac{A}{K} = .212$, we have

$$t = (1.26 \times 10^9)\frac{\ln[1 + 8.33(.212)]}{\ln 2} \approx 1.85 \times 10^9.$$

The granite is about 1.85 billion yr old.

Now try Exercise 53.

Logarithms to Other Bases We can use a calculator to find the values of either natural logarithms (base e) or common logarithms (base 10). However, sometimes we must use logarithms to other bases. The following theorem can be used to convert logarithms from one base to another.

Looking Ahead to Calculus

In calculus, natural logarithms are more convenient to work with than logarithms to other bases. The change-of-base theorem enables us to convert any logarithmic function to a *natural* logarithmic function.

Change-of-Base Theorem

For any positive real numbers x, a, and b, where $a \neq 1$ and $b \neq 1$:

$$\log_a x = \frac{\log_b x}{\log_b a}.$$

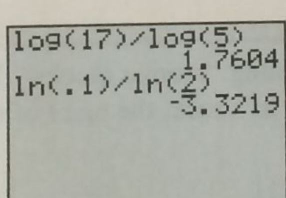
TEACHING TIP Show students that $\log_2 10$ can be evaluated using $\frac{\log 10}{\log 2}$ or $\frac{\ln 10}{\ln 2}$.

Any positive number other than 1 can be used for base b in the change-of-base theorem, but usually the only practical bases are e and 10 since calculators give logarithms only for these two bases.

EXAMPLE 5 Using the Change-of-Base Theorem

Use the change-of-base theorem to find an approximation to four decimal places for each logarithm.

(a) $\log_5 17$ **(b)** $\log_2 .1$



The screen shows how the result of Example 5(a) can be found using *common* logarithms, and how the result of Example 5(b) can be found using *natural* logarithms. The results are the same as those in Example 5.

Solution

(a) $\log_5 17 = \frac{\ln 17}{\ln 5} \approx \frac{2.8332}{1.6094} \approx 1.7604$ Using natural logarithms

(b) $\log_2 .1 = \frac{\log .1}{\log 2} \approx \frac{-1.0000}{.3010} \approx -3.3219$ Using common logarithms

Now try Exercises 35 and 37.

NOTE In Example 5, we showed logarithms such as ln 17 and ln 5 evaluated in intermediate steps to four decimal places. However, the final answers were obtained *without* rounding these intermediate values, using all the digits obtained with the calculator. In general, wait until the final step to round off the answer; otherwise, a build-up of round-off errors may cause the final answer to have an incorrect final decimal place digit.

EXAMPLE 6 Modeling Diversity of Species

One measure of the diversity of the species in an ecological community is modeled by the formula

$$H = -[P_1 \log_2 P_1 + P_2 \log_2 P_2 + \cdots + P_n \log_2 P_n],$$

where $P_1, P_2, \ldots, P_n$ are the proportions of a sample that belong to each of n species found in the sample. (*Source:* Ludwig, J. and J. Reynolds, *Statistical Ecology: A Primer on Methods and Computing,* New York, Wiley, 1988, p. 92.)

Find the measure of diversity in a community with two species where there are 90 of one species and 10 of the other.

Solution Since there are 100 members in the community, $P_1 = \frac{90}{100} = .9$ and $P_2 = \frac{10}{100} = .1$, so

$$H = -[.9 \log_2 .9 + .1 \log_2 .1].$$

In Example 5(b), we found that $\log_2 .1 \approx -3.32$. Now we find $\log_2 .9$.

$$\log_2 .9 = \frac{\log .9}{\log 2} \approx \frac{-.0458}{.3010} \approx -.152$$

Therefore,

$$\begin{aligned} H &= -[.9 \log_2 .9 + .1 \log_2 .1] \\ &\approx -[.9(-.152) + .1(-3.32)] \approx .469. \end{aligned}$$

Verify that $H \approx .971$ if there are 60 of one species and 40 of the other. As the proportions of n species get closer to $\frac{1}{n}$ each, the measure of diversity increases to a maximum of $\log_2 n$.

Now try Exercise 57.

Exponential and Logarithmic Equations We solved exponential equations in earlier sections. General methods for solving these equations depend on the property below, which follows from the fact that logarithmic functions are one-to-one.

TEACHING TIP Point out that, in practice, x and y are each algebraic expressions. The process of going from $x = y$ to $\log_a x = \log_a y$ is referred to as "taking logarithms on both sides of the equation."

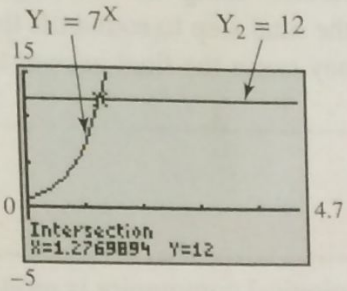
Property of Logarithms

If $x > 0$, $y > 0$, $a > 0$, and $a \neq 1$, then

$$x = y \quad \text{if and only if} \quad \log_a x = \log_a y.$$

EXAMPLE 7 Solving an Exponential Equation

Solve $7^x = 12$. Give the solution to four decimal places.



As seen in the display at the bottom of the screen, when rounded to four decimal places, the solution agrees with that found in Example 7.

Solution The properties of exponents given in Section 9.1 cannot be used to solve this equation, so we apply the preceding property of logarithms. While any appropriate base b can be used, the best practical base is base 10 or base e. We choose base e (natural) logarithms here.

$$7^x = 12$$

$$\ln 7^x = \ln 12 \quad \text{Property of logarithms}$$

$$x \ln 7 = \ln 12 \quad \text{Power property (Section 9.2)}$$

$$x = \frac{\ln 12}{\ln 7} \quad \text{Divide by } \ln 7.$$

$$x \approx 1.2770 \quad \text{Use a calculator.}$$

The solution set is $\{1.2770\}$.

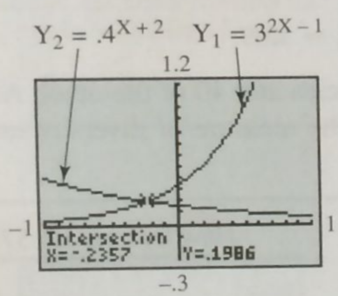
Now try Exercise 63.

CAUTION Be careful when evaluating a quotient like $\frac{\ln 12}{\ln 7}$ in Example 7. Do not confuse this quotient with $\ln \frac{12}{7}$, which can be written as $\ln 12 - \ln 7$. You *cannot* change the quotient of *two logarithms* to a difference of logarithms.

$$\frac{\ln 12}{\ln 7} \neq \ln \frac{12}{7}$$

EXAMPLE 8 Solving an Exponential Equation

Solve $3^{2x-1} = .4^{x+2}$. Give the solution to four decimal places.



This screen supports the solution found in Example 8.

TEACHING TIP Caution students against applying the quotient property of logarithms incorrectly here.

Solution

$$3^{2x-1} = .4^{x+2}$$

$$\ln 3^{2x-1} = \ln .4^{x+2} \quad \text{Take natural logarithms on both sides.}$$

$$(2x - 1) \ln 3 = (x + 2) \ln .4 \quad \text{Power property}$$

$$2x \ln 3 - \ln 3 = x \ln .4 + 2 \ln .4 \quad \text{Distributive property}$$

$$2x \ln 3 - x \ln .4 = 2 \ln .4 + \ln 3 \quad \text{Write the terms with } x \text{ on one side.}$$

$$x(2 \ln 3 - \ln .4) = 2 \ln .4 + \ln 3 \quad \text{Factor out } x.$$

$$x = \frac{2 \ln .4 + \ln 3}{2 \ln 3 - \ln .4} \quad \text{Divide by } 2 \ln 3 - \ln .4.$$

$$x = \frac{\ln .16 + \ln 3}{\ln 9 - \ln .4} \quad \text{Power property}$$

$$x = \frac{\ln .48}{\ln \frac{9}{.4}} \quad \text{Product property, Quotient property (Section 9.2)}$$

$$x \approx -.2357 \quad \text{Use a calculator.}$$

The solution set is $\{-.2357\}$.

Now try Exercise 67.

EXAMPLE 9 Solving Base e Exponential Equations

Solve each equation. Give solutions to four decimal places.

(a) $e^{x^2} = 200$ **(b)** $e^{2x+1} \cdot e^{-4x} = 3e$

Solution

(a)

$$e^{x^2} = 200$$

$$\ln e^{x^2} = \ln 200 \quad \text{Take natural logarithms on both sides.}$$

$$x^2 = \ln 200 \quad \ln e^{x^2} = x^2 \text{ (Section 9.2)}$$

$$x = \pm\sqrt{\ln 200} \quad \text{Square root property (Appendix A)}$$

$$x \approx \pm 2.3018 \quad \text{Use a calculator.}$$

The solution set is $\{\pm 2.3018\}$.

(b)

$$e^{2x+1} \cdot e^{-4x} = 3e$$

$$e^{-2x+1} = 3e \quad a^m \cdot a^n = a^{m+n}$$

$$e^{-2x} = 3 \quad \text{Divide by } e; \frac{a^m}{a^n} = a^{m-n}.$$

$$\ln e^{-2x} = \ln 3 \quad \text{Take natural logarithms on both sides.}$$

$$-2x \ln e = \ln 3 \quad \text{Power property}$$

$$-2x = \ln 3 \quad \ln e = 1$$

$$x = -\frac{1}{2}\ln 3 \quad \text{Multiply by } -\tfrac{1}{2}.$$

$$x \approx -.5493$$

The solution set is $\{-.5493\}$.

Now try Exercises 69 and 75.

EXAMPLE 10 Solving a Logarithmic Equation

TEACHING TIP Tell students that a logarithmic equation cannot be solved with two logarithms on the same side of the equation. The equation must be transformed so that each side contains an expression with at most one logarithm.

Solve $\log_a(x + 6) - \log_a(x + 2) = \log_a x$.

Solution

$$\log_a \frac{x+6}{x+2} = \log_a x \quad \text{Rewrite the equation using the quotient property.}$$

$$\frac{x+6}{x+2} = x \quad \text{Property of logarithms}$$

$$x + 6 = x(x + 2) \quad \text{Multiply by } x + 2.$$

$$x + 6 = x^2 + 2x \quad \text{Distributive property}$$

$$x^2 + x - 6 = 0 \quad \text{Standard form}$$

$$(x + 3)(x - 2) = 0 \quad \text{Factor.}$$

$$x = -3 \quad \text{or} \quad x = 2 \quad \text{Zero-factor property (Appendix A)}$$

The negative solution ($x = -3$) is not in the domain of $\log_a x$ in the original equation, so the only valid solution is the number 2, giving the solution set $\{2\}$.

Now try Exercise 81.

CAUTION Recall that the domain of $y = \log_b x$ is $(0, \infty)$. For this reason, ***it is always necessary to check that apparent solutions of a logarithmic equation result in logarithms of positive numbers in the original equation.***

TEACHING TIP Point out that the definition of logarithm could have been used in Example 11 by first writing

$$\log(3x + 2) + \log(x - 1) = 1$$
$$\log_{10}[(3x + 2)(x - 1)] = 1$$
$$(3x + 2)(x - 1) = 10^1,$$

then continuing as shown in the example.

EXAMPLE 11 Solving a Logarithmic Equation

Solve $\log(3x + 2) + \log(x - 1) = 1$.

Solution The notation $\log x$ is an abbreviation for $\log_{10} x$, and $1 = \log_{10} 10$.

$$\log(3x + 2) + \log(x - 1) = 1$$
$$\log(3x + 2) + \log(x - 1) = \log 10 \quad \text{Substitute.}$$
$$\log[(3x + 2)(x - 1)] = \log 10 \quad \text{Product property}$$
$$(3x + 2)(x - 1) = 10 \quad \text{Property of logarithms}$$
$$3x^2 - x - 2 = 10 \quad \text{Multiply.}$$
$$3x^2 - x - 12 = 0 \quad \text{Subtract 10.}$$
$$x = \frac{1 \pm \sqrt{1 + 144}}{6} \quad \text{Quadratic formula (Appendix A)}$$

The number $\frac{1 - \sqrt{145}}{6}$ is negative, so $x - 1$ is negative. Therefore, $\log(x - 1)$ is not defined and this proposed solution must be discarded. Since $\frac{1 + \sqrt{145}}{6} > 1$, both $3x + 2$ and $x - 1$ are positive and the solution set is $\left\{\frac{1 + \sqrt{145}}{6}\right\}$.

Now try Exercise 85.

EXAMPLE 12 Solving a Base e Logarithmic Equation

Solve $\ln e^{\ln x} - \ln(x - 3) = \ln 2$.

Solution

$$\ln e^{\ln x} - \ln(x - 3) = \ln 2$$
$$\ln x - \ln(x - 3) = \ln 2 \quad e^{\ln x} = x \text{ (Section 9.2)}$$
$$\ln \frac{x}{x - 3} = \ln 2 \quad \text{Quotient property}$$
$$\frac{x}{x - 3} = 2 \quad \text{Property of logarithms}$$
$$x = 2(x - 3) \quad \text{Multiply by } x - 3.$$
$$x = 2x - 6 \quad \text{Distributive property}$$
$$6 = x \quad \text{Solve for } x. \text{ (Appendix A)}$$

Verify that the solution set is $\{6\}$.

Now try Exercise 91.

A summary of the methods used for solving equations in this section follows.

Solving Exponential or Logarithmic Equations

To solve an exponential or logarithmic equation, change the given equation into one of the following forms, where a and b are real numbers, $a > 0$, and $a \neq 1$.

1. $a^{f(x)} = b$
Solve by taking logarithms on both sides.

2. $\log_a f(x) = b$
Solve by changing to exponential form $a^b = f(x)$.

3. $\log_a f(x) = \log_a g(x)$
The given equation is equivalent to the equation $f(x) = g(x)$. Solve algebraically.

4. In a more complicated equation, such as the one in Example 9(b), it may be necessary to first solve for $a^{f(x)}$ or $\log_a f(x)$ and then solve the resulting equation using one of the methods given above.

9.3 Exercises

1. increasing **2.** increasing
3. $f^{-1}(x) = \log_5 x$ **4.** $\log_4 11$
5. natural; common
6. $\frac{\ln 12}{\ln 3}$ **7.** There is no power of 2 that yields a result of 0.
8. 3 and 4
9. $\log 8 = .90308999$

Concept Check Answer each of the following.

1. For the exponential function defined by $f(x) = a^x$, where $a > 1$, is the function increasing or is it decreasing over its entire domain?

2. For the logarithmic function defined by $g(x) = \log_a x$, where $a > 1$, is the function increasing or is it decreasing over its entire domain?

3. If $f(x) = 5^x$, what is the rule for $f^{-1}(x)$?

4. What is the name given to the exponent to which 4 must be raised in order to obtain 11?

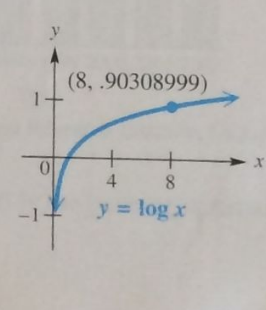
5. A base e logarithm is called a(n) ________ logarithm; a base 10 logarithm is called a(n) ________ logarithm.

6. How is $\log_3 12$ written in terms of natural logarithms?

7. Why is $\log_2 0$ undefined?

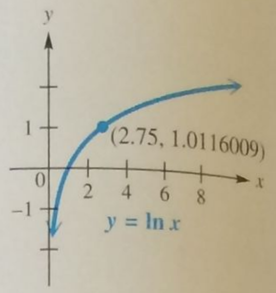
8. Between what two consecutive integers must $\log_2 12$ lie?

9. The graph of $y = \log x$ shows a point on the graph. Write the logarithmic equation associated with that point.



10. $\ln 2.75 = 1.0116009$
11. 1.5563 **12.** 1.8573
13. −1.3768 **14.** −.4962
15. 4.3010 **16.** −3.6990
17. 3.5835 **18.** 4.2767
19. −3.1701 **20.** −1.1426
21. 4.6931 **22.** −3.3069
23. 3.2 **24.** 8.4 **25.** 1.8
26. 13.5 **27.** 2.0×10^{-3}
28. 4.0×10^{-4} **29.** 1.6×10^{-5}
30. 3.2×10^{-7} **31.** poor fen
32. bog **33.** rich fen
34. **(a)** 2.60031933
(b) 1.60031933 **(c)** .6003193298
(d) The whole number parts will vary, but the decimal parts will be the same. **35.** 2.3219
36. 3.1699 **37.** −.2537
38. −.1647 **39.** 1.9376
40. 1.0932 **41.** −1.4125
42. −22.0488 **43.** **(a)** 3
(b) 5^2 or 25 **(c)** $\frac{1}{e}$
44. **(a)** 7 **(b)** ln 3 **(c)** 2 ln 3 or ln 9 **45.** **(a)** 5 **(b)** ln 3
(c) 2 ln 3 or ln 9 **46.** **(a)** 3
(b) 1 **(c)** 2 **47.** **(a)** 20 **(b)** 30
(c) 50 **(d)** 60 **(e)** about 3 decibels

10. The graph of $y = \ln x$ shows a point on the graph. Write the logarithmic equation associated with that point.



Use a calculator with logarithm keys to find an approximation to four decimal places for each expression. See Figures 20 and 23.

11. log 36 **12.** log 72 **13.** log .042 **14.** log .319
15. $\log(2 \times 10^4)$ **16.** $\log(2 \times 10^{-4})$ **17.** ln 36 **18.** ln 72
19. ln .042 **20.** ln .319 **21.** $\ln(2 \times e^4)$ **22.** $\ln(2 \times e^{-4})$

For each substance, find the pH from the given hydronium ion concentration. See Example 1(a).

23. grapefruit, 6.3×10^{-4} **24.** crackers, 3.9×10^{-9}
25. limes, 1.6×10^{-2} **26.** sodium hydroxide (lye), 3.2×10^{-14}

Find the $[H_3O^+]$ *for each substance with the given pH. See Example 1(b).*

27. soda pop, 2.7 **28.** wine, 3.4
29. beer, 4.8 **30.** drinking water, 6.5

In Exercises 31–33, suppose that water from a wetland area is sampled and found to have the given hydronium ion concentration. Determine whether the wetland is a rich fen, poor fen, or bog. See Example 2.

31. 2.49×10^{-5} **32.** 2.49×10^{-2} **33.** 2.49×10^{-7}

34. Use your calculator to find an approximation for each logarithm.
(a) log 398.4 **(b)** log 39.84 **(c)** log 3.984
(d) From your answers to parts (a)–(c), make a conjecture concerning the decimal values in the approximations of common logarithms of numbers greater than 1 that have the same digits.

Use the change-of-base theorem to find an approximation to four decimal places for each logarithm. See Example 5.

35. $\log_2 5$ **36.** $\log_2 9$ **37.** $\log_8 .59$ **38.** $\log_8 .71$
39. $\log_{\sqrt{13}} 12$ **40.** $\log_{\sqrt{19}} 5$ **41.** $\log_{.32} 5$ **42.** $\log_{.91} 8$

43. Given $g(x) = e^x$, evaluate **(a)** $g(\ln 3)$ **(b)** $g[\ln(5^2)]$ **(c)** $g\left[\ln\left(\frac{1}{e}\right)\right]$.

44. Given $f(x) = 3^x$, evaluate **(a)** $f(\log_3 7)$ **(b)** $f[\log_3(\ln 3)]$ **(c)** $f[\log_3(2 \ln 3)]$.

45. Given $f(x) = \ln x$, evaluate **(a)** $f(e^5)$ **(b)** $f(e^{\ln 3})$ **(c)** $f(e^{2 \ln 3})$.

46. Given $f(x) = \log_2 x$, evaluate **(a)** $f(2^3)$ **(b)** $f(2^{\log_2 2})$ **(c)** $f(2^{2 \log_2 2})$.

Solve each application of logarithms. See Examples 3 and 4.

47. **Decibel Levels** Find the decibel ratings of sounds having the following intensities.
(a) $100I_0$ **(b)** $1000I_0$ **(c)** $100{,}000I_0$ **(d)** $1{,}000{,}000I_0$
(e) If the intensity of a sound is doubled, by how much is the decibel rating increased?

48. (a) 21 (b) 70 (c) 91 (d) 120 (e) 140 49. (a) 3 (b) 6 (c) 8 50. about $5{,}000{,}000I_0$ 51. about $126{,}000{,}000I_0$ 52. It was more than 25 times greater in magnitude. 53. about 70 million visitors; We must assume that the rate of increase continues to be logarithmic.

48. ***Decibel Levels*** Find the decibel ratings of the following sounds, having intensities as given. Round each answer to the nearest whole number.

(a) whisper, $115I_0$ (b) busy street, $9{,}500{,}000I_0$

(c) heavy truck, 20 m away, $1{,}200{,}000{,}000I_0$

(d) rock music, $895{,}000{,}000{,}000I_0$

(e) jetliner at takeoff, $109{,}000{,}000{,}000{,}000I_0$

49. ***Earthquake Intensity*** The magnitude of an earthquake, measured on the Richter scale, is $\log_{10} \frac{I}{I_0}$, where I is the amplitude registered on a seismograph 100 km from the epicenter of the earthquake, and I_0 is the amplitude of an earthquake of a certain (small) size. Find the Richter scale ratings for earthquakes having the following amplitudes.

(a) $1000I_0$ (b) $1{,}000{,}000I_0$ (c) $100{,}000{,}000I_0$

50. ***Earthquake Intensity*** On June 16, 1999, the city of Puebla in central Mexico was shaken by an earthquake that measured 6.7 on the Richter scale. Express this reading in terms of I_0. See Exercise 49. *(Source: Times Picayune.)*

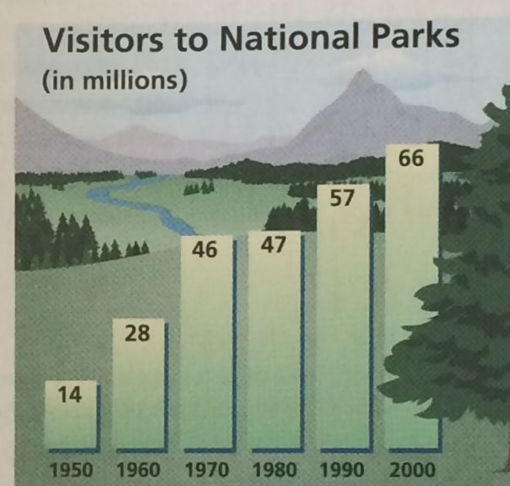
51. ***Earthquake Intensity*** On September 19, 1985, Mexico's largest recent earthquake, measuring 8.1 on the Richter scale, killed about 9500 people. Express the magnitude of an 8.1 reading in terms of I_0. *(Source: Times Picayune.)*

52. Compare your answers to Exercises 50 and 51. How much greater was the force of the 1985 earthquake than the 1999 earthquake?

53. ***(Modeling) Visitors to U.S. National Parks*** The heights of the bars in the graph represent the number of visitors (in millions) to U.S. National Parks from 1950–2000. Suppose x represents the number of years since 1900—thus, 1950 is represented by 50, 1960 is represented by 60, and so on. The logarithmic function defined by

$$f(x) = -269 + 73 \ln x$$

closely models the data. Use this function to estimate the number of visitors in the year 2004. What assumption must we make to estimate the number of visitors in years beyond 2000?

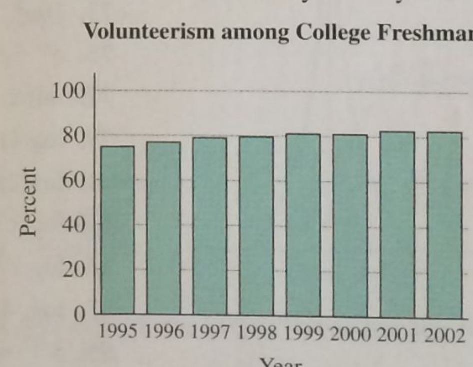


Source: Statistical Abstract of the United States, 2000.

54. ***(Modeling) Volunteerism among College Freshmen*** The growth in the percentage of college freshmen who reported involvement in volunteer work during their last year of high school is shown in the bar graph. Connecting the tops of the bars with a continuous curve would give a graph that indicates logarithmic growth. The function defined by

$$f(t) = 74.61 + 3.84 \ln t, \quad t \geq 1,$$

where t represents the number of years since 1994 and $f(t)$ is the percent, approximates the curve reasonably well.



Source: Higher Education Research Institute, UCLA.

54. **(a)** approximately 82.595%; This is exceptionally close to the percent shown in the graph and to the actual percent. **55.** **(a)** 2 **(b)** 2 **(c)** 2 **(d)** 1
56. **(a)** 4 **(b)** 4 **(c)** 5
57. 1 **58.** 1.59
59. $\log_7 19$; $\frac{\log 19}{\log 7}$; $\frac{\ln 19}{\ln 7}$
60. $\log_3 10$; $\frac{\log 10}{\log 3}$ or $\frac{1}{\log 3}$; $\frac{\ln 10}{\ln 3}$
61. $\log_{1/2} 12$; $\frac{\log 12}{\log\left(\frac{1}{2}\right)}$; $\frac{\ln 12}{\ln\left(\frac{1}{2}\right)}$
62. $\log_{1/3} 4$; $\frac{\log 4}{\log\left(\frac{1}{3}\right)}$; $\frac{\ln 4}{\ln\left(\frac{1}{3}\right)}$
63. {1.6309} **64.** {1.7925}
65. {−.0803} **66.** {3.6674}
67. {2.2694} **68.** {−13.2571}
69. {2.3863} **70.** {−.4849}
71. {−.1227} **72.** {2.1023}
73. ∅ **74.** ∅ **75.** {2} **76.** {3}
77. $\left\{\frac{1}{3}\right\}$ **78.** {−5} **79.** ∅
80. ∅ **81.** {1} **82.** $\left\{\frac{4}{3}\right\}$
83. {5} **84.** {4}
85. {5} **86.** {1}
87. $\left\{\frac{9}{2}\right\}$ **88.** $\left\{\frac{15}{7}\right\}$
89. {−17.5314} **90.** {−2.4874}
91. {8} **92.** {6} **93.** {4}
94. {1, 10} **95.** {1, 100}
96. {−2, 2}

$\left(\frac{1}{2}\right)^x = 12$

$\log_{\frac{1}{2}} 12 = x$

$x = \frac{\ln 12}{\ln 1/2}$

(a) What does the function predict for the percent of freshmen entering college in 2002 who performed volunteer work during their last year of high school? How does this compare to the actual percent of 82.6?

(b) Explain why an exponential function would *not* provide a good model for these data.

55. ***(Modeling) Diversity of Species*** The number of species in a sample is given by

$$S(n) = a \ln\left(1 + \frac{n}{a}\right).$$

Here n is the number of individuals in the sample, and a is a constant that indicates the diversity of species in the community. If $a = .36$, find $S(n)$ for each value of n. (*Hint:* $S(n)$ must be a whole number.)

(a) 100 **(b)** 200 **(c)** 150 **(d)** 10

56. ***(Modeling) Diversity of Species*** In Exercise 55, find $S(n)$ if a changes to .88. Use the following values of n.

(a) 50 **(b)** 100 **(c)** 250

57. ***(Modeling) Diversity of Species*** Suppose a sample of a small community shows two species with 50 individuals each. Find the measure of diversity H. (See Example 6.)

58. ***(Modeling) Diversity of Species*** A virgin forest in northwestern Pennsylvania has 4 species of large trees with the following proportions of each: hemlock, .521; beech, .324; birch, .081; maple, .074. Find the measure of diversity H. (See Example 6.)

Concept Check *An exponential equation such as $5^x = 9$ can be solved for its exact solution using the meaning of logarithm and the change-of-base theorem. Since x is the exponent to which 5 must be raised in order to obtain 9, the exact solution is $\log_5 9$, or $\frac{\log 9}{\log 5}$ or $\frac{\ln 9}{\ln 5}$. For the following equations, give the exact solution in three forms similar to the forms explained here.*

59. $7^x = 19$ **60.** $3^x = 10$ **61.** $\left(\frac{1}{2}\right)^x = 12$ **62.** $\left(\frac{1}{3}\right)^x = 4$

Solve each equation. When solutions are irrational, give them as decimals correct to four decimal places. See Examples 7–12.

63. $3^x = 6$ **64.** $4^x = 12$ **65.** $6^{1-2x} = 8$

66. $3^{2x-5} = 13$ **67.** $2^{x+3} = 5^x$ **68.** $6^{x+3} = 4^x$

69. $e^{x-1} = 4$ **70.** $e^{2-x} = 12$ **71.** $2e^{5x+2} = 8$

72. $10e^{3x-7} = 5$ **73.** $2^x = -3$ **74.** $3^x = -6$

75. $e^{8x} \cdot e^{2x} = e^{20}$ **76.** $e^{6x} \cdot e^x = e^{21}$

77. $\ln(6x + 1) = \ln 3$ **78.** $\ln(7 - x) = \ln 12$

79. $\log 4x - \log(x - 3) = \log 2$ **80.** $\ln(-x) + \ln 3 = \ln(2x - 15)$

81. $\log(2x - 1) + \log 10x = \log 10$ **82.** $\ln 5x - \ln(2x - 1) = \ln 4$

83. $\log(x + 25) = 1 + \log(2x - 7)$ **84.** $\ln(5 + 4x) - \ln(3 + x) = \ln 3$

85. $\log x + \log(3x - 13) = 1$ **86.** $\ln(2x + 5) + \ln x = \ln 7$

87. $\log_6 4x - \log_6(x - 3) = \log_6 12$ **88.** $\log_2 3x + \log_2 3 = \log_2(2x + 15)$

89. $5^{x+2} = 2^{2x-1}$ **90.** $6^{x-3} = 3^{4x+1}$

91. $\ln e^x - \ln e^3 = \ln e^5$ **92.** $\ln e^x - 2 \ln e = \ln e^4$

93. $\log_2(\log_2 x) = 1$ **94.** $\log x = \sqrt{\log x}$

95. $\log x^2 = (\log x)^2$ **96.** $\log_2 \sqrt{2x^2} = \frac{3}{2}$

97. during 2011
98. (a) 11.6451 m per sec
(b) 2.4823 sec 99. (a) about
24% (b) 1963 100. (b) 984 ft
(c) 39 ft

(Modeling) *Solve each application.*

97. ***Average Annual Public University Costs*** The table shows the cost of a year's tuition, room and board, and fees at a public university from 2000–2008. (*Note:* The amounts for 2004–2008 are projections.) Letting y represent the cost and x represent the number of years since 2000, we find that the function defined by

$$f(x) = 8160(1.06)^x$$

models the data quite well. According to this function, when will the cost in 2000 be doubled?

Year	Average Annual Cost
2000	\$7,990
2001	\$8,470
2002	\$9,338
2003	\$9,805
2004	\$10,295
2005	\$10,810
2006	\$11,351
2007	\$11,918
2008	\$12,514

Source: www.princetonreview.com

98. ***Race Speed*** At the World Championship races held at Rome's Olympic Stadium in 1987, American sprinter Carl Lewis ran the 100-m race in 9.86 sec. His speed in meters per second after t seconds is closely modeled by the function defined by

$$f(t) = 11.65(1 - e^{-t/1.27}).$$

(*Source:* Banks, R. B., *Towing Icebergs, Falling Dominoes, and Other Adventures in Applied Mathematics,* Princeton University Press, 1998.)

(a) How fast was he running as he crossed the finish line?

(b) After how many seconds was he running at the rate of 10 m per sec?

99. ***Fatherless Children*** The percent of U.S. children growing up without a father has increased rapidly since 1950. If x represents the number of years since 1900, the function defined by

$$f(x) = \frac{25}{1 + 1364.3e^{-x/9.316}}$$

models the percent fairly well. (*Source:* National Longitudinal Survey of Youth; U.S. Department of Commerce; U.S. Bureau of the Census.)

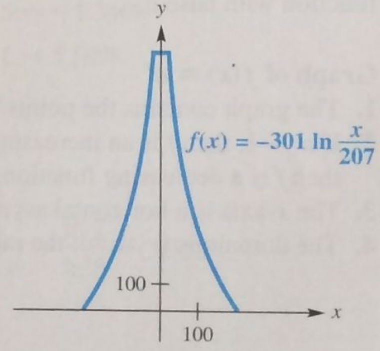
(a) What percent of U.S. children lived in a home without a father in 1997?

(b) In what year were 10% of these children living in a home without a father?

100. ***Height of the Eiffel Tower*** The right side of the Eiffel Tower in Paris, France has a shape that can be approximated by the graph of the function defined by

$$f(x) = -301 \ln \frac{x}{207}.$$

See the figure. (*Source:* Banks, R. B., *Towing Icebergs, Falling Dominoes, and Other Adventures in Applied Mathematics,* Princeton University Press, 1998.)



(a) Explain why the shape of the left side of the Eiffel Tower has the formula given by $f(-x)$.

(b) The short horizontal line at the top of the figure has length 15.7488 ft. Approximately how tall is the Eiffel Tower?

(c) Approximately how far from the center of the tower is the point on the right side that is 500 ft above the ground?