1 Preparation for Calculus

Automobile Aerodynamics *(Exercise 96, p. 30)*

Hours of Daylight
(Example 3, p. 33)

Conveyor Design *(Exercise 23, p. 16)*

Cell Phone Subscribers
(Exercise 68, p. 9)

Modeling Carbon Dioxide Concentration *(Example 6, p. 7)*

Clockwise from top left, Gyi nesa/iStockphoto.com; hjschneider/iStockphoto.com;
Andy Dean Photography/Shutterstock.com; Gavriel Jecan/Terra/CORBIS; xtrekx/Shutterstock.com

1.1 Graphs and Models

- **Sketch the graph of an equation.**
- **Find the intercepts of a graph.**
- **Test a graph for symmetry with respect to an axis and the origin.**
- **Find the points of intersection of two graphs.**
- **Interpret mathematical models for real-life data.**

RENÉ DESCARTES (1596–1650)

Descartes made many contributions to philosophy, science, and mathematics. The idea of representing points in the plane by pairs of real numbers and representing curves in the plane by equations was described by Descartes in his book *La Géométrie*, published in 1637.

See LarsonCalculus.com to read more of this biography.

The Graph of an Equation

In 1637, the French mathematician René Descartes revolutionized the study of mathematics by combining its two major fields—algebra and geometry. With Descartes's coordinate plane, geometric concepts could be formulated analytically and algebraic concepts could be viewed graphically. The power of this approach was such that within a century of its introduction, much of calculus had been developed.

The same approach can be followed in your study of calculus. That is, by viewing calculus from multiple perspectives—*graphically*, *analytically*, and *numerically*—you will increase your understanding of core concepts.

Consider the equation $3x + y = 7$. The point $(2, 1)$ is a **solution point** of the equation because the equation is satisfied (is true) when 2 is substituted for x and 1 is substituted for y. This equation has many other solutions, such as $(1, 4)$ and $(0, 7)$. To find other solutions systematically, solve the original equation for y.

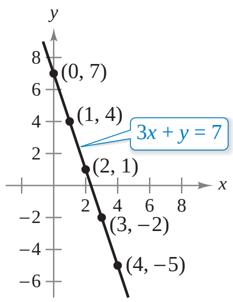
$$y = 7 - 3x$$ Analytic approach

Then construct a **table of values** by substituting several values of x.

x	0	1	2	3	4
y	7	4	1	-2	-5

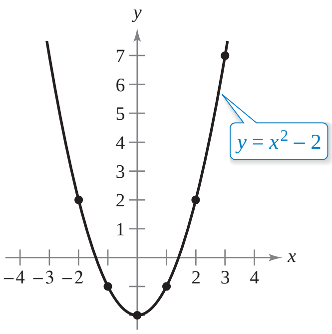
Numerical approach

From the table, you can see that $(0, 7)$, $(1, 4)$, $(2, 1)$, $(3, -2)$, and $(4, -5)$ are solutions of the original equation $3x + y = 7$. Like many equations, this equation has an infinite number of solutions. The set of all solution points is the **graph** of the equation, as shown in Figure 1.1. Note that the sketch shown in Figure 1.1 is referred to as the graph of $3x + y = 7$, even though it really represents only a *portion* of the graph. The entire graph would extend beyond the page.



Graphical approach: $3x + y = 7$
Figure 1.1

In this course, you will study many sketching techniques. The simplest is point plotting—that is, you plot points until the basic shape of the graph seems apparent.



The parabola $y = x^2 - 2$
Figure 1.2

EXAMPLE 1 **Sketching a Graph by Point Plotting**

To sketch the graph of $y = x^2 - 2$, first construct a table of values. Next, plot the points shown in the table. Then connect the points with a smooth curve, as shown in Figure 1.2. This graph is a **parabola.** It is one of the conics you will study in Chapter 10.

x	-2	-1	0	1	2	3
y	2	-1	-2	-1	2	7

The Granger Collection, New York

One disadvantage of point plotting is that to get a good idea about the shape of a graph, you may need to plot many points. With only a few points, you could badly misrepresent the graph. For instance, to sketch the graph of

$$y = \frac{1}{30}x(39 - 10x^2 + x^4)$$

you plot five points:

$$(-3, -3),\quad (-1, -1),\quad (0, 0),\quad (1, 1),\quad \text{and} \quad (3, 3)$$

as shown in Figure 1.3(a). From these five points, you might conclude that the graph is a line. This, however, is not correct. By plotting several more points, you can see that the graph is more complicated, as shown in Figure 1.3(b).

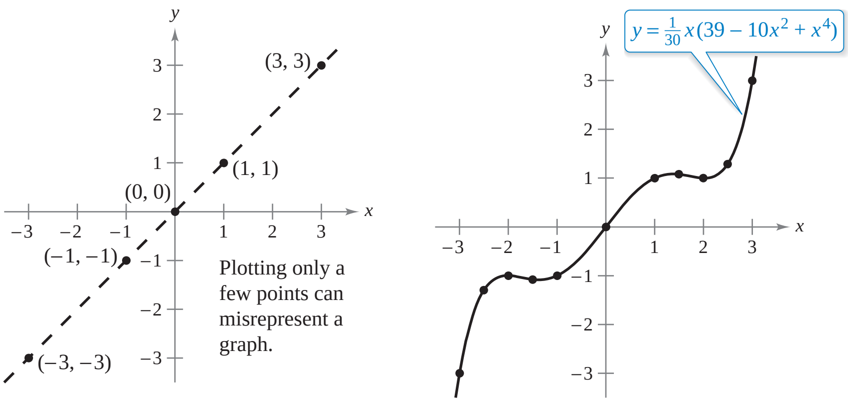


Figure 1.3

Exploration

Comparing Graphical and Analytic Approaches Use a graphing utility to graph each equation. In each case, find a viewing window that shows the important characteristics of the graph.

a. $y = x^3 - 3x^2 + 2x + 5$

b. $y = x^3 - 3x^2 + 2x + 25$

c. $y = -x^3 - 3x^2 + 20x + 5$

d. $y = 3x^3 - 40x^2 + 50x - 45$

e. $y = -(x + 12)^3$

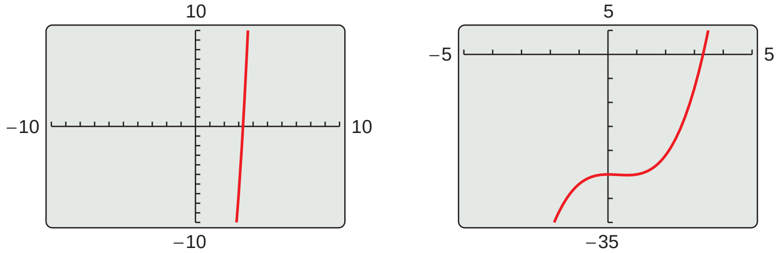
f. $y = (x - 2)(x - 4)(x - 6)$

A purely graphical approach to this problem would involve a simple "guess, check, and revise" strategy. What types of things do you think an analytic approach might involve? For instance, does the graph have symmetry? Does the graph have turns? If so, where are they? As you proceed through Chapters 2, 3, and 4 of this text, you will study many new analytic tools that will help you analyze graphs of equations such as these.

TECHNOLOGY Graphing an equation has been made easier by technology. Even with technology, however, it is possible to misrepresent a graph badly. For instance, each of the graphing utility* screens in Figure 1.4 shows a portion of the graph of

$$y = x^3 - x^2 - 25.$$

From the screen on the left, you might assume that the graph is a line. From the screen on the right, however, you can see that the graph is not a line. So, whether you are sketching a graph by hand or using a graphing utility, you must realize that different "viewing windows" can produce very different views of a graph. In choosing a viewing window, your goal is to show a view of the graph that fits well in the context of the problem.



Graphing utility screens of $y = x^3 - x^2 - 25$
Figure 1.4

*In this text, the term *graphing utility* means either a graphing calculator, such as the *TI-Nspire*, or computer graphing software, such as *Maple* or *Mathematica*.

Intercepts of a Graph

REMARK Some texts denote the x-intercept as the x-coordinate of the point $(a, 0)$ rather than the point itself. Unless it is necessary to make a distinction, when the term *intercept* is used in this text, it will mean either the point or the coordinate.

Two types of solution points that are especially useful in graphing an equation are those having zero as their x- or y-coordinate. Such points are called **intercepts** because they are the points at which the graph intersects the x- or y-axis. The point $(a, 0)$ is an **x-intercept** of the graph of an equation when it is a solution point of the equation. To find the x-intercepts of a graph, let y be zero and solve the equation for x. The point $(0, b)$ is a **y-intercept** of the graph of an equation when it is a solution point of the equation. To find the y-intercepts of a graph, let x be zero and solve the equation for y.

It is possible for a graph to have no intercepts, or it might have several. For instance, consider the four graphs shown in Figure 1.5.

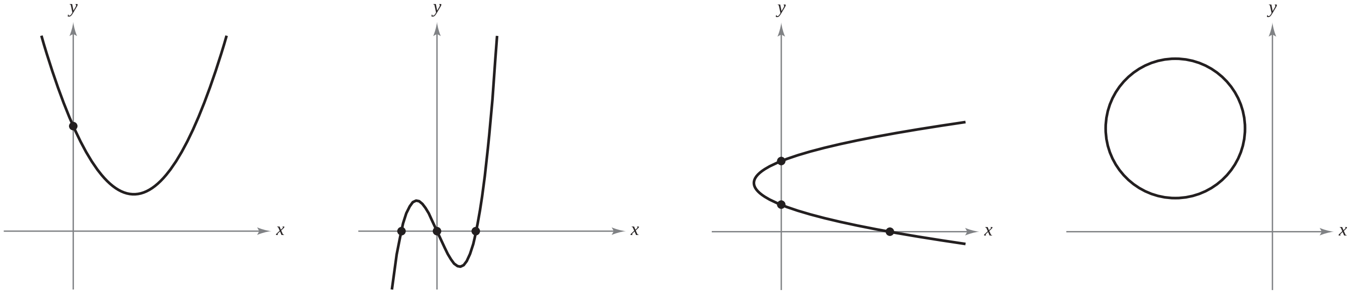


Figure 1.5

EXAMPLE 2 Finding *x*- and *y*-Intercepts

Find the x- and y-intercepts of the graph of $y = x^3 - 4x$.

Solution To find the x-intercepts, let y be zero and solve for x.

$$x^3 - 4x = 0 \quad \text{Let } y \text{ be zero.}$$
$$x(x - 2)(x + 2) = 0 \quad \text{Factor.}$$
$$x = 0, 2, \text{ or } -2 \quad \text{Solve for } x.$$

Because this equation has three solutions, you can conclude that the graph has three x-intercepts:

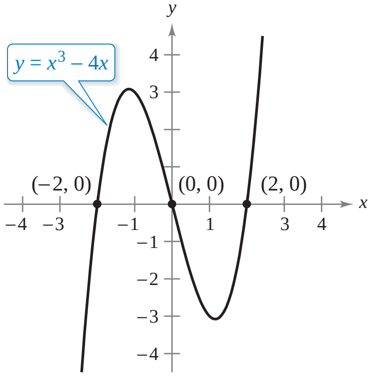
$$(0, 0),\quad (2, 0),\quad \text{and}\quad (-2, 0). \quad x\text{-intercepts}$$

To find the y-intercepts, let x be zero. Doing this produces $y = 0$. So, the y-intercept is

$$(0, 0). \quad y\text{-intercept}$$

(See Figure 1.6.)

TECHNOLOGY Example 2 uses an analytic approach to finding intercepts. When an analytic approach is not possible, you can use a graphical approach by finding the points at which the graph intersects the axes. Use the *trace* feature of a graphing utility to approximate the intercepts of the graph of the equation in Example 2. Note that your utility may have a built-in program that can find the x-intercepts of a graph. (Your utility may call this the *root* or *zero* feature.) If so, use the program to find the x-intercepts of the graph of the equation in Example 2.



Intercepts of a graph
Figure 1.6

Symmetry of a Graph

Knowing the symmetry of a graph before attempting to sketch it is useful because you need only half as many points to sketch the graph. The three types of symmetry listed below can be used to help sketch the graphs of equations (see Figure 1.7).

1. A graph is **symmetric with respect to the *y*-axis** if, whenever (x, y) is a point on the graph, then $(-x, y)$ is also a point on the graph. This means that the portion of the graph to the left of the *y*-axis is a mirror image of the portion to the right of the *y*-axis.
2. A graph is **symmetric with respect to the *x*-axis** if, whenever (x, y) is a point on the graph, then $(x, -y)$ is also a point on the graph. This means that the portion of the graph below the *x*-axis is a mirror image of the portion above the *x*-axis.
3. A graph is **symmetric with respect to the origin** if, whenever (x, y) is a point on the graph, then $(-x, -y)$ is also a point on the graph. This means that the graph is unchanged by a rotation of 180° about the origin.

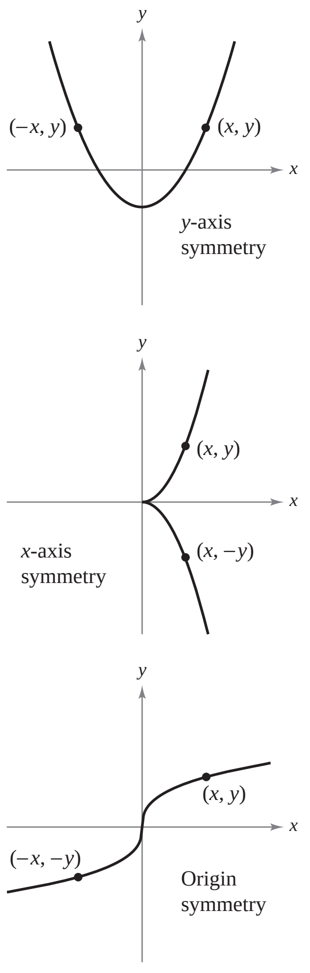


Figure 1.7

Tests for Symmetry

1. The graph of an equation in x and y is symmetric with respect to the y-axis when replacing x by $-x$ yields an equivalent equation.
2. The graph of an equation in x and y is symmetric with respect to the x-axis when replacing y by $-y$ yields an equivalent equation.
3. The graph of an equation in x and y is symmetric with respect to the origin when replacing x by $-x$ and y by $-y$ yields an equivalent equation.

The graph of a polynomial has symmetry with respect to the y-axis when each term has an even exponent (or is a constant). For instance, the graph of

$$y = 2x^4 - x^2 + 2$$

has symmetry with respect to the y-axis. Similarly, the graph of a polynomial has symmetry with respect to the origin when each term has an odd exponent, as illustrated in Example 3.

EXAMPLE 3 Testing for Symmetry

Test the graph of $y = 2x^3 - x$ for symmetry with respect to (a) the y-axis and (b) the origin.

Solution

a.

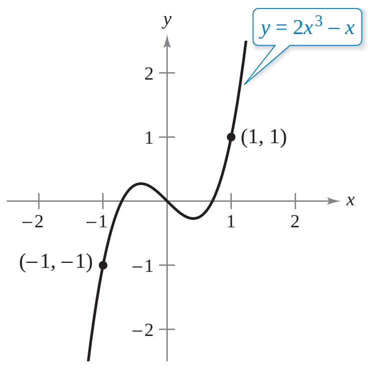
$y = 2x^3 - x$	Write original equation.
$y = 2(-x)^3 - (-x)$	Replace x by $-x$.
$y = -2x^3 + x$	Simplify. It is not an equivalent equation.

Because replacing x by $-x$ does *not* yield an equivalent equation, you can conclude that the graph of $y = 2x^3 - x$ is *not* symmetric with respect to the y-axis.

b.

$y = 2x^3 - x$	Write original equation.
$-y = 2(-x)^3 - (-x)$	Replace x by $-x$ and y by $-y$.
$-y = -2x^3 + x$	Simplify.
$y = 2x^3 - x$	Equivalent equation

Because replacing x by $-x$ and y by $-y$ yields an equivalent equation, you can conclude that the graph of $y = 2x^3 - x$ is symmetric with respect to the origin, as shown in Figure 1.8.



Origin symmetry
Figure 1.8

EXAMPLE 4 Using Intercepts and Symmetry to Sketch a Graph

See LarsonCalculus.com for an interactive version of this type of example.

Sketch the graph of $x - y^2 = 1$.

Solution The graph is symmetric with respect to the x-axis because replacing y by $-y$ yields an equivalent equation.

$$x - y^2 = 1 \quad \text{Write original equation.}$$
$$x - (-y)^2 = 1 \quad \text{Replace } y \text{ by } -y.$$
$$x - y^2 = 1 \quad \text{Equivalent equation}$$

This means that the portion of the graph below the x-axis is a mirror image of the portion above the x-axis. To sketch the graph, first plot the x-intercept and the points above the x-axis. Then reflect in the x-axis to obtain the entire graph, as shown in Figure 1.9.

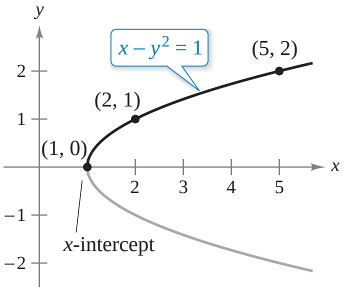


Figure 1.9

TECHNOLOGY Graphing utilities are designed so that they most easily graph equations in which y is a function of x (see Section 1.3 for a definition of **function**). To graph other types of equations, you need to split the graph into two or more parts *or* you need to use a different graphing mode. For instance, to graph the equation in Example 4, you can split it into two parts.

$$y_1 = \sqrt{x - 1} \quad \text{Top portion of graph}$$
$$y_2 = -\sqrt{x - 1} \quad \text{Bottom portion of graph}$$

Points of Intersection

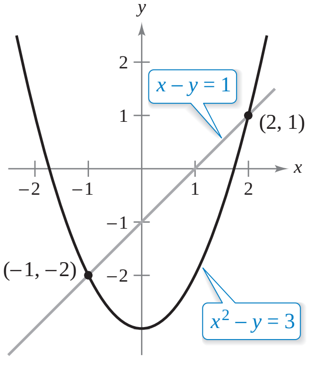
A **point of intersection** of the graphs of two equations is a point that satisfies both equations. You can find the point(s) of intersection of two graphs by solving their equations simultaneously.

EXAMPLE 5 Finding Points of Intersection

Find all points of intersection of the graphs of

$$x^2 - y = 3 \quad \text{and} \quad x - y = 1.$$

Solution Begin by sketching the graphs of both equations in the *same* rectangular coordinate system, as shown in Figure 1.10. From the figure, it appears that the graphs have two points of intersection. You can find these two points as follows.



Two points of intersection
Figure 1.10

$$y = x^2 - 3 \quad \text{Solve first equation for } y.$$
$$y = x - 1 \quad \text{Solve second equation for } y.$$
$$x^2 - 3 = x - 1 \quad \text{Equate } y\text{-values.}$$
$$x^2 - x - 2 = 0 \quad \text{Write in general form.}$$
$$(x - 2)(x + 1) = 0 \quad \text{Factor.}$$
$$x = 2 \text{ or } -1 \quad \text{Solve for } x.$$

The corresponding values of y are obtained by substituting $x = 2$ and $x = -1$ into either of the original equations. Doing this produces two points of intersection:

$$(2, 1) \quad \text{and} \quad (-1, -2). \quad \text{Points of intersection}$$

You can check the points of intersection in Example 5 by substituting into *both* of the original equations or by using the *intersect* feature of a graphing utility.

Mathematical Models

Real-life applications of mathematics often use equations as **mathematical models.** In developing a mathematical model to represent actual data, you should strive for two (often conflicting) goals: accuracy and simplicity. That is, you want the model to be simple enough to be workable, yet accurate enough to produce meaningful results. Section 1.4 explores these goals more completely.

EXAMPLE 6 Comparing Two Mathematical Models

The Mauna Loa Observatory in Hawaii has been measuring the increasing concentration of carbon dioxide in Earth's atmosphere since 1958.

The Mauna Loa Observatory in Hawaii records the carbon dioxide concentration y (in parts per million) in Earth's atmosphere. The January readings for various years are shown in Figure 1.11. In the July 1990 issue of *Scientific American*, these data were used to predict the carbon dioxide level in Earth's atmosphere in the year 2035, using the quadratic model

$$y = 0.018t^2 + 0.70t + 316.2$$ Quadratic model for 1960–1990 data

where $t = 0$ represents 1960, as shown in Figure 1.11(a). The data shown in Figure 1.11(b) represent the years 1980 through 2010 and can be modeled by

$$y = 1.68t + 303.5$$ Linear model for 1980–2010 data

where $t = 0$ represents 1960. What was the prediction given in the *Scientific American* article in 1990? Given the new data for 1990 through 2010, does this prediction for the year 2035 seem accurate?

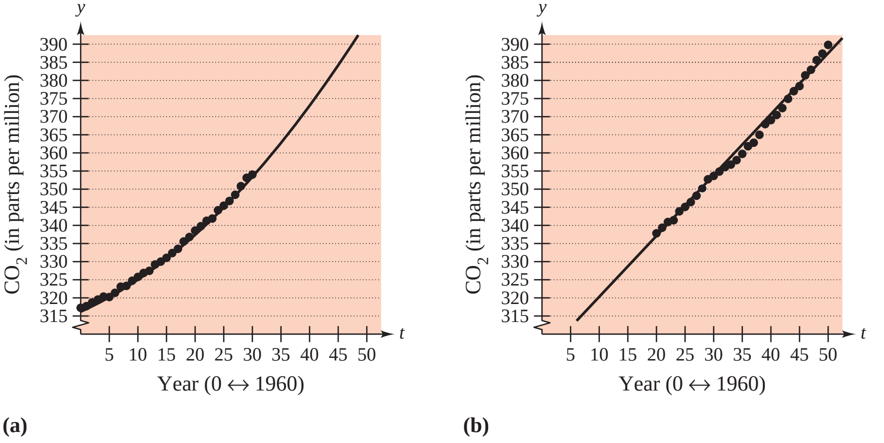


Figure 1.11

Solution To answer the first question, substitute $t = 75$ (for 2035) into the quadratic model.

$$y = 0.018(75)^2 + 0.70(75) + 316.2 = 469.95$$ Quadratic model

So, the prediction in the *Scientific American* article was that the carbon dioxide concentration in Earth's atmosphere would reach about 470 parts per million in the year 2035. Using the linear model for the 1980–2010 data, the prediction for the year 2035 is

$$y = 1.68(75) + 303.5 = 429.5.$$ Linear model

So, based on the linear model for 1980–2010, it appears that the 1990 prediction was too high.

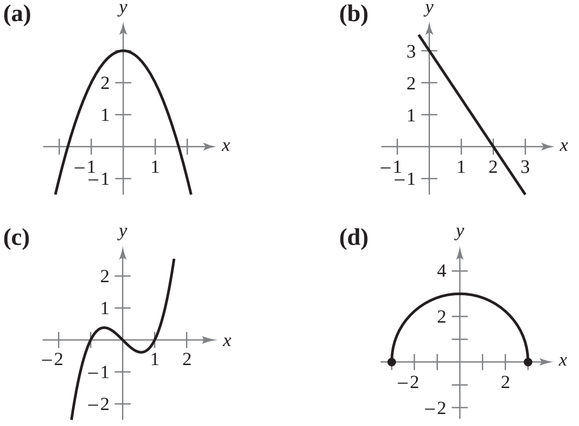
The models in Example 6 were developed using a procedure called *least squares regression* (see Section 13.9). The quadratic and linear models have correlations given by $r^2 \approx 0.997$ and $r^2 \approx 0.994$, respectively. The closer r^2 is to 1, the "better" the model.

Gavriel Jecan/Terra/CORBIS

1.1 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Matching In Exercises 1–4, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]



1. $y = -\frac{3}{2}x + 3$
2. $y = \sqrt{9 - x^2}$
3. $y = 3 - x^2$
4. $y = x^3 - x$

Sketching a Graph by Point Plotting In Exercises 5–14, sketch the graph of the equation by point plotting.

5. $y = \frac{1}{2}x + 2$
6. $y = 5 - 2x$
7. $y = 4 - x^2$
8. $y = (x - 3)^2$
9. $y = |x + 2|$
10. $y = |x| - 1$
11. $y = \sqrt{x} - 6$
12. $y = \sqrt{x + 2}$
13. $y = \dfrac{3}{x}$
14. $y = \dfrac{1}{x + 2}$

Approximating Solution Points In Exercises 15 and 16, use a graphing utility to graph the equation. Move the cursor along the curve to approximate the unknown coordinate of each solution point accurate to two decimal places.

15. $y = \sqrt{5 - x}$
 (a) $(2, y)$
 (b) $(x, 3)$
16. $y = x^5 - 5x$
 (a) $(-0.5, y)$
 (b) $(x, -4)$

Finding Intercepts In Exercises 17–26, find any intercepts.

17. $y = 2x - 5$
18. $y = 4x^2 + 3$
19. $y = x^2 + x - 2$
20. $y^2 = x^3 - 4x$
21. $y = x\sqrt{16 - x^2}$
22. $y = (x - 1)\sqrt{x^2 + 1}$
23. $y = \dfrac{2 - \sqrt{x}}{5x + 1}$
24. $y = \dfrac{x^2 + 3x}{(3x + 1)^2}$
25. $x^2y - x^2 + 4y = 0$
26. $y = 2x - \sqrt{x^2 + 1}$

Testing for Symmetry In Exercises 27–38, test for symmetry with respect to each axis and to the origin.

27. $y = x^2 - 6$
28. $y = x^2 - x$
29. $y^2 = x^3 - 8x$
30. $y = x^3 + x$
31. $xy = 4$
32. $xy^2 = -10$
33. $y = 4 - \sqrt{x + 3}$
34. $xy - \sqrt{4 - x^2} = 0$
35. $y = \dfrac{x}{x^2 + 1}$
36. $y = \dfrac{x^2}{x^2 + 1}$
37. $y = |x^3 + x|$
38. $|y| - x = 3$

Using Intercepts and Symmetry to Sketch a Graph In Exercises 39–56, find any intercepts and test for symmetry. Then sketch the graph of the equation.

39. $y = 2 - 3x$
40. $y = \frac{2}{3}x + 1$
41. $y = 9 - x^2$
42. $y = 2x^2 + x$
43. $y = x^3 + 2$
44. $y = x^3 - 4x$
45. $y = x\sqrt{x + 5}$
46. $y = \sqrt{25 - x^2}$
47. $x = y^3$
48. $x = y^2 - 4$
49. $y = \dfrac{8}{x}$
50. $y = \dfrac{10}{x^2 + 1}$
51. $y = 6 - |x|$
52. $y = |6 - x|$
53. $y^2 - x = 9$
54. $x^2 + 4y^2 = 4$
55. $x + 3y^2 = 6$
56. $3x - 4y^2 = 8$

Finding Points of Intersection In Exercises 57–62, find the points of intersection of the graphs of the equations.

57. $x + y = 8$
 $4x - y = 7$
58. $3x - 2y = -4$
 $4x + 2y = -10$
59. $x^2 + y = 6$
 $x + y = 4$
60. $x = 3 - y^2$
 $y = x - 1$
61. $x^2 + y^2 = 5$
 $x - y = 1$
62. $x^2 + y^2 = 25$
 $-3x + y = 15$

Finding Points of Intersection In Exercises 63–66, use a graphing utility to find the points of intersection of the graphs. Check your results analytically.

63. $y = x^3 - 2x^2 + x - 1$
 $y = -x^2 + 3x - 1$
64. $y = x^4 - 2x^2 + 1$
 $y = 1 - x^2$
65. $y = \sqrt{x + 6}$
 $y = \sqrt{-x^2 - 4x}$
66. $y = -|2x - 3| + 6$
 $y = 6 - x$

The symbol indicates an exercise in which you are instructed to use graphing technology or a symbolic computer algebra system. The solutions of other exercises may also be facilitated by the use of appropriate technology.

67. Modeling Data The table shows the Gross Domestic Product, or GDP (in trillions of dollars), for selected years. *(Source: U.S. Bureau of Economic Analysis)*

Year	1980	1985	1990	1995	2000	2005	2010
GDP	2.8	4.2	5.8	7.4	10.0	12.6	14.5

(a) Use the regression capabilities of a graphing utility to find a mathematical model of the form $y = at^2 + bt + c$ for the data. In the model, y represents the GDP (in trillions of dollars) and t represents the year, with $t = 0$ corresponding to 1980.

(b) Use a graphing utility to plot the data and graph the model. Compare the data with the model.

(c) Use the model to predict the GDP in the year 2020.

68. Modeling Data

The table shows the numbers of cellular phone subscribers (in millions) in the United States for selected years. *(Source: CTIA-The Wireless)*

Year	1995	1998	2001	2004	2007	2010
Number	34	69	128	182	255	303

(a) Use the regression capabilities of a graphing utility to find a mathematical model of the form $y = at^2 + bt + c$ for the data. In the model, y represents the number of subscribers (in millions) and t represents the year, with $t = 5$ corresponding to 1995.

(b) Use a graphing utility to plot the data and graph the model. Compare the data with the model.

(c) Use the model to predict the number of cellular phone subscribers in the United States in the year 2020.

69. Break-Even Point Find the sales necessary to break even $(R = C)$ when the cost C of producing x units is $C = 2.04x + 5600$ and the revenue R from selling x units is $R = 3.29x$.

70. Copper Wire The resistance y in ohms of 1000 feet of solid copper wire at 77°F can be approximated by the model

$$y = \frac{10{,}770}{x^2} - 0.37, \quad 5 \le x \le 100$$

where x is the diameter of the wire in mils (0.001 in.). Use a graphing utility to graph the model. By about what factor is the resistance changed when the diameter of the wire is doubled?

71. Using Solution Points For what values of k does the graph of $y = kx^3$ pass through the point?

(a) $(1, 4)$ (b) $(-2, 1)$ (c) $(0, 0)$ (d) $(-1, -1)$

72. Using Solution Points For what values of k does the graph of $y^2 = 4kx$ pass through the point?

(a) $(1, 1)$ (b) $(2, 4)$ (c) $(0, 0)$ (d) $(3, 3)$

WRITING ABOUT CONCEPTS

Writing Equations **In Exercises 73 and 74, write an equation whose graph has the indicated property. (There may be more than one correct answer.)**

73. The graph has intercepts at $x = -4$, $x = 3$, and $x = 8$.

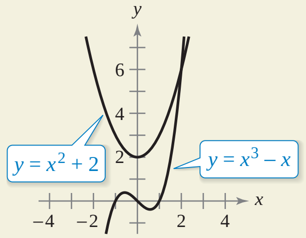
74. The graph has intercepts at $x = -\frac{3}{2}$, $x = 4$, and $x = \frac{5}{2}$.

75. Proof

(a) Prove that if a graph is symmetric with respect to the x-axis and to the y-axis, then it is symmetric with respect to the origin. Give an example to show that the converse is not true.

(b) Prove that if a graph is symmetric with respect to one axis and to the origin, then it is symmetric with respect to the other axis.

76. HOW DO YOU SEE IT? Use the graphs of the two equations to answer the questions below.



(a) What are the intercepts for each equation?

(b) Determine the symmetry for each equation.

(c) Determine the point of intersection of the two equations.

True or False? **In Exercises 77–80, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

77. If $(-4, -5)$ is a point on a graph that is symmetric with respect to the x-axis, then $(4, -5)$ is also a point on the graph.

78. If $(-4, -5)$ is a point on a graph that is symmetric with respect to the y-axis, then $(4, -5)$ is also a point on the graph.

79. If $b^2 - 4ac > 0$ and $a \neq 0$, then the graph of $y = ax^2 + bx + c$ has two x-intercepts.

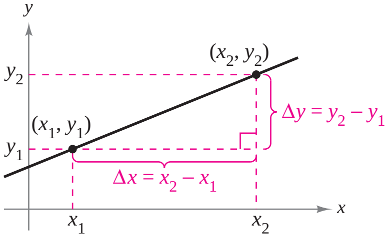
80. If $b^2 - 4ac = 0$ and $a \neq 0$, then the graph of $y = ax^2 + bx + c$ has only one x-intercept.

Andy Dean Photography/Shutterstock.com

1.2 Linear Models and Rates of Change

- **Find the slope of a line passing through two points.**
- **Write the equation of a line with a given point and slope.**
- **Interpret slope as a ratio or as a rate in a real-life application.**
- **Sketch the graph of a linear equation in slope-intercept form.**
- **Write equations of lines that are parallel or perpendicular to a given line.**

The Slope of a Line



$\Delta y = y_2 - y_1 =$ change in y
$\Delta x = x_2 - x_1 =$ change in x
Figure 1.12

The **slope** of a nonvertical line is a measure of the number of units the line rises (or falls) vertically for each unit of horizontal change from left to right. Consider the two points (x_1, y_1) and (x_2, y_2) on the line in Figure 1.12. As you move from left to right along this line, a vertical change of

$$\Delta y = y_2 - y_1 \qquad \text{Change in } y$$

units corresponds to a horizontal change of

$$\Delta x = x_2 - x_1 \qquad \text{Change in } x$$

units. (Δ is the Greek uppercase letter *delta*, and the symbols Δy and Δx are read "delta y" and "delta x.")

Definition of the Slope of a Line

The **slope** m of the nonvertical line passing through (x_1, y_1) and (x_2, y_2) is

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1}, \quad x_1 \neq x_2.$$

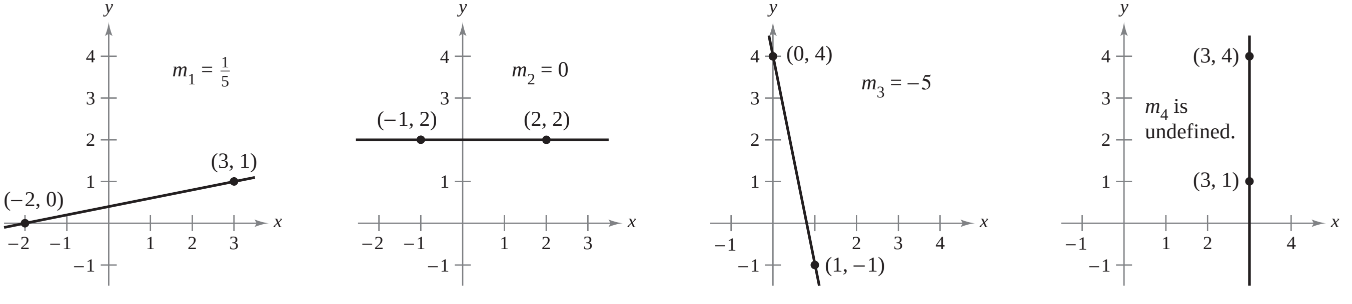
Slope is not defined for vertical lines.

When using the formula for slope, note that

$$\frac{y_2 - y_1}{x_2 - x_1} = \frac{-(y_1 - y_2)}{-(x_1 - x_2)} = \frac{y_1 - y_2}{x_1 - x_2}.$$

So, it does not matter in which order you subtract *as long as* you are consistent and both "subtracted coordinates" come from the same point.

Figure 1.13 shows four lines: one has a positive slope, one has a slope of zero, one has a negative slope, and one has an "undefined" slope. In general, the greater the absolute value of the slope of a line, the steeper the line. For instance, in Figure 1.13, the line with a slope of -5 is steeper than the line with a slope of $\frac{1}{5}$.



If m is positive, then the line rises from left to right.
If m is zero, then the line is horizontal.
If m is negative, then the line falls from left to right.
If m is undefined, then the line is vertical.
Figure 1.13

Exploration

Investigating Equations of Lines Use a graphing utility to graph each of the linear equations. Which point is common to all seven lines? Which value in the equation determines the slope of each line?

a. $y - 4 = -2(x + 1)$

b. $y - 4 = -1(x + 1)$

c. $y - 4 = -\frac{1}{2}(x + 1)$

d. $y - 4 = 0(x + 1)$

e. $y - 4 = \frac{1}{2}(x + 1)$

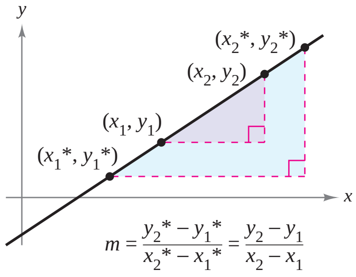
f. $y - 4 = 1(x + 1)$

g. $y - 4 = 2(x + 1)$

Use your results to write an equation of a line passing through $(-1, 4)$ with a slope of m.

Equations of Lines

Any two points on a nonvertical line can be used to calculate its slope. This can be verified from the similar triangles shown in Figure 1.14. (Recall that the ratios of corresponding sides of similar triangles are equal.)



Any two points on a nonvertical line can be used to determine its slope.
Figure 1.14

If (x_1, y_1) is a point on a nonvertical line that has a slope of m and (x, y) is *any other* point on the line, then

$$\frac{y - y_1}{x - x_1} = m.$$

This equation in the variables x and y can be rewritten in the form

$$y - y_1 = m(x - x_1)$$

which is the **point-slope form** of the equation of a line.

Point-Slope Form of the Equation of a Line

The **point-slope form** of the equation of the line that passes through the point (x_1, y_1) and has a slope of m is

$$y - y_1 = m(x - x_1).$$

REMARK Remember that only nonvertical lines have a slope. Consequently, vertical lines cannot be written in point-slope form. For instance, the equation of the vertical line passing through the point $(1, -2)$ is $x = 1$.

EXAMPLE 1 Finding an Equation of a Line

Find an equation of the line that has a slope of 3 and passes through the point $(1, -2)$. Then sketch the line.

Solution

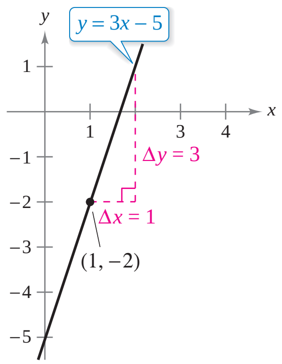
$y - y_1 = m(x - x_1)$ Point-slope form

$y - (-2) = 3(x - 1)$ Substitute -2 for y_1, 1 for x_1, and 3 for m.

$y + 2 = 3x - 3$ Simplify.

$y = 3x - 5$ Solve for y.

To sketch the line, first plot the point $(1, -2)$. Then, because the slope is $m = 3$, you can locate a second point on the line by moving one unit to the right and three units upward, as shown in Figure 1.15.



The line with a slope of 3 passing through the point $(1, -2)$
Figure 1.15

Ratios and Rates of Change

The slope of a line can be interpreted as either a *ratio* or a *rate*. If the x- and y-axes have the same unit of measure, then the slope has no units and is a **ratio.** If the x- and y-axes have different units of measure, then the slope is a rate or **rate of change.** In your study of calculus, you will encounter applications involving both interpretations of slope.

EXAMPLE 2 Using Slope as a Ratio

The maximum recommended slope of a wheelchair ramp is $\frac{1}{12}$. A business installs a wheelchair ramp that rises to a height of 22 inches over a length of 24 feet, as shown in Figure 1.16. Is the ramp steeper than recommended? *(Source: ADA Standards for Accessible Design)*

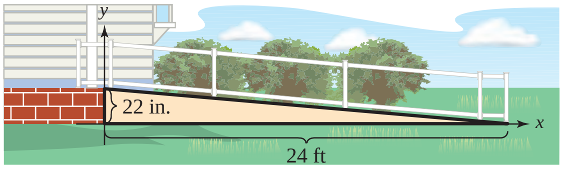


Figure 1.16

Solution The length of the ramp is 24 feet or $12(24) = 288$ inches. The slope of the ramp is the ratio of its height (the rise) to its length (the run).

$$\begin{aligned}\text{Slope of ramp} &= \frac{\text{rise}}{\text{run}}\\ &= \frac{22 \text{ in.}}{288 \text{ in.}}\\ &\approx 0.076\end{aligned}$$

Because the slope of the ramp is less than $\frac{1}{12} \approx 0.083$, the ramp is not steeper than recommended. Note that the slope is a ratio and has no units.

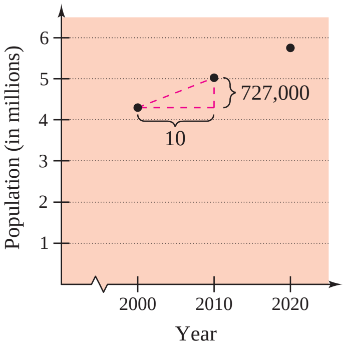
EXAMPLE 3 Using Slope as a Rate of Change

The population of Colorado was about 4,302,000 in 2000 and about 5,029,000 in 2010. Find the average rate of change of the population over this 10-year period. What will the population of Colorado be in 2020? *(Source: U.S. Census Bureau)*

Solution Over this 10-year period, the average rate of change of the population of Colorado was

$$\begin{aligned}\text{Rate of change} &= \frac{\text{change in population}}{\text{change in years}}\\ &= \frac{5{,}029{,}000 - 4{,}302{,}000}{2010 - 2000}\\ &= 72{,}700 \text{ people per year.}\end{aligned}$$

Assuming that Colorado's population continues to increase at this same rate for the next 10 years, it will have a 2020 population of about 5,756,000 (see Figure 1.17).



Population of Colorado
Figure 1.17

The rate of change found in Example 3 is an **average rate of change.** An average rate of change is always calculated over an interval. In this case, the interval is [2000, 2010]. In Chapter 3, you will study another type of rate of change called an *instantaneous rate of change.*

Graphing Linear Models

Many problems in coordinate geometry can be classified into two basic categories.

1. Given a graph (or parts of it), find its equation.
2. Given an equation, sketch its graph.

For lines, problems in the first category can be solved by using the point-slope form. The point-slope form, however, is not especially useful for solving problems in the second category. The form that is better suited to sketching the graph of a line is the **slope-intercept** form of the equation of a line.

The Slope-Intercept Form of the Equation of a Line

The graph of the linear equation

$$y = mx + b \qquad \text{Slope-intercept form}$$

is a line whose slope is m and whose y-intercept is $(0, b)$.

EXAMPLE 4 Sketching Lines in the Plane

Sketch the graph of each equation.

a. $y = 2x + 1$

b. $y = 2$

c. $3y + x - 6 = 0$

Solution

a. Because $b = 1$, the y-intercept is $(0, 1)$. Because the slope is $m = 2$, you know that the line rises two units for each unit it moves to the right, as shown in Figure 1.18(a).

b. By writing the equation $y = 2$ in slope-intercept form

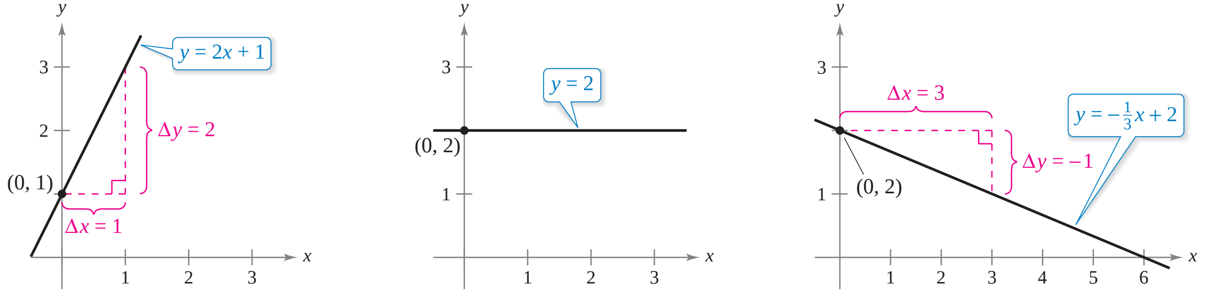
$$y = (0)x + 2$$

you can see that the slope is $m = 0$ and the y-intercept is $(0, 2)$. Because the slope is zero, you know that the line is horizontal, as shown in Figure 1.18(b).

c. Begin by writing the equation in slope-intercept form.

$$3y + x - 6 = 0 \qquad \text{Write original equation.}$$
$$3y = -x + 6 \qquad \text{Isolate } y\text{-term on the left.}$$
$$y = -\tfrac{1}{3}x + 2 \qquad \text{Slope-intercept form}$$

In this form, you can see that the y-intercept is $(0, 2)$ and the slope is $m = -\frac{1}{3}$. This means that the line falls one unit for every three units it moves to the right, as shown in Figure 1.18(c).



(a) $m = 2$; line rises (b) $m = 0$; line is horizontal (c) $m = -\frac{1}{3}$; line falls

Figure 1.18

Because the slope of a vertical line is not defined, its equation cannot be written in slope-intercept form. However, the equation of any line can be written in the **general form**

$$Ax + By + C = 0$$ General form of the equation of a line

where A and B are not *both* zero. For instance, the vertical line

$$x = a$$ Vertical line

can be represented by the general form

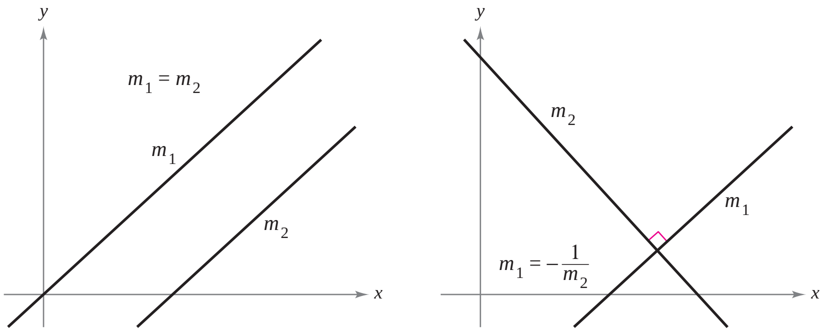
$$x - a = 0.$$ General form

SUMMARY OF EQUATIONS OF LINES

1. General form: $Ax + By + C = 0$
2. Vertical line: $x = a$
3. Horizontal line: $y = b$
4. Slope-intercept form: $y = mx + b$
5. Point-slope form: $y - y_1 = m(x - x_1)$

Parallel and Perpendicular Lines

The slope of a line is a convenient tool for determining whether two lines are parallel or perpendicular, as shown in Figure 1.19. Specifically, nonvertical lines with the same slope are parallel, and nonvertical lines whose slopes are negative reciprocals are perpendicular.



Parallel lines Perpendicular lines

Figure 1.19

REMARK In mathematics, the phrase "if and only if" is a way of stating two implications in one statement. For instance, the first statement at the right could be rewritten as the following two implications.

a. If two distinct nonvertical lines are parallel, then their slopes are equal.

b. If two distinct nonvertical lines have equal slopes, then they are parallel.

Parallel and Perpendicular Lines

1. Two distinct nonvertical lines are **parallel** if and only if their slopes are equal—that is, if and only if

$$m_1 = m_2.$$ Parallel ⟺ Slopes are equal.

2. Two nonvertical lines are **perpendicular** if and only if their slopes are negative reciprocals of each other—that is, if and only if

$$m_1 = -\frac{1}{m_2}.$$ Perpendicular ⟺ Slopes are negative reciprocals.

EXAMPLE 5 Finding Parallel and Perpendicular Lines

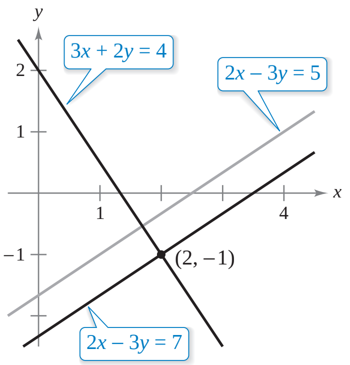
See LarsonCalculus.com for an interactive version of this type of example.

Find the general forms of the equations of the lines that pass through the point $(2, -1)$ and are (a) parallel to and (b) perpendicular to the line $2x - 3y = 5$.

Solution Begin by writing the linear equation $2x - 3y = 5$ in slope-intercept form.

$$2x - 3y = 5 \quad \text{Write original equation.}$$
$$y = \frac{2}{3}x - \frac{5}{3} \quad \text{Slope-intercept form}$$

So, the given line has a slope of $m = \frac{2}{3}$. (See Figure 1.20.)



Lines parallel and perpendicular to $2x - 3y = 5$

Figure 1.20

a. The line through $(2, -1)$ that is parallel to the given line also has a slope of $\frac{2}{3}$.

$$y - y_1 = m(x - x_1) \quad \text{Point-slope form}$$
$$y - (-1) = \frac{2}{3}(x - 2) \quad \text{Substitute.}$$
$$3(y + 1) = 2(x - 2) \quad \text{Simplify.}$$
$$3y + 3 = 2x - 4 \quad \text{Distributive Property}$$
$$2x - 3y - 7 = 0 \quad \text{General form}$$

Note the similarity to the equation of the given line, $2x - 3y = 5$.

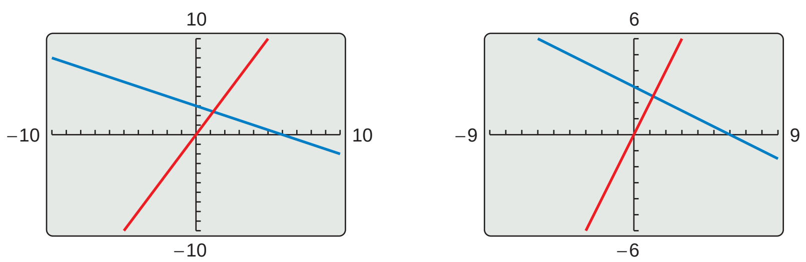
b. Using the negative reciprocal of the slope of the given line, you can determine that the slope of a line perpendicular to the given line is $-\frac{3}{2}$.

$$y - y_1 = m(x - x_1) \quad \text{Point-slope form}$$
$$y - (-1) = -\frac{3}{2}(x - 2) \quad \text{Substitute.}$$
$$2(y + 1) = -3(x - 2) \quad \text{Simplify.}$$
$$2y + 2 = -3x + 6 \quad \text{Distributive Property}$$
$$3x + 2y - 4 = 0 \quad \text{General form}$$

TECHNOLOGY PITFALL The slope of a line will appear distorted if you use different tick-mark spacing on the x- and y-axes. For instance, the graphing utility screens in Figures 1.21(a) and 1.21(b) both show the lines

$$y = 2x \quad \text{and} \quad y = -\frac{1}{2}x + 3.$$

Because these lines have slopes that are negative reciprocals, they must be perpendicular. In Figure 1.21(a), however, the lines don't appear to be perpendicular because the tick-mark spacing on the x-axis is not the same as that on the y-axis. In Figure 1.21(b), the lines appear perpendicular because the tick-mark spacing on the x-axis is the same as on the y-axis. This type of viewing window is said to have a *square setting*.



(a) Tick-mark spacing on the x-axis is not the same as tick-mark spacing on the y-axis.

(b) Tick-mark spacing on the x-axis is the same as tick-mark spacing on the y-axis.

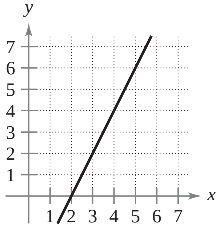
Figure 1.21

1.2 Exercises

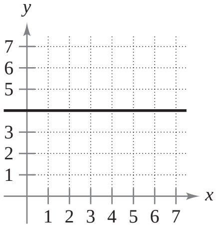
See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Estimating Slope **In Exercises 1–4, estimate the slope of the line from its graph. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

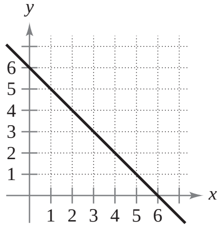
1.



2.



3.



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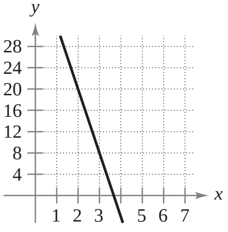


Finding the Slope of a Line **In Exercises 5–10, plot the pair of points and find the slope of the line passing through them.**

5. $(3, -4), (5, 2)$
6. $(1, 1), (-2, 7)$
7. $(4, 6), (4, 1)$
8. $(3, -5), (5, -5)$
9. $\left(-\frac{1}{2}, \frac{2}{3}\right), \left(-\frac{3}{4}, \frac{1}{6}\right)$
10. $\left(\frac{7}{8}, \frac{3}{4}\right), \left(\frac{5}{4}, -\frac{1}{4}\right)$

Sketching Lines **In Exercises 11 and 12, sketch the lines through the point with the indicated slopes. Make the sketches on the same set of coordinate axes.**

	Point	Slopes			
11.	$(3, 4)$	(a) 1	(b) -2	(c) $-\frac{3}{2}$	(d) Undefined
12.	$(-2, 5)$	(a) 3	(b) -3	(c) $\frac{1}{3}$	(d) 0

Finding Points on a Line **In Exercises 13–16, use the point on the line and the slope of the line to find three additional points that the line passes through. (There is more than one correct answer.)**

	Point	Slope		Point	Slope
13.	$(6, 2)$	$m = 0$	**14.**	$(-4, 3)$	m is undefined.
15.	$(1, 7)$	$m = -3$	**16.**	$(-2, -2)$	$m = 2$

Finding an Equation of a Line **In Exercises 17–22, find an equation of the line that passes through the point and has the indicated slope. Then sketch the line.**

	Point	Slope		Point	Slope
17.	$(0, 3)$	$m = \frac{3}{4}$	**18.**	$(-5, -2)$	m is undefined.
19.	$(0, 0)$	$m = \frac{2}{3}$	**20.**	$(0, 4)$	$m = 0$
21.	$(3, -2)$	$m = 3$	**22.**	$(-2, 4)$	$m = -\frac{3}{5}$

23. Conveyor Design

A moving conveyor is built to rise 1 meter for each 3 meters of horizontal change.

(a) Find the slope of the conveyor.

(b) Suppose the conveyor runs between two floors in a factory. Find the length of the conveyor when the vertical distance between floors is 10 feet.

24. Modeling Data The table shows the populations y (in millions) of the United States for 2004 through 2009. The variable t represents the time in years, with $t = 4$ corresponding to 2004. *(Source: U.S. Census Bureau)*

t	4	5	6	7	8	9
y	293.0	295.8	298.6	301.6	304.4	307.0

(a) Plot the data by hand and connect adjacent points with a line segment.

(b) Use the slope of each line segment to determine the year when the population increased least rapidly.

(c) Find the average rate of change of the population of the United States from 2004 through 2009.

(d) Use the average rate of change of the population to predict the population of the United States in 2020.

Finding the Slope and *y*-Intercept **In Exercises 25–30, find the slope and the *y*-intercept (if possible) of the line.**

25. $y = 4x - 3$
26. $-x + y = 1$
27. $x + 5y = 20$
28. $6x - 5y = 15$
29. $x = 4$
30. $y = -1$

Sketching a Line in the Plane **In Exercises 31–38, sketch a graph of the equation.**

31. $y = -3$
32. $x = 4$
33. $y = -2x + 1$
34. $y = \frac{1}{3}x - 1$
35. $y - 2 = \frac{3}{2}(x - 1)$
36. $y - 1 = 3(x + 4)$
37. $2x - y - 3 = 0$
38. $x + 2y + 6 = 0$

Finding an Equation of a Line **In Exercises 39–46, find an equation of the line that passes through the points. Then sketch the line.**

39. $(0, 0), (4, 8)$
40. $(-2, -2), (1, 7)$

xtrekx/Shutterstock.com

41. (2, 8), (5, 0) **42.** (−3, 6), (1, 2)

43. (6, 3), (6, 8) **44.** (1, −2), (3, −2)

45. $\left(\frac{1}{2}, \frac{7}{2}\right), \left(0, \frac{3}{4}\right)$ **46.** $\left(\frac{7}{8}, \frac{3}{4}\right), \left(\frac{5}{4}, -\frac{1}{4}\right)$

47. Finding an Equation of a Line Find an equation of the vertical line with x-intercept at 3.

48. Equation of a Line Show that the line with intercepts $(a, 0)$ and $(0, b)$ has the following equation.

$$\frac{x}{a} + \frac{y}{b} = 1, \quad a \neq 0, b \neq 0$$

Writing an Equation in General Form **In Exercises 49–54, use the result of Exercise 48 to write an equation of the line in general form.**

49. x-intercept: (2, 0)
y-intercept: (0, 3)

50. x-intercept: $\left(-\frac{2}{3}, 0\right)$
y-intercept: (0, −2)

51. Point on line: (1, 2)
x-intercept: $(a, 0)$
y-intercept: $(0, a)$
$(a \neq 0)$

52. Point on line: (−3, 4)
x-intercept: $(a, 0)$
y-intercept: $(0, a)$
$(a \neq 0)$

53. Point on line: (9, −2)
x-intercept: $(2a, 0)$
y-intercept: $(0, a)$
$(a \neq 0)$

54. Point on line: $\left(-\frac{2}{3}, -2\right)$
x-intercept: $(a, 0)$
y-intercept: $(0, -a)$
$(a \neq 0)$

Finding Parallel and Perpendicular Lines **In Exercises 55–62, write the general forms of the equations of the lines through the point (a) parallel to the given line and (b) perpendicular to the given line.**

	Point	Line		Point	Line
55.	(−7, −2)	$x = 1$	**56.**	(−1, 0)	$y = -3$
57.	(2, 5)	$x - y = -2$	**58.**	(−3, 2)	$x + y = 7$
59.	(2, 1)	$4x - 2y = 3$	**60.**	$\left(\frac{5}{6}, -\frac{1}{2}\right)$	$7x + 4y = 8$
61.	$\left(\frac{3}{4}, \frac{7}{8}\right)$	$5x - 3y = 0$	**62.**	(4, −5)	$3x + 4y = 7$

Rate of Change **In Exercises 63–66, you are given the dollar value of a product in 2012 *and* the rate at which the value of the product is expected to change during the next 5 years. Write a linear equation that gives the dollar value V of the product in terms of the year t. (Let $t = 0$ represent 2010.)**

	2012 Value	Rate
63.	$1850	$250 increase per year
64.	$156	$4.50 increase per year
65.	$17,200	$1600 decrease per year
66.	$245,000	$5600 decrease per year

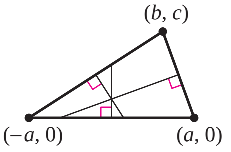
Collinear Points **In Exercises 67 and 68, determine whether the points are collinear. (Three points are *collinear* if they lie on the same line.)**

67. (−2, 1), (−1, 0), (2, −2) **68.** (0, 4), (7, −6), (−5, 11)

WRITING ABOUT CONCEPTS

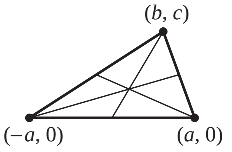
Finding Points of Intersection **In Exercises 69–71, find the coordinates of the point of intersection of the given segments. Explain your reasoning.**

69.



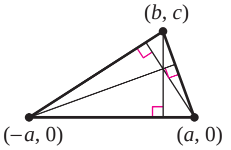
Perpendicular bisectors

70.



Medians

71.



Altitudes

72. Collinear Points Show that the points of intersection in Exercises 69, 70, and 71 are collinear.

73. Analyzing a Line A line is represented by the equation $ax + by = 4$.

(a) When is the line parallel to the x-axis?

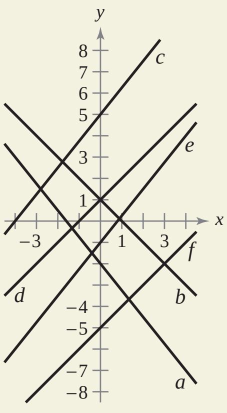
(b) When is the line parallel to the y-axis?

(c) Give values for a and b such that the line has a slope of $\frac{5}{8}$.

(d) Give values for a and b such that the line is perpendicular to $y = \frac{2}{5}x + 3$.

(e) Give values for a and b such that the line coincides with the graph of $5x + 6y = 8$.

74. HOW DO YOU SEE IT? Several lines (labeled a–f) are shown in the figure below.



(a) Which lines have a positive slope?

(b) Which lines have a negative slope?

(c) Which lines appear parallel?

(d) Which lines appear perpendicular?

75. Temperature Conversion Find a linear equation that expresses the relationship between the temperature in degrees Celsius C and degrees Fahrenheit F. Use the fact that water freezes at 0°C (32°F) and boils at 100°C (212°F). Use the equation to convert 72°F to degrees Celsius.

76. Reimbursed Expenses A company reimburses its sales representatives \$200 per day for lodging and meals plus \$0.51 per mile driven. Write a linear equation giving the daily cost C to the company in terms of x, the number of miles driven. How much does it cost the company if a sales representative drives 137 miles on a given day?

77. Choosing a Job As a salesperson, you receive a monthly salary of \$2000, plus a commission of 7% of sales. You are offered a new job at \$2300 per month, plus a commission of 5% of sales.

(a) Write linear equations for your monthly wage W in terms of your monthly sales s for your current job and your job offer.

(b) Use a graphing utility to graph each equation and find the point of intersection. What does it signify?

(c) You think you can sell \$20,000 worth of a product per month. Should you change jobs? Explain.

78. Straight-Line Depreciation A small business purchases a piece of equipment for \$875. After 5 years, the equipment will be outdated, having no value.

(a) Write a linear equation giving the value y of the equipment in terms of the time x (in years), $0 \le x \le 5$.

(b) Find the value of the equipment when $x = 2$.

(c) Estimate (to two-decimal-place accuracy) the time when the value of the equipment is \$200.

79. Apartment Rental A real estate office manages an apartment complex with 50 units. When the rent is \$780 per month, all 50 units are occupied. However, when the rent is \$825, the average number of occupied units drops to 47. Assume that the relationship between the monthly rent p and the demand x is linear. (*Note:* The term *demand* refers to the number of occupied units.)

(a) Write a linear equation giving the demand x in terms of the rent p.

(b) *Linear extrapolation* Use a graphing utility to graph the demand equation and use the *trace* feature to predict the number of units occupied when the rent is raised to \$855.

(c) *Linear interpolation* Predict the number of units occupied when the rent is lowered to \$795. Verify graphically.

80. Modeling Data An instructor gives regular 20-point quizzes and 100-point exams in a mathematics course. Average scores for six students, given as ordered pairs (x, y), where x is the average quiz score and y is the average exam score, are (18, 87), (10, 55), (19, 96), (16, 79), (13, 76), and (15, 82).

(a) Use the regression capabilities of a graphing utility to find the least squares regression line for the data.

(b) Use a graphing utility to plot the points and graph the regression line in the same viewing window.

(c) Use the regression line to predict the average exam score for a student with an average quiz score of 17.

(d) Interpret the meaning of the slope of the regression line.

(e) The instructor adds 4 points to the average exam score of everyone in the class. Describe the changes in the positions of the plotted points and the change in the equation of the line.

81. Tangent Line Find an equation of the line tangent to the circle $x^2 + y^2 = 169$ at the point $(5, 12)$.

82. Tangent Line Find an equation of the line tangent to the circle $(x - 1)^2 + (y - 1)^2 = 25$ at the point $(4, -3)$.

Distance In Exercises 83–86, find the distance between the point and line, or between the lines, using the formula for the distance between the point (x_1, y_1) and the line $Ax + By + C = 0$.

$$\textbf{Distance} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}$$

83. Point: $(-2, 1)$
Line: $x - y - 2 = 0$

84. Point: $(2, 3)$
Line: $4x + 3y = 10$

85. Line: $x + y = 1$
Line: $x + y = 5$

86. Line: $3x - 4y = 1$
Line: $3x - 4y = 10$

87. Distance Show that the distance between the point (x_1, y_1) and the line $Ax + By + C = 0$ is

$$\text{Distance} = \frac{|Ax_1 + By_1 + C|}{\sqrt{A^2 + B^2}}.$$

88. Distance Write the distance d between the point $(3, 1)$ and the line $y = mx + 4$ in terms of m. Use a graphing utility to graph the equation. When is the distance 0? Explain the result geometrically.

89. Proof Prove that the diagonals of a rhombus intersect at right angles. (A rhombus is a quadrilateral with sides of equal lengths.)

90. Proof Prove that the figure formed by connecting consecutive midpoints of the sides of any quadrilateral is a parallelogram.

91. Proof Prove that if the points (x_1, y_1) and (x_2, y_2) lie on the same line as (x_1^*, y_1^*) and (x_2^*, y_2^*), then

$$\frac{y_2^* - y_1^*}{x_2^* - x_1^*} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Assume $x_1 \ne x_2$ and $x_1^* \ne x_2^*$.

92. Proof Prove that if the slopes of two nonvertical lines are negative reciprocals of each other, then the lines are perpendicular.

True or False? In Exercises 93–96, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

93. The lines represented by $ax + by = c_1$ and $bx - ay = c_2$ are perpendicular. Assume $a \ne 0$ and $b \ne 0$.

94. It is possible for two lines with positive slopes to be perpendicular to each other.

95. If a line contains points in both the first and third quadrants, then its slope must be positive.

96. The equation of any line can be written in general form.

1.3 Functions and Their Graphs

- Use function notation to represent and evaluate a function.
- Find the domain and range of a function.
- Sketch the graph of a function.
- Identify different types of transformations of functions.
- Classify functions and recognize combinations of functions.

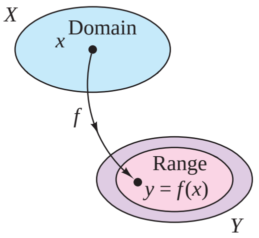
Functions and Function Notation

A **relation** between two sets X and Y is a set of ordered pairs, each of the form (x, y), where x is a member of X and y is a member of Y. A **function** from X to Y is a relation between X and Y that has the property that any two ordered pairs with the same x-value also have the same y-value. The variable x is the **independent variable,** and the variable y is the **dependent variable.**

Many real-life situations can be modeled by functions. For instance, the area A of a circle is a function of the circle's radius r.

$$A = \pi r^2 \qquad \text{A is a function of r.}$$

In this case, r is the independent variable and A is the dependent variable.



A real-valued function f of a real variable
Figure 1.22

Definition of a Real-Valued Function of a Real Variable

Let X and Y be sets of real numbers. A **real-valued function f of a real variable x** from X to Y is a correspondence that assigns to each number x in X exactly one number y in Y.

The **domain** of f is the set X. The number y is the **image** of x under f and is denoted by $f(x)$, which is called the **value of f at x.** The **range** of f is a subset of Y and consists of all images of numbers in X (see Figure 1.22).

FUNCTION NOTATION

The word *function* was first used by Gottfried Wilhelm Leibniz in 1694 as a term to denote any quantity connected with a curve, such as the coordinates of a point on a curve or the slope of a curve. Forty years later, Leonhard Euler used the word "function" to describe any expression made up of a variable and some constants. He introduced the notation $y = f(x)$.

Functions can be specified in a variety of ways. In this text, however, you will concentrate primarily on functions that are given by equations involving the dependent and independent variables. For instance, the equation

$$x^2 + 2y = 1 \qquad \text{Equation in implicit form}$$

defines y, the dependent variable, as a function of x, the independent variable. To **evaluate** this function (that is, to find the y-value that corresponds to a given x-value), it is convenient to isolate y on the left side of the equation.

$$y = \tfrac{1}{2}(1 - x^2) \qquad \text{Equation in explicit form}$$

Using f as the name of the function, you can write this equation as

$$f(x) = \tfrac{1}{2}(1 - x^2). \qquad \text{Function notation}$$

The original equation

$$x^2 + 2y = 1$$

implicitly defines y as a function of x. When you solve the equation for y, you are writing the equation in **explicit** form.

Function notation has the advantage of clearly identifying the dependent variable as $f(x)$ while at the same time telling you that x is the independent variable and that the function itself is "f." The symbol $f(x)$ is read "f of x." Function notation allows you to be less wordy. Instead of asking "What is the value of y that corresponds to $x = 3$?" you can ask "What is $f(3)$?"

In an equation that defines a function of x, the role of the variable x is simply that of a placeholder. For instance, the function

$$f(x) = 2x^2 - 4x + 1$$

can be described by the form

$$f(\square) = 2(\square)^2 - 4(\square) + 1$$

where rectangles are used instead of x. To evaluate $f(-2)$, replace each rectangle with -2.

$$\begin{aligned} f(-2) &= 2(-2)^2 - 4(-2) + 1 && \text{Substitute } -2 \text{ for } x. \\ &= 2(4) + 8 + 1 && \text{Simplify.} \\ &= 17 && \text{Simplify.} \end{aligned}$$

Although f is often used as a convenient function name and x as the independent variable, you can use other symbols. For instance, these three equations all define the same function.

$$\begin{aligned} f(x) &= x^2 - 4x + 7 && \text{Function name is } f\text{, independent variable is } x. \\ f(t) &= t^2 - 4t + 7 && \text{Function name is } f\text{, independent variable is } t. \\ g(s) &= s^2 - 4s + 7 && \text{Function name is } g\text{, independent variable is } s. \end{aligned}$$

EXAMPLE 1 Evaluating a Function

For the function f defined by $f(x) = x^2 + 7$, evaluate each expression.

a. $f(3a)$ **b.** $f(b - 1)$ **c.** $\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

Solution

a.
$$\begin{aligned} f(3a) &= (3a)^2 + 7 && \text{Substitute } 3a \text{ for } x. \\ &= 9a^2 + 7 && \text{Simplify.} \end{aligned}$$

b.
$$\begin{aligned} f(b - 1) &= (b - 1)^2 + 7 && \text{Substitute } b - 1 \text{ for } x. \\ &= b^2 - 2b + 1 + 7 && \text{Expand binomial.} \\ &= b^2 - 2b + 8 && \text{Simplify.} \end{aligned}$$

c.
$$\begin{aligned} \frac{f(x + \Delta x) - f(x)}{\Delta x} &= \frac{[(x + \Delta x)^2 + 7] - (x^2 + 7)}{\Delta x} \\ &= \frac{x^2 + 2x\Delta x + (\Delta x)^2 + 7 - x^2 - 7}{\Delta x} \\ &= \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= \frac{\Delta x(2x + \Delta x)}{\Delta x} \\ &= 2x + \Delta x, \quad \Delta x \neq 0 \end{aligned}$$

REMARK The expression in Example 1(c) is called a *difference quotient* and has a special significance in calculus. You will learn more about this in Chapter 3.

In calculus, it is important to specify the domain of a function or expression clearly. For instance, in Example 1(c), the two expressions

$$\frac{f(x + \Delta x) - f(x)}{\Delta x} \quad \text{and} \quad 2x + \Delta x, \quad \Delta x \neq 0$$

are equivalent because $\Delta x = 0$ is excluded from the domain of each expression. Without a stated domain restriction, the two expressions would not be equivalent.

The Domain and Range of a Function

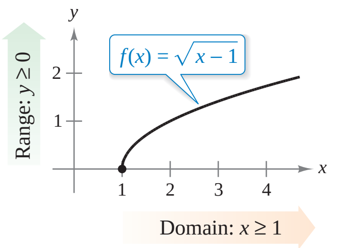
The domain of a function can be described explicitly, or it may be described *implicitly* by an equation used to define the function. The implied domain is the set of all real numbers for which the equation is defined, whereas an explicitly defined domain is one that is given along with the function. For example, the function

$$f(x) = \frac{1}{x^2 - 4}, \quad 4 \le x \le 5$$

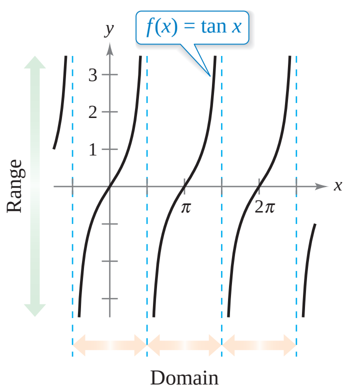
has an explicitly defined domain given by $\{x: 4 \le x \le 5\}$. On the other hand, the function

$$g(x) = \frac{1}{x^2 - 4}$$

has an implied domain that is the set $\{x: x \ne \pm 2\}$.



(a) The domain of f is $[1, \infty)$, and the range is $[0, \infty)$.



(b) The domain of f is all x-values such that $x \ne \frac{\pi}{2} + n\pi$, and the range is $(-\infty, \infty)$.

Figure 1.23

EXAMPLE 2 Finding the Domain and Range of a Function

a. The domain of the function

$$f(x) = \sqrt{x - 1}$$

is the set of all x-values for which $x - 1 \ge 0$, which is the interval $[1, \infty)$. To find the range, observe that $f(x) = \sqrt{x - 1}$ is never negative. So, the range is the interval $[0, \infty)$, as shown in Figure 1.23(a).

b. The domain of the tangent function

$$f(x) = \tan x$$

is the set of all x-values such that

$$x \ne \frac{\pi}{2} + n\pi, \quad n \text{ is an integer.}$$ Domain of tangent function

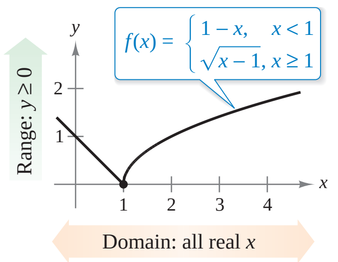
The range of this function is the set of all real numbers, as shown in Figure 1.23(b). For a review of the characteristics of this and other trigonometric functions, see Appendix C.

EXAMPLE 3 A Function Defined by More than One Equation

For the piecewise-defined function

$$f(x) = \begin{cases} 1 - x, & x < 1 \\ \sqrt{x - 1}, & x \ge 1 \end{cases}$$

f is defined for $x < 1$ and $x \ge 1$. So, the domain is the set of all real numbers. On the portion of the domain for which $x \ge 1$, the function behaves as in Example 2(a). For $x < 1$, the values of $1 - x$ are positive. So, the range of the function is the interval $[0, \infty)$. (See Figure 1.24.)

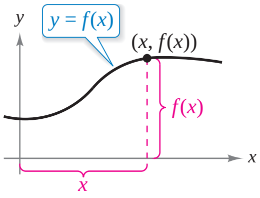


The domain of f is $(-\infty, \infty)$, and the range is $[0, \infty)$.

Figure 1.24

A function from X to Y is **one-to-one** when to each y-value in the range there corresponds exactly one x-value in the domain. For instance, the function in Example 2(a) is one-to-one, whereas the functions in Examples 2(b) and 3 are not one-to-one. A function from X to Y is **onto** when its range consists of all of Y.

The Graph of a Function



The graph of a function
Figure 1.25

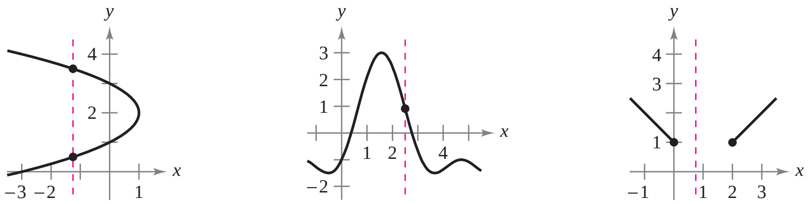
The graph of the function $y = f(x)$ consists of all points $(x, f(x))$, where x is in the domain of f. In Figure 1.25, note that

$$x = \text{the directed distance from the } y\text{-axis}$$

and

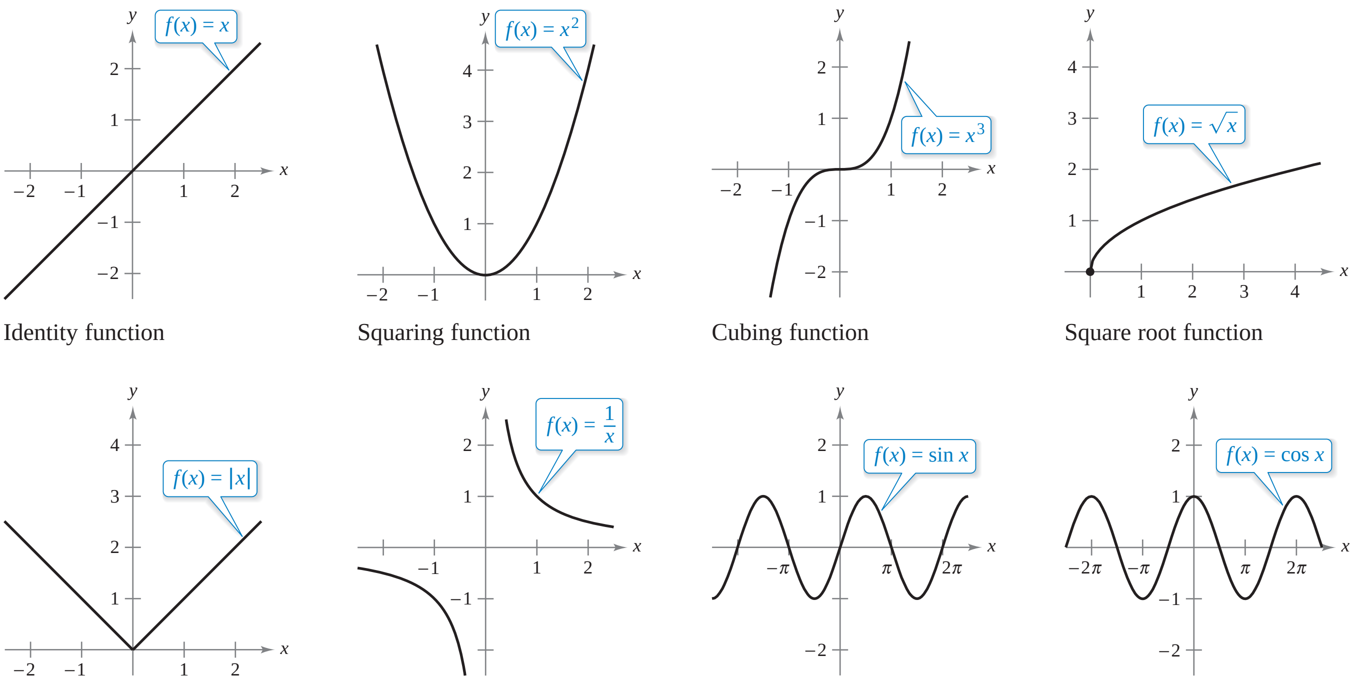
$$f(x) = \text{the directed distance from the } x\text{-axis.}$$

A vertical line can intersect the graph of a function of x at most *once*. This observation provides a convenient visual test, called the **Vertical Line Test,** for functions of x. That is, a graph in the coordinate plane is the graph of a function of x if and only if no vertical line intersects the graph at more than one point. For example, in Figure 1.26(a), you can see that the graph does not define y as a function of x because a vertical line intersects the graph twice, whereas in Figures 1.26(b) and (c), the graphs do define y as a function of x.



(a) Not a function of x **(b)** A function of x **(c)** A function of x
Figure 1.26

Figure 1.27 shows the graphs of eight basic functions. You should be able to recognize these graphs. (Graphs of the other four basic trigonometric functions are shown in Appendix C.)



Identity function | Squaring function | Cubing function | Square root function

Absolute value function | Rational function | Sine function | Cosine function

The graphs of eight basic functions
Figure 1.27

Transformations of Functions

Some families of graphs have the same basic shape. For example, compare the graph of $y = x^2$ with the graphs of the four other quadratic functions shown in Figure 1.28.

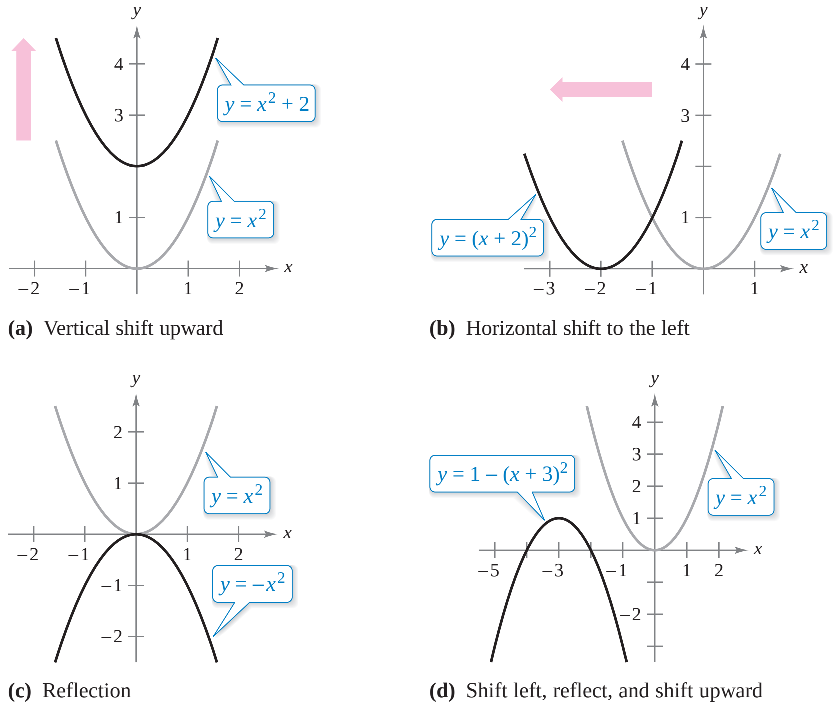


Figure 1.28

Each of the graphs in Figure 1.28 is a **transformation** of the graph of $y = x^2$. The three basic types of transformations illustrated by these graphs are vertical shifts, horizontal shifts, and reflections. Function notation lends itself well to describing transformations of graphs in the plane. For instance, using

$f(x) = x^2$ Original function

as the original function, the transformations shown in Figure 1.28 can be represented by these equations.

a. $y = f(x) + 2$ Vertical shift up two units

b. $y = f(x + 2)$ Horizontal shift to the left two units

c. $y = -f(x)$ Reflection about the x-axis

d. $y = -f(x + 3) + 1$ Shift left three units, reflect about the x-axis, and shift up one unit

Basic Types of Transformations ($c > 0$)

Original graph:	$y = f(x)$
Horizontal shift c units to the **right:**	$y = f(x - c)$
Horizontal shift c units to the **left:**	$y = f(x + c)$
Vertical shift c units **downward:**	$y = f(x) - c$
Vertical shift c units **upward:**	$y = f(x) + c$
Reflection (about the x-axis):	$y = -f(x)$
Reflection (about the y-axis):	$y = f(-x)$
Reflection (about the origin):	$y = -f(-x)$

Classifications and Combinations of Functions

LEONHARD EULER (1707–1783)

In addition to making major contributions to almost every branch of mathematics, Euler was one of the first to apply calculus to real-life problems in physics. His extensive published writings include such topics as shipbuilding, acoustics, optics, astronomy, mechanics, and magnetism.

See LarsonCalculus.com to read more of this biography.

The modern notion of a function is derived from the efforts of many seventeenth- and eighteenth-century mathematicians. Of particular note was Leonhard Euler, who introduced the function notation $y = f(x)$. By the end of the eighteenth century, mathematicians and scientists had concluded that many real-world phenomena could be represented by mathematical models taken from a collection of functions called **elementary functions.** Elementary functions fall into three categories.

1. Algebraic functions (polynomial, radical, rational)
2. Trigonometric functions (sine, cosine, tangent, and so on)
3. Exponential and logarithmic functions

You can review the trigonometric functions in Appendix C. The other nonalgebraic functions, such as the inverse trigonometric functions and the exponential and logarithmic functions, are introduced in Sections 1.5 and 1.6.

The most common type of algebraic function is a **polynomial function**

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$$

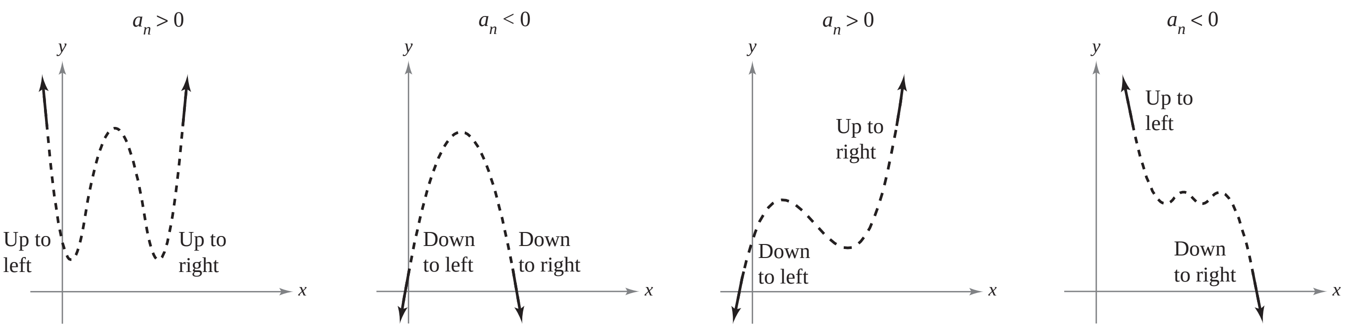
where n is a nonnegative integer. The numbers a_i are **coefficients,** with a_n the **leading coefficient** and a_0 the **constant term** of the polynomial function. If $a_n \neq 0$, then n is the **degree** of the polynomial function. The zero polynomial $f(x) = 0$ is not assigned a degree. It is common practice to use subscript notation for coefficients of general polynomial functions, but for polynomial functions of low degree, these simpler forms are often used. (Note that $a \neq 0$.)

Zeroth degree:	$f(x) = a$	Constant function
First degree:	$f(x) = ax + b$	Linear function
Second degree:	$f(x) = ax^2 + bx + c$	Quadratic function
Third degree:	$f(x) = ax^3 + bx^2 + cx + d$	Cubic function

Although the graph of a nonconstant polynomial function can have several turns, eventually the graph will rise or fall without bound as x moves to the right or left. Whether the graph of

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdot \cdot \cdot + a_2x^2 + a_1x + a_0$$

eventually rises or falls can be determined by the function's degree (odd or even) and by the leading coefficient a_n, as indicated in Figure 1.29. Note that the dashed portions of the graphs indicate that the **Leading Coefficient Test** determines *only* the right and left behavior of the graph.



Graphs of polynomial functions of even degree

Graphs of polynomial functions of odd degree

The Leading Coefficient Test for polynomial functions
Figure 1.29

North Wind Picture Archives/Alamy

FOR FURTHER INFORMATION For more on the history of the concept of a function, see the article "Evolution of the Function Concept: A Brief Survey" by Israel Kleiner in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

Just as a rational number can be written as the quotient of two integers, a **rational function** can be written as the quotient of two polynomials. Specifically, a function f is rational when it has the form

$$f(x) = \frac{p(x)}{q(x)}, \quad q(x) \neq 0$$

where $p(x)$ and $q(x)$ are polynomials.

Polynomial functions and rational functions are examples of **algebraic functions.** An algebraic function of x is one that can be expressed as a finite number of sums, differences, multiples, quotients, and radicals involving x^n. For example,

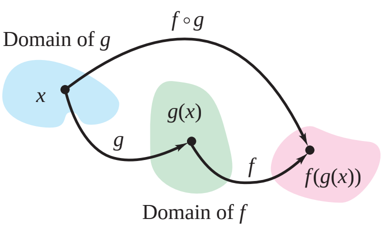
$$f(x) = \sqrt{x + 1}$$

is algebraic. Functions that are not algebraic are **transcendental.** For instance, the trigonometric functions are transcendental.

Two functions can be combined in various ways to create new functions. For example, given $f(x) = 2x - 3$ and $g(x) = x^2 + 1$, you can form the functions shown.

$$(f + g)(x) = f(x) + g(x) = (2x - 3) + (x^2 + 1) \quad \text{Sum}$$
$$(f - g)(x) = f(x) - g(x) = (2x - 3) - (x^2 + 1) \quad \text{Difference}$$
$$(fg)(x) = f(x)g(x) = (2x - 3)(x^2 + 1) \quad \text{Product}$$
$$(f/g)(x) = \frac{f(x)}{g(x)} = \frac{2x - 3}{x^2 + 1} \quad \text{Quotient}$$

You can combine two functions in yet another way, called **composition.** The resulting function is called a **composite function.**



The domain of the composite function $f \circ g$
Figure 1.30

Definition of Composite Function

Let f and g be functions. The function $(f \circ g)(x) = f(g(x))$ is the **composite** of f with g. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f (see Figure 1.30).

The composite of f with g is generally not the same as the composite of g with f. This is shown in the next example.

EXAMPLE 4 Finding Composite Functions

See LarsonCalculus.com for an interactive version of this type of example.

For $f(x) = 2x - 3$ and $g(x) = \cos x$, find each composite function.

a. $f \circ g$ **b.** $g \circ f$

Solution

a.
$$\begin{aligned}(f \circ g)(x) &= f(g(x)) && \text{Definition of } f \circ g\\ &= f(\cos x) && \text{Substitute } \cos x \text{ for } g(x).\\ &= 2(\cos x) - 3 && \text{Definition of } f(x)\\ &= 2\cos x - 3 && \text{Simplify.}\end{aligned}$$

b.
$$\begin{aligned}(g \circ f)(x) &= g(f(x)) && \text{Definition of } g \circ f\\ &= g(2x - 3) && \text{Substitute } 2x - 3 \text{ for } f(x).\\ &= \cos(2x - 3) && \text{Definition of } g(x)\end{aligned}$$

Note that $(f \circ g)(x) \neq (g \circ f)(x)$.

> **Exploration**
>
> Use a graphing utility to graph each function. Determine whether the function is *even, odd,* or *neither*.
>
> $f(x) = x^2 - x^4$
>
> $g(x) = 2x^3 + 1$
>
> $h(x) = x^5 - 2x^3 + x$
>
> $j(x) = 2 - x^6 - x^8$
>
> $k(x) = x^5 - 2x^4 + x - 2$
>
> $p(x) = x^9 + 3x^5 - x^3 + x$
>
> Describe a way to identify a function as odd or even by inspecting the equation.

In Section 1.1, an x-intercept of a graph was defined to be a point $(a, 0)$ at which the graph crosses the x-axis. If the graph represents a function f, then the number a is a **zero** of f. In other words, *the zeros of a function f are the solutions of the equation* $f(x) = 0$. For example, the function

$$f(x) = x - 4$$

has a zero at $x = 4$ because $f(4) = 0$.

In Section 1.1, you also studied different types of symmetry. In the terminology of functions, a function is **even** when its graph is symmetric with respect to the y-axis, and is **odd** when its graph is symmetric with respect to the origin. The symmetry tests in Section 1.1 yield the following test for even and odd functions.

> **Test for Even and Odd Functions**
>
> The function $y = f(x)$ is **even** when
>
> $$f(-x) = f(x).$$
>
> The function $y = f(x)$ is **odd** when
>
> $$f(-x) = -f(x).$$

EXAMPLE 5 Even and Odd Functions and Zeros of Functions

Determine whether each function is even, odd, or neither. Then find the zeros of the function.

a. $f(x) = x^3 - x$ **b.** $g(x) = 1 + \cos x$

Solution

a. This function is odd because

$$f(-x) = (-x)^3 - (-x) = -x^3 + x = -(x^3 - x) = -f(x).$$

The zeros of f are

$$\begin{aligned} x^3 - x &= 0 && \text{Let } f(x) = 0.\\ x(x^2 - 1) &= 0 && \text{Factor.}\\ x(x - 1)(x + 1) &= 0 && \text{Factor.}\\ x &= 0, 1, -1. && \text{Zeros of } f \end{aligned}$$

See Figure 1.31(a).

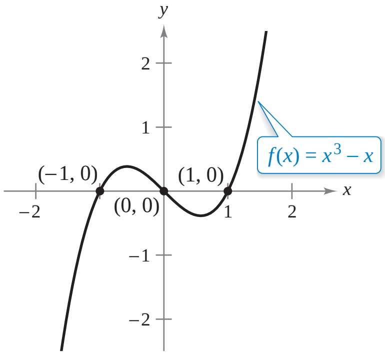
b. This function is even because

$$g(-x) = 1 + \cos(-x) = 1 + \cos x = g(x). \qquad \cos(-x) = \cos(x)$$

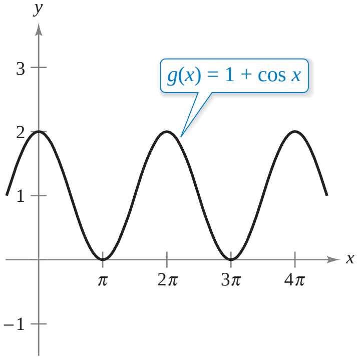
The zeros of g are

$$\begin{aligned} 1 + \cos x &= 0 && \text{Let } g(x) = 0.\\ \cos x &= -1 && \text{Subtract 1 from each side.}\\ x &= (2n + 1)\pi, \ n \text{ is an integer.} && \text{Zeros of } g \end{aligned}$$

See Figure 1.31(b).



(a) Odd function



(b) Even function

Figure 1.31

Each function in Example 5 is either even or odd. However, some functions, such as

$$f(x) = x^2 + x + 1$$

are neither even nor odd.

1.3 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Evaluating a Function **In Exercises 1–10, evaluate the function at the given value(s) of the independent variable. Simplify the results.**

1. $f(x) = 7x - 4$
(a) $f(0)$ (b) $f(-3)$ (c) $f(b)$ (d) $f(x - 1)$

2. $f(x) = \sqrt{x + 5}$
(a) $f(-4)$ (b) $f(11)$ (c) $f(4)$ (d) $f(x + \Delta x)$

3. $g(x) = 5 - x^2$
(a) $g(0)$ (b) $g(\sqrt{5})$ (c) $g(-2)$ (d) $g(t - 1)$

4. $g(x) = x^2(x - 4)$
(a) $g(4)$ (b) $g(\frac{3}{2})$ (c) $g(c)$ (d) $g(t + 4)$

5. $f(x) = \cos 2x$
(a) $f(0)$ (b) $f\left(-\frac{\pi}{4}\right)$ (c) $f\left(\frac{\pi}{3}\right)$ (d) $f(\pi)$

6. $f(x) = \sin x$
(a) $f(\pi)$ (b) $f\left(\frac{5\pi}{4}\right)$ (c) $f\left(\frac{2\pi}{3}\right)$ (d) $f\left(-\frac{\pi}{6}\right)$

7. $f(x) = x^3$
$\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

8. $f(x) = 3x - 1$
$\dfrac{f(x) - f(1)}{x - 1}$

9. $f(x) = \dfrac{1}{\sqrt{x - 1}}$
$\dfrac{f(x) - f(2)}{x - 2}$

10. $f(x) = x^3 - x$
$\dfrac{f(x) - f(1)}{x - 1}$

Finding the Domain and Range of a Function **In Exercises 11–22, find the domain and range of the function.**

11. $f(x) = 4x^2$

12. $g(x) = x^2 - 5$

13. $f(x) = x^3$

14. $h(x) = 4 - x^2$

15. $g(x) = \sqrt{6x}$

16. $h(x) = -\sqrt{x + 3}$

17. $f(x) = \sqrt{16 - x^2}$

18. $f(x) = |x - 3|$

19. $f(t) = \sec \dfrac{\pi t}{4}$

20. $h(t) = \cot t$

21. $f(x) = \dfrac{3}{x}$

22. $f(x) = \dfrac{x - 2}{x + 4}$

Finding the Domain of a Function **In Exercises 23–28, find the domain of the function.**

23. $f(x) = \sqrt{x} + \sqrt{1 - x}$

24. $f(x) = \sqrt{x^2 - 3x + 2}$

25. $g(x) = \dfrac{2}{1 - \cos x}$

26. $h(x) = \dfrac{1}{\sin x - (1/2)}$

27. $f(x) = \dfrac{1}{|x + 3|}$

28. $g(x) = \dfrac{1}{|x^2 - 4|}$

Finding the Domain and Range of a Piecewise Function **In Exercises 29–32, evaluate the function as indicated. Determine its domain and range.**

29. $f(x) = \begin{cases} 2x + 1, & x < 0 \\ 2x + 2, & x \geq 0 \end{cases}$
(a) $f(-1)$ (b) $f(0)$ (c) $f(2)$ (d) $f(t^2 + 1)$

30. $f(x) = \begin{cases} x^2 + 2, & x \leq 1 \\ 2x^2 + 2, & x > 1 \end{cases}$
(a) $f(-2)$ (b) $f(0)$ (c) $f(1)$ (d) $f(s^2 + 2)$

31. $f(x) = \begin{cases} |x| + 1, & x < 1 \\ -x + 1, & x \geq 1 \end{cases}$
(a) $f(-3)$ (b) $f(1)$ (c) $f(3)$ (d) $f(b^2 + 1)$

32. $f(x) = \begin{cases} \sqrt{x + 4}, & x \leq 5 \\ (x - 5)^2, & x > 5 \end{cases}$
(a) $f(-3)$ (b) $f(0)$ (c) $f(5)$ (d) $f(10)$

Sketching a Graph of a Function **In Exercises 33–40, sketch a graph of the function and find its domain and range. Use a graphing utility to verify your graph.**

33. $f(x) = 4 - x$

34. $g(x) = \dfrac{4}{x}$

35. $h(x) = \sqrt{x - 6}$

36. $f(x) = \frac{1}{4}x^3 + 3$

37. $f(x) = \sqrt{9 - x^2}$

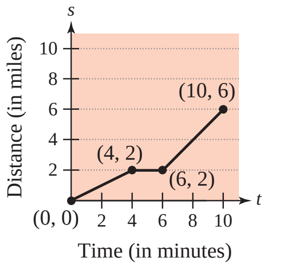
38. $f(x) = x + \sqrt{4 - x^2}$

39. $g(t) = 3 \sin \pi t$

40. $h(\theta) = -5 \cos \dfrac{\theta}{2}$

WRITING ABOUT CONCEPTS

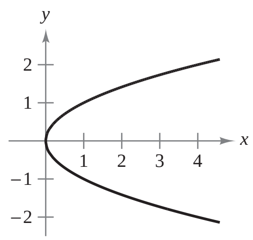
41. Describing a Graph The graph of the distance that a student drives in a 10-minute trip to school is shown in the figure. Give a verbal description of the characteristics of the student's drive to school.



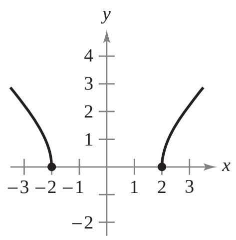
42. Sketching a Graph A student who commutes 27 miles to attend college remembers, after driving a few minutes, that a term paper that is due has been forgotten. Driving faster than usual, the student returns home, picks up the paper, and once again starts toward school. Sketch a possible graph of the student's distance from home as a function of time.

Using the Vertical Line Test **In Exercises 43–46, use the Vertical Line Test to determine whether y is a function of x. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

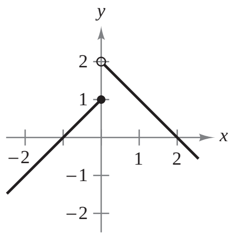
43. $x - y^2 = 0$



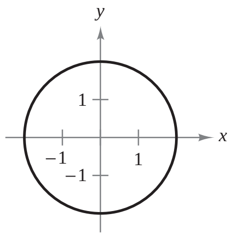
44. $\sqrt{x^2 - 4} - y = 0$



45. $y = \begin{cases} x + 1, & x \le 0 \\ -x + 2, & x > 0 \end{cases}$



46. $x^2 + y^2 = 4$



Deciding Whether an Equation Is a Function **In Exercises 47–50, determine whether y is a function of x.**

47. $x^2 + y^2 = 16$

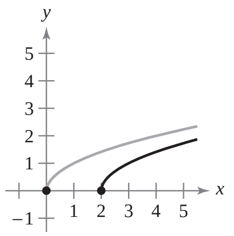
48. $x^2 + y = 16$

49. $y^2 = x^2 - 1$

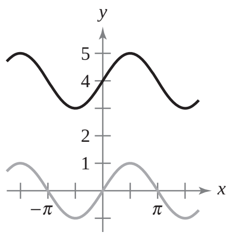
50. $x^2y - x^2 + 4y = 0$

Transformation of a Function **In Exercises 51–54, the graph shows one of the eight basic functions on page 22 and a transformation of the function. Describe the transformation. Then use your description to write an equation for the transformation.**

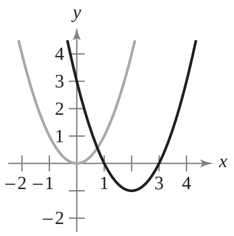
51.



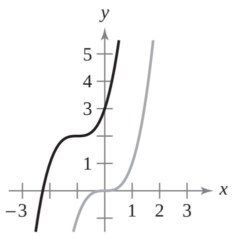
52.



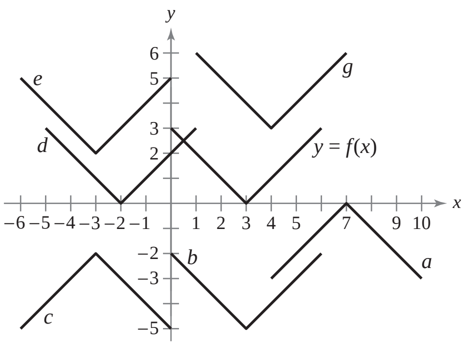
53.



54.



Matching **In Exercises 55–60, use the graph of $y = f(x)$ to match the function with its graph.**



55. $y = f(x + 5)$

56. $y = f(x) - 5$

57. $y = -f(-x) - 2$

58. $y = -f(x - 4)$

59. $y = f(x + 6) + 2$

60. $y = f(x - 1) + 3$

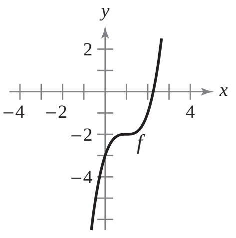
61. Sketching Transformations Use the graph of f shown in the figure to sketch the graph of each function. To print an enlarged copy of the graph, go to *MathGraphs.com*.

(a) $f(x + 3)$ (b) $f(x - 1)$

(c) $f(x) + 2$ (d) $f(x) - 4$

(e) $3f(x)$ (f) $\frac{1}{4}f(x)$

(g) $-f(x)$ (h) $-f(-x)$



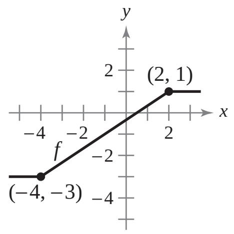
62. Sketching Transformations Use the graph of f shown in the figure to sketch the graph of each function. To print an enlarged copy of the graph, go to *MathGraphs.com*.

(a) $f(x - 4)$ (b) $f(x + 2)$

(c) $f(x) + 4$ (d) $f(x) - 1$

(e) $2f(x)$ (f) $\frac{1}{2}f(x)$

(g) $f(-x)$ (h) $-f(x)$



Combinations of Functions **In Exercises 63 and 64, find (a) $f(x) + g(x)$, (b) $f(x) - g(x)$, (c) $f(x) \cdot g(x)$, and (d) $f(x)/g(x)$.**

63. $f(x) = 3x - 4$
$g(x) = 4$

64. $f(x) = x^2 + 5x + 4$
$g(x) = x + 1$

65. Evaluating Composite Functions Given $f(x) = \sqrt{x}$ and $g(x) = x^2 - 1$, evaluate each expression.

(a) $f(g(1))$ (b) $g(f(1))$ (c) $g(f(0))$

(d) $f(g(-4))$ (e) $f(g(x))$ (f) $g(f(x))$

66. Evaluating Composite Functions Given $f(x) = \sin x$ and $g(x) = \pi x$, evaluate each expression.

(a) $f(g(2))$ (b) $f\left(g\left(\frac{1}{2}\right)\right)$ (c) $g(f(0))$

(d) $g\left(f\left(\frac{\pi}{4}\right)\right)$ (e) $f(g(x))$ (f) $g(f(x))$

Finding Composite Functions **In Exercises 67–70, find the composite functions $f \circ g$ and $g \circ f$. Find the domain of each composite function. Are the two composite functions equal?**

67. $f(x) = x^2,\ g(x) = \sqrt{x}$

68. $f(x) = x^2 - 1,\ g(x) = \cos x$

69. $f(x) = \dfrac{3}{x},\ g(x) = x^2 - 1$

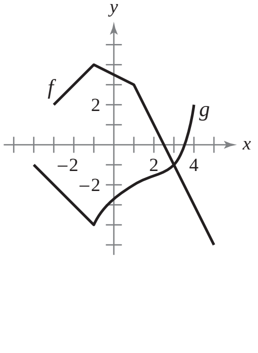
70. $f(x) = \dfrac{1}{x},\ g(x) = \sqrt{x + 2}$

71. Evaluating Composite Functions Use the graphs of f and g to evaluate each expression. If the result is undefined, explain why.

(a) $(f \circ g)(3)$ (b) $g(f(2))$

(c) $g(f(5))$ (d) $(f \circ g)(-3)$

(e) $(g \circ f)(-1)$ (f) $f(g(-1))$



72. **Ripples** A pebble is dropped into a calm pond, causing ripples in the form of concentric circles. The radius (in feet) of the outer ripple is given by $r(t) = 0.6t$, where t is the time in seconds after the pebble strikes the water. The area of the circle is given by the function $A(r) = \pi r^2$. Find and interpret $(A \circ r)(t)$.

Think About It **In Exercises 73 and 74, $F(x) = f \circ g \circ h$. Identify functions for f, g, and h. (There are many correct answers.)**

73. $F(x) = \sqrt{2x - 2}$

74. $F(x) = -4 \sin(1 - x)$

Think About It **In Exercises 75 and 76, find the coordinates of a second point on the graph of a function f when the given point is on the graph and the function is (a) even and (b) odd.**

75. $\left(-\frac{3}{2}, 4\right)$

76. $(4, 9)$

77. **Even and Odd Functions** The graphs of f, g, and h are shown in the figure. Decide whether each function is even, odd, or neither.

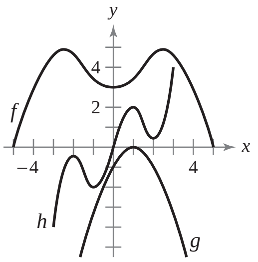


Figure for 77

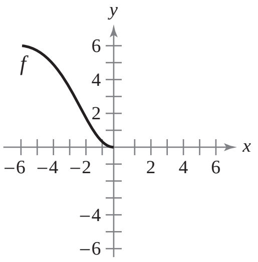


Figure for 78

78. **Even and Odd Functions** The domain of the function f shown in the figure is $-6 \le x \le 6$.

(a) Complete the graph of f given that f is even.

(b) Complete the graph of f given that f is odd.

Even and Odd Functions and Zeros of Functions **In Exercises 79–82, determine whether the function is even, odd, or neither. Then find the zeros of the function. Use a graphing utility to verify your result.**

79. $f(x) = x^2(4 - x^2)$

80. $f(x) = \sqrt[3]{x}$

81. $f(x) = x \cos x$

82. $f(x) = \sin^2 x$

Writing Functions **In Exercises 83–86, write an equation for a function that has the given graph.**

83. Line segment connecting $(-2, 4)$ and $(0, -6)$

84. Line segment connecting $(3, 1)$ and $(5, 8)$

85. The bottom half of the parabola $x + y^2 = 0$

86. The bottom half of the circle $x^2 + y^2 = 36$

Sketching a Graph **In Exercises 87–90, sketch a possible graph of the situation.**

87. The speed of an airplane as a function of time during a 5-hour flight

88. The height of a baseball as a function of horizontal distance during a home run

89. The amount of a certain brand of sneaker sold by a sporting goods store as a function of the price of the sneaker

90. The value of a new car as a function of time over a period of 8 years

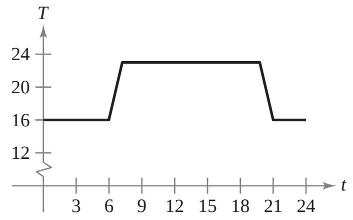
91. **Domain** Find the value of c such that the domain of $f(x) = \sqrt{c - x^2}$ is $[-5, 5]$.

92. **Domain** Find all values of c such that the domain of

$$f(x) = \frac{x + 3}{x^2 + 3cx + 6}$$

is the set of all real numbers.

93. **Graphical Reasoning** An electronically controlled thermostat is programmed to lower the temperature during the night automatically (see figure). The temperature T in degrees Celsius is given in terms of t, the time in hours on a 24-hour clock.

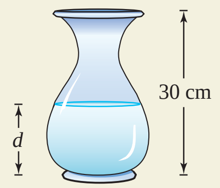


(a) Approximate $T(4)$ and $T(15)$.

(b) The thermostat is reprogrammed to produce a temperature $H(t) = T(t - 1)$. How does this change the temperature? Explain.

(c) The thermostat is reprogrammed to produce a temperature $H(t) = T(t) - 1$. How does this change the temperature? Explain.

94. **HOW DO YOU SEE IT?** Water runs into a vase of height 30 centimeters at a constant rate. The vase is full after 5 seconds. Use this information and the shape of the vase shown to answer the questions when d is the depth of the water in centimeters and t is the time in seconds (see figure).



(a) Explain why d is a function of t.

(b) Determine the domain and range of the function.

(c) Sketch a possible graph of the function.

(d) Use the graph in part (c) to approximate $d(4)$. What does this represent?

95. Modeling Data The table shows the average numbers of acres per farm in the United States for selected years. *(Source: U.S. Department of Agriculture)*

Year	1960	1970	1980	1990	2000	2010
Acreage	297	374	429	460	436	418

(a) Plot the data, where A is the acreage and t is the time in years, with $t = 0$ corresponding to 1960. Sketch a freehand curve that approximates the data.

(b) Use the curve in part (a) to approximate $A(25)$.

96. Automobile Aerodynamics

The horsepower H required to overcome wind drag on a certain automobile is approximated by

$$H(x) = 0.002x^2 + 0.005x - 0.029, \quad 10 \le x \le 100$$

where x is the speed of the car in miles per hour.

(a) Use a graphing utility to graph H.

(b) Rewrite the power function so that x represents the speed in kilometers per hour. [Find $H(x/1.6)$.]

97. Think About It Write the function $f(x) = |x| + |x - 2|$ without using absolute value signs. (For a review of absolute value, see Appendix C.)

98. Writing Use a graphing utility to graph the polynomial functions $p_1(x) = x^3 - x + 1$ and $p_2(x) = x^3 - x$. How many zeros does each function have? Is there a cubic polynomial that has no zeros? Explain.

99. Proof Prove that the function is odd.

$$f(x) = a_{2n+1}x^{2n+1} + \cdot \cdot \cdot + a_3x^3 + a_1x$$

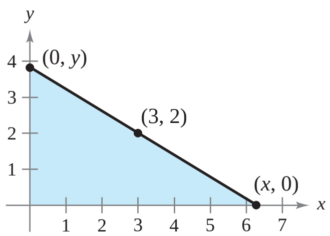
100. Proof Prove that the function is even.

$$f(x) = a_{2n}x^{2n} + a_{2n-2}x^{2n-2} + \cdot \cdot \cdot + a_2x^2 + a_0$$

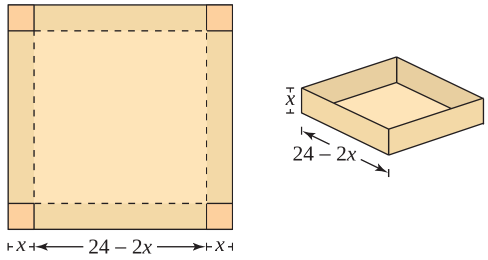
101. Proof Prove that the product of two even (or two odd) functions is even.

102. Proof Prove that the product of an odd function and an even function is odd.

103. Length A right triangle is formed in the first quadrant by the x- and y-axes and a line through the point $(3, 2)$ (see figure). Write the length L of the hypotenuse as a function of x.



104. Volume An open box of maximum volume is to be made from a square piece of material 24 centimeters on a side by cutting equal squares from the corners and turning up the sides (see figure).



(a) Write the volume V as a function of x, the length of the corner squares. What is the domain of the function?

(b) Use a graphing utility to graph the volume function and approximate the dimensions of the box that yield a maximum volume.

(c) Use the *table* feature of a graphing utility to verify your answer in part (b). (The first two rows of the table are shown.)

Height, x	Length and Width	Volume, V
1	$24 - 2(1)$	$1[24 - 2(1)]^2 = 484$
2	$24 - 2(2)$	$2[24 - 2(2)]^2 = 800$

True or False? **In Exercises 105–110, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

105. If $f(a) = f(b)$, then $a = b$.

106. A vertical line can intersect the graph of a function at most once.

107. If $f(x) = f(-x)$ for all x in the domain of f, then the graph of f is symmetric with respect to the y-axis.

108. If f is a function, then

$$f(ax) = af(x).$$

109. The graph of a function of x cannot have symmetry with respect to the x-axis.

110. If the domain of a function consists of a single number, then its range must also consist of only one number.

PUTNAM EXAM CHALLENGE

111. Let R be the region consisting of the points (x, y) of the Cartesian plane satisfying both $|x| - |y| \le 1$ and $|y| \le 1$. Sketch the region R and find its area.

112. Consider a polynomial $f(x)$ with real coefficients having the property $f(g(x)) = g(f(x))$ for every polynomial $g(x)$ with real coefficients. Determine and prove the nature of $f(x)$.

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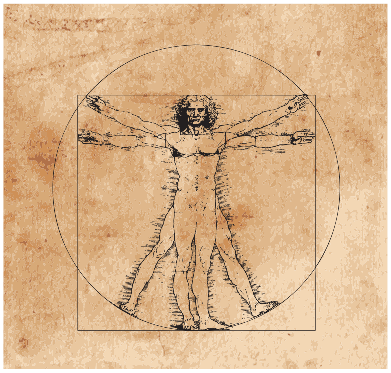
1.4 Fitting Models to Data

- **Fit a linear model to a real-life data set.**
- **Fit a quadratic model to a real-life data set.**
- **Fit a trigonometric model to a real-life data set.**

Fitting a Linear Model to Data

A basic premise of science is that much of the physical world can be described mathematically and that many physical phenomena are predictable. This scientific outlook was part of the scientific revolution that took place in Europe during the late 1500s. Two early publications connected with this revolution were *On the Revolutions of the Heavenly Spheres* by the Polish astronomer Nicolaus Copernicus and *On the Fabric of the Human Body* by the Belgian anatomist Andreas Vesalius. Each of these books was published in 1543, and each broke with prior tradition by suggesting the use of a scientific method rather than unquestioned reliance on authority.

One basic technique of modern science is gathering data and then describing the data with a mathematical model. For instance, the data in Example 1 are inspired by Leonardo da Vinci's famous drawing that indicates that a person's height and arm span are equal.



A computer graphics drawing based on the pen and ink drawing of Leonardo da Vinci's famous study of human proportions, called *Vitruvian Man*

EXAMPLE 1 Fitting a Linear Model to Data

See LarsonCalculus.com for an interactive version of this type of example.

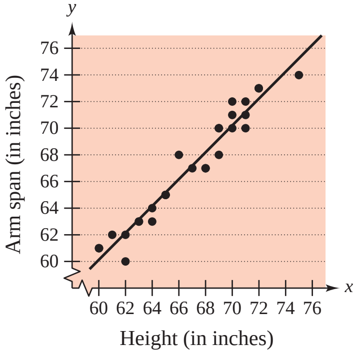
A class of 28 people collected the data shown below, which represent their heights x and arm spans y (rounded to the nearest inch).

$$(60, 61), (65, 65), (68, 67), (72, 73), (61, 62), (63, 63), (70, 71),$$
$$(75, 74), (71, 72), (62, 60), (65, 65), (66, 68), (62, 62), (72, 73),$$
$$(70, 70), (69, 68), (69, 70), (60, 61), (63, 63), (64, 64), (71, 71),$$
$$(68, 67), (69, 70), (70, 72), (65, 65), (64, 63), (71, 70), (67, 67)$$

Find a linear model to represent these data.

Solution There are different ways to model these data with an equation. The simplest would be to observe that x and y are about the same and list the model as simply $y = x$. A more careful analysis would be to use a procedure from statistics called linear regression. (You will study this procedure in Section 13.9.) The least squares regression line for these data is

$$y = 1.006x - 0.23. \qquad \text{Least squares regression line}$$



Linear model and data
Figure 1.32

The graph of the model and the data are shown in Figure 1.32. From this model, you can see that a person's arm span tends to be about the same as his or her height.

TECHNOLOGY Many graphing utilities have built-in least squares regression programs. Typically, you enter the data into the calculator and then run the linear regression program. The program usually displays the slope and y-intercept of the best-fitting line and the *correlation coefficient* r. The correlation coefficient gives a measure of how well the data can be modeled by a line. The closer $|r|$ is to 1, the better the data can be modeled by a line. For instance, the correlation coefficient for the model in Example 1 is $r \approx 0.97$, which indicates that the linear model is a good fit for the data. If the r-value is positive, then the variables have a positive correlation, as in Example 1. If the r-value is negative, then the variables have a negative correlation.

Hal_P/Shutterstock.com

Fitting a Quadratic Model to Data

A function that gives the height s of a falling object in terms of the time t is called a *position function*. If air resistance is not considered, then the position of a falling object can be modeled by

$$s(t) = \frac{1}{2}gt^2 + v_0t + s_0$$

where g is the acceleration due to gravity, v_0 is the initial velocity, and s_0 is the initial height. The value of g depends on where the object is dropped. On Earth, g is approximately -32 feet per second per second, or -9.8 meters per second per second.

To discover the value of g experimentally, you could record the heights of a falling object at several increments, as shown in Example 2.

EXAMPLE 2 Fitting a Quadratic Model to Data

A basketball is dropped from a height of about $5\frac{1}{4}$ feet. The height of the basketball is recorded 23 times at intervals of about 0.02 second. The results are shown in the table.

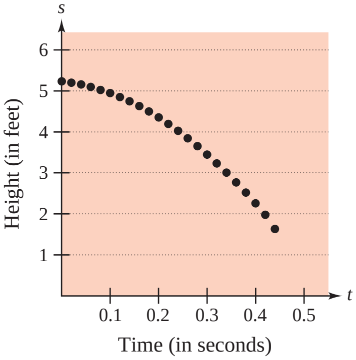
Time	0.0	0.02	0.04	0.06	0.08	0.099996
Height	5.23594	5.20353	5.16031	5.0991	5.02707	4.95146

Time	0.119996	0.139992	0.159988	0.179988	0.199984	0.219984
Height	4.85062	4.74979	4.63096	4.50132	4.35728	4.19523

Time	0.23998	0.25993	0.27998	0.299976	0.319972	0.339961
Height	4.02958	3.84593	3.65507	3.44981	3.23375	3.01048

Time	0.359961	0.379951	0.399941	0.419941	0.439941
Height	2.76921	2.52074	2.25786	1.98058	1.63488

Find a model to fit these data. Then use the model to predict the time when the basketball will hit the ground.



Scatter plot of data
Figure 1.33

Solution Begin by sketching a scatter plot of the data, as shown in Figure 1.33. From the scatter plot, you can see that the data do not appear to be linear. It does appear, however, that they might be quadratic. To check this, enter the data into a graphing utility that has a quadratic regression program. You should obtain the model

$$s = -15.45t^2 - 1.302t + 5.2340. \quad \text{Least squares regression quadratic}$$

Using this model, you can predict the time when the basketball hits the ground by substituting 0 for s and solving the resulting equation for t.

$$0 = -15.45t^2 - 1.302t + 5.2340 \quad \text{Let } s = 0.$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{Quadratic Formula}$$

$$t = \frac{-(-1.302) \pm \sqrt{(-1.302)^2 - 4(-15.45)(5.2340)}}{2(-15.45)} \quad \text{Substitute } a = -15.45, b = -1.302, \text{ and } c = 5.2340.$$

$$t \approx 0.54 \quad \text{Choose positive solution.}$$

The solution is about 0.54 second. In other words, the basketball will continue to fall for about 0.1 second more before hitting the ground. (Note that the experimental value of g is $\frac{1}{2}g = -15.45$, or $g = -30.90$ feet per second per second.)

Fitting a Trigonometric Model to Data

What is mathematical modeling? This is one of the questions that is asked in the book *Guide to Mathematical Modelling*. Here is part of the answer.*

1. Mathematical modeling consists of applying your mathematical skills to obtain useful answers to real problems.
2. Learning to apply mathematical skills is very different from learning mathematics itself.
3. Models are used in a very wide range of applications, some of which do not appear initially to be mathematical in nature.
4. Models often allow quick and cheap evaluation of alternatives, leading to optimal solutions that are not otherwise obvious.
5. There are no precise rules in mathematical modeling and no "correct" answers.
6. Modeling can be learned only by *doing*.

The amount of daylight received by locations on Earth varies with the time of year.

REMARK For a review of trigonometric functions, see Appendix C.

EXAMPLE 3 Fitting a Trigonometric Model to Data

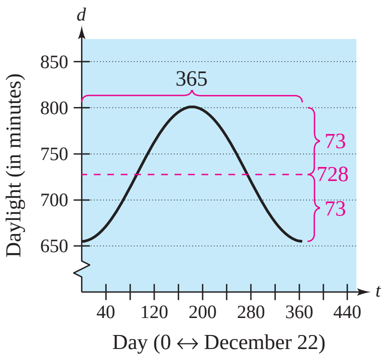
The number of hours of daylight on a given day on Earth depends on the latitude and the time of year. Here are the numbers of minutes of daylight at a location of 20°N latitude on the longest and shortest days of the year: June 21, 801 minutes; December 22, 655 minutes. Use these data to write a model for the amount of daylight d (in minutes) on each day of the year at a location of 20°N latitude. How could you check the accuracy of your model?

Solution Here is one way to create a model. You can hypothesize that the model is a sine function whose period is 365 days. Using the given data, you can conclude that the amplitude of the graph is $(801 - 655)/2$, or 73. So, one possible model is

$$d = 728 - 73 \sin\left(\frac{2\pi t}{365} + \frac{\pi}{2}\right).$$

In this model, t represents the number of each day of the year, with December 22 represented by $t = 0$. A graph of this model is shown in Figure 1.34. To check the accuracy of this model, a weather almanac was used to find the numbers of minutes of daylight on different days of the year at the location of 20°N latitude.

Date	Value of t	Actual Daylight	Daylight Given by Model
Dec 22	0	655 min	655 min
Jan 1	10	657 min	656 min
Feb 1	41	676 min	672 min
Mar 1	69	705 min	701 min
Apr 1	100	740 min	739 min
May 1	130	772 min	773 min
Jun 1	161	796 min	796 min
Jun 21	181	801 min	801 min
Jul 1	191	799 min	800 min
Aug 1	222	782 min	785 min
Sep 1	253	752 min	754 min
Oct 1	283	718 min	716 min
Nov 1	314	685 min	681 min
Dec 1	344	661 min	660 min



Graph of model
Figure 1.34

You can see that the model is fairly accurate.

* Text from Dilwyn Edwards and Mike Hamson, *Guide to Mathematical Modelling* (Boca Raton: CRC Press, 1990), p. 4. Used by permission of the authors.

hjschneider/iStockphoto.com

1.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

1. Wages Each ordered pair gives the average weekly wage x for federal government workers and the average weekly wage y for state government workers for 2001 through 2009. *(Source: U.S. Bureau of Labor Statistics)*

(941, 727), (1001, 754), (1043, 770), (1111, 791), (1151, 812), (1198, 844), (1248, 883), (1275, 923), (1303, 937)

(a) Plot the data. From the graph, do the data appear to be approximately linear?

(b) Visually find a linear model for the data. Graph the model.

(c) Use the model to approximate y when $x = 1075$.

2. Quiz Scores The ordered pairs represent the scores on two consecutive 15-point quizzes for a class of 15 students.

(7, 13), (9, 7), (14, 14), (15, 15), (10, 15), (9, 7), (11, 14), (7, 14), (14, 11), (14, 15), (8, 10), (15, 9), (10, 11), (9, 10), (11, 10)

(a) Plot the data. From the graph, does the relationship between consecutive scores appear to be approximately linear?

(b) If the data appear to be approximately linear, find a linear model for the data. If not, give some possible explanations.

3. Hooke's Law Hooke's Law states that the force F required to compress or stretch a spring (within its elastic limits) is proportional to the distance d that the spring is compressed or stretched from its original length. That is, $F = kd$, where k is a measure of the stiffness of the spring and is called the *spring constant*. The table shows the elongation d in centimeters of a spring when a force of F newtons is applied.

F	20	40	60	80	100
d	1.4	2.5	4.0	5.3	6.6

(a) Use the regression capabilities of a graphing utility to find a linear model for the data.

(b) Use a graphing utility to plot the data and graph the model. How well does the model fit the data? Explain.

(c) Use the model to estimate the elongation of the spring when a force of 55 newtons is applied.

4. Falling Object In an experiment, students measured the speed s (in meters per second) of a falling object t seconds after it was released. The results are shown in the table.

t	0	1	2	3	4
s	0	11.0	19.4	29.2	39.4

(a) Use the regression capabilities of a graphing utility to find a linear model for the data.

(b) Use a graphing utility to plot the data and graph the model. How well does the model fit the data? Explain.

(c) Use the model to estimate the speed of the object after 2.5 seconds.

5. Energy Consumption and Gross National Product The data show the per capita energy consumptions (in millions of Btu) and the per capita gross national incomes (in thousands of U.S. dollars) for several countries in 2008. *(Source: U.S. Energy Information Administration and The World Bank)*

Argentina	(81, 7.19)	India	(17, 1.04)
Australia	(274, 40.24)	Italy	(136, 35.46)
Bangladesh	(6, 0.52)	Japan	(172, 38.13)
Brazil	(54, 7.30)	Mexico	(66, 9.99)
Canada	(422, 43.64)	Poland	(101, 11.73)
Ecuador	(35, 3.69)	Turkey	(57, 9.02)
Hungary	(110, 12.81)	Venezuela	(121, 9.23)

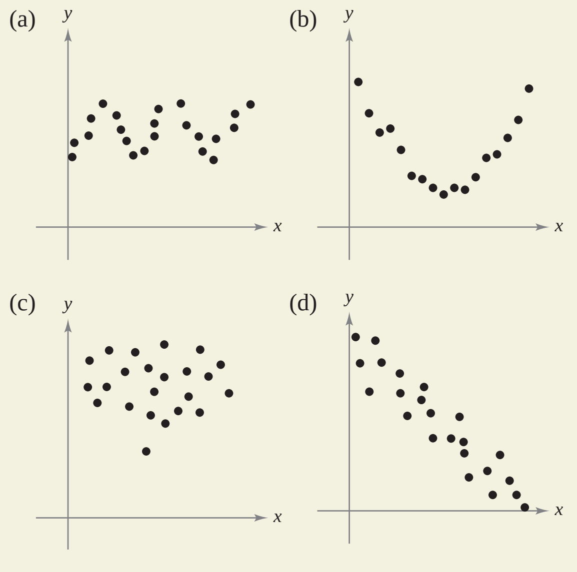
(a) Use the regression capabilities of a graphing utility to find a linear model for the data. What is the correlation coefficient?

(b) Use a graphing utility to plot the data and graph the model.

(c) Interpret the graph in part (b). Use the graph to identify the three countries that differ most from the linear model.

(d) Delete the data for the three countries identified in part (c). Fit a linear model to the remaining data and give the correlation coefficient.

6. HOW DO YOU SEE IT? Determine whether the data can be modeled by a linear function, a quadratic function, or a trigonometric function, or that there appears to be no relationship between x and y.



7. Beam Strength Students in a lab measured the breaking strength S (in pounds) of wood 2 inches thick, x inches high, and 12 inches long. The results are shown in the table.

x	4	6	8	10	12
S	2370	5460	10,310	16,250	23,860

(a) Use the regression capabilities of a graphing utility to find a quadratic model for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the model to approximate the breaking strength when $x = 2$.

(d) How many times greater is the breaking strength for a 4-inch-high board than for a 2-inch-high board?

(e) How many times greater is the breaking strength for a 12-inch-high board than for a 6-inch-high board? When the height of a board increases by a factor, does the breaking strength increase by the same factor? Explain.

8. Car Performance The time t (in seconds) required to attain a speed of s miles per hour from a standing start for a Volkswagen Passat is shown in the table. *(Source: Car & Driver)*

s	30	40	50	60	70	80	90
t	2.7	3.8	4.9	6.3	8.0	9.9	12.2

(a) Use the regression capabilities of a graphing utility to find a quadratic model for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the graph in part (b) to state why the model is not appropriate for determining the times required to attain speeds of less than 20 miles per hour.

(d) Because the test began from a standing start, add the point (0, 0) to the data. Fit a quadratic model to the revised data and graph the new model.

(e) Does the quadratic model in part (d) more accurately model the behavior of the car? Explain.

9. Engine Performance A V8 car engine is coupled to a dynamometer, and the horsepower y is measured at different engine speeds x (in thousands of revolutions per minute). The results are shown in the table.

x	1	2	3	4	5	6
y	40	85	140	200	225	245

(a) Use the regression capabilities of a graphing utility to find a cubic model for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the model to approximate the horsepower when the engine is running at 4500 revolutions per minute.

10. Boiling Temperature The table shows the temperatures T (in degrees Fahrenheit) at which water boils at selected pressures p (in pounds per square inch). *(Source: Standard Handbook for Mechanical Engineers)*

p	5	10	14.696 (1 atmosphere)	20
T	162.24°	193.21°	212.00°	227.96°

p	30	40	60	80	100
T	250.33°	267.25°	292.71°	312.03°	327.81°

(a) Use the regression capabilities of a graphing utility to find a cubic model for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the graph to estimate the pressure required for the boiling point of water to exceed 300°F.

(d) Explain why the model would not be accurate for pressures exceeding 100 pounds per square inch.

11. Automobile Costs The data in the table show the variable costs of operating an automobile in the United States for 2000 through 2010, where t is the year, with $t = 0$ corresponding to 2000. The functions y_1, y_2, and y_3 represent the costs in cents per mile for gas, maintenance, and tires, respectively. *(Source: Bureau of Transportation Statistics)*

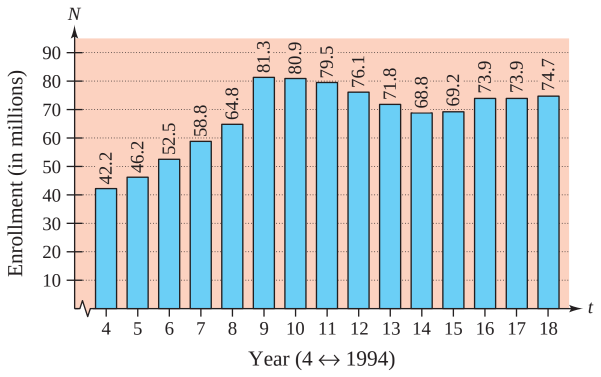
t	y_1	y_2	y_3
0	6.9	3.6	1.7
1	7.9	3.9	1.8
2	5.9	4.1	1.8
3	7.2	4.1	1.8
4	6.5	5.4	0.7
5	9.5	4.9	0.7
6	8.9	4.9	0.7
7	11.7	4.6	0.7
8	10.1	4.6	0.8
9	11.4	4.5	0.8
10	12.3	4.4	1.0

(a) Use the regression capabilities of a graphing utility to find cubic models for y_1 and y_3, and a quadratic model for y_2.

(b) Use a graphing utility to graph y_1, y_2, y_3, and $y_1 + y_2 + y_3$ in the same viewing window. Use the model to estimate the total variable cost per mile in 2014.

12. Health Maintenance Organizations The bar graph shows the numbers of people N (in millions) receiving care in HMOs for the years 1994 through 2008. *(Source: HealthLeaders-InterStudy)*

HMO Enrollment



(a) Let t be the time in years, with $t = 4$ corresponding to 1994. Use the regression capabilities of a graphing utility to find linear and cubic models for the data.

(b) Use a graphing utility to plot the data and graph the linear and cubic models.

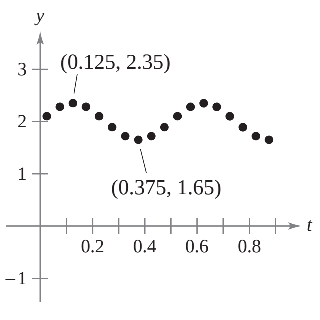
(c) Use the graphs in part (b) to determine which is the better model.

(d) Use a graphing utility to find and graph a quadratic model for the data. How well does the model fit the data? Explain.

(e) Use the linear and cubic models to estimate the number of people receiving care in HMOs in the year 2014. What do you notice?

(f) Use a graphing utility to find other models for the data. Which models do you think best represent the data? Explain.

13. Harmonic Motion The motion of an oscillating weight suspended by a spring was measured by a motion detector. The data collected and the approximate maximum (positive and negative) displacements from equilibrium are shown in the figure. The displacement y is measured in centimeters, and the time t is measured in seconds.



(a) Is y a function of t? Explain.

(b) Approximate the amplitude and period of the oscillations.

(c) Find a model for the data.

(d) Use a graphing utility to graph the model in part (c). Compare the result with the data in the figure.

14. Temperature The table shows the normal daily high temperatures for Miami M and Syracuse S (in degrees Fahrenheit) for month t, with $t = 1$ corresponding to January. *(Source: National Oceanic and Atmospheric Administration)*

t	1	2	3	4	5	6
M	76.5	77.7	80.7	83.8	87.2	89.5
S	31.4	33.5	43.1	55.7	68.5	77.0

t	7	8	9	10	11	12
M	90.9	90.6	89.0	85.4	81.2	77.5
S	81.7	79.6	71.4	59.8	47.4	36.3

(a) A model for Miami is

$$M(t) = 83.70 + 7.46 \sin(0.4912t - 1.95).$$

Find a model for Syracuse.

(b) Use a graphing utility to plot the data and graph the model for Miami. How well does the model fit?

(c) Use a graphing utility to plot the data and graph the model for Syracuse. How well does the model fit?

(d) Use the models to estimate the average annual temperature in each city. Which term of the model did you use? Explain.

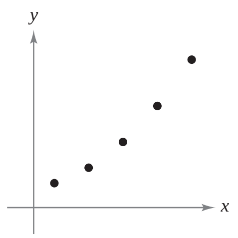
(e) What is the period of each model? Is it what you expected? Explain.

(f) Which city has a greater variability in temperature throughout the year? Which factor of the models determines this variability? Explain.

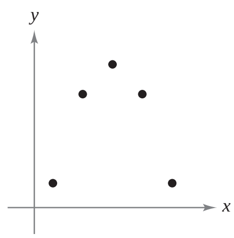
WRITING ABOUT CONCEPTS

Modeling Data **In Exercises 15 and 16, describe a possible real-life situation for each data set. Then describe how a model could be used in the real-life setting.**

15.



16.



PUTNAM EXAM CHALLENGE

17. For $i = 1, 2$, let T_i be a triangle with side lengths a_i, b_i, c_i, and area A_i. Suppose that $a_1 \le a_2$, $b_1 \le b_2$, $c_1 \le c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \le A_2$?

This problem was composed by the Committee on the Putnam Prize Competition.

1.5 Inverse Functions

- **Verify that one function is the inverse function of another function.**
- **Determine whether a function has an inverse function.**
- **Develop properties of the six inverse trigonometric functions.**

Inverse Functions

Recall from Section 1.3 that a function can be represented by a set of ordered pairs. For instance, the function $f(x) = x + 3$ from $A = \{1, 2, 3, 4\}$ to $B = \{4, 5, 6, 7\}$ can be written as

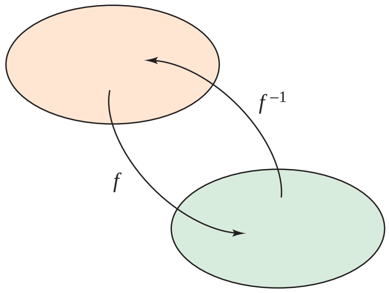
$$f: \{(1, 4), (2, 5), (3, 6), (4, 7)\}.$$

By interchanging the first and second coordinates of each ordered pair, you can form the **inverse function** of f. This function is denoted by f^{-1}. It is a function from B to A, and can be written as

$$f^{-1}: \{(4, 1), (5, 2), (6, 3), (7, 4)\}.$$

Note that the domain of f is equal to the range of f^{-1}, and vice versa, as shown in Figure 1.35. The functions f and f^{-1} have the effect of "undoing" each other. That is, when you form the composition of f with f^{-1} or the composition of f^{-1} with f, you obtain the identity function.

$$f(f^{-1}(x)) = x \qquad \text{and} \qquad f^{-1}(f(x)) = x$$



Domain of f = range of f^{-1}
Domain of f^{-1} = range of f
Figure 1.35

REMARK Although the notation used to denote an inverse function resembles *exponential notation*, it is a different use of -1 as a superscript. That is, in general,

$$f^{-1}(x) \neq \frac{1}{f(x)}.$$

Definition of Inverse Function

A function g is the **inverse function** of the function f when

$f(g(x)) = x$ for each x in the domain of g

and

$g(f(x)) = x$ for each x in the domain of f.

The function g is denoted by f^{-1} (read "f inverse").

Exploration

Finding Inverse Functions Explain how to "undo" each of the functions below. Then use your explanation to write the inverse function of f.

a. $f(x) = x - 5$
b. $f(x) = 6x$
c. $f(x) = \dfrac{x}{2}$
d. $f(x) = 3x + 2$
e. $f(x) = x^3$
f. $f(x) = 4(x - 2)$

Use a graphing utility to graph each function and its inverse function in the same "square" viewing window. What observation can you make about each pair of graphs?

Here are some important observations about inverse functions.

1. If g is the inverse function of f, then f is the inverse function of g.
2. The domain of f^{-1} is equal to the range of f, and the range of f^{-1} is equal to the domain of f.
3. A function need not have an inverse function, but when it does, the inverse function is unique (see Exercise 140).

You can think of f^{-1} as undoing what has been done by f. For example, subtraction can be used to undo addition, and division can be used to undo multiplication. So,

$$f(x) = x + c \qquad \text{and} \qquad f^{-1}(x) = x - c$$

Subtraction can be used to undo addition.

are inverse functions of each other and

$$f(x) = cx \qquad \text{and} \qquad f^{-1}(x) = \frac{x}{c},\ c \neq 0$$

Division can be used to undo multiplication.

are inverse functions of each other.

EXAMPLE 1 **Verifying Inverse Functions**

Show that the functions are inverse functions of each other.

$$f(x) = 2x^3 - 1 \quad \text{and} \quad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

REMARK In Example 1, try comparing the functions f and g verbally.

For f: First cube x, then multiply by 2, then subtract 1.

For g: First add 1, then divide by 2, then take the cube root.

Do you see the "undoing pattern"?

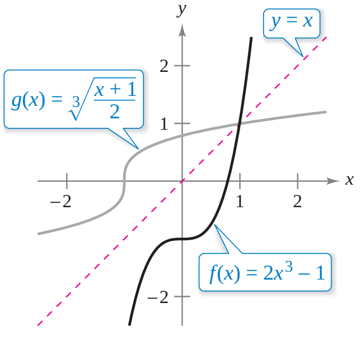
Solution Because the domains and ranges of both f and g consist of all real numbers, you can conclude that both composite functions exist for all x. The composition of f with g is

$$\begin{aligned} f(g(x)) &= 2\left(\sqrt[3]{\frac{x+1}{2}}\right)^3 - 1 \\ &= 2\left(\frac{x+1}{2}\right) - 1 \\ &= x + 1 - 1 \\ &= x. \end{aligned}$$

The composition of g with f is

$$g(f(x)) = \sqrt[3]{\frac{(2x^3 - 1) + 1}{2}} = \sqrt[3]{\frac{2x^3}{2}} = \sqrt[3]{x^3} = x.$$

Because $f(g(x)) = x$ and $g(f(x)) = x$, you can conclude that f and g are inverse functions of each other (see Figure 1.36).



f and g are inverse functions of each other.
Figure 1.36

In Figure 1.36, the graphs of f and $g = f^{-1}$ appear to be mirror images of each other with respect to the line $y = x$. The graph of f^{-1} is a **reflection** of the graph of f in the line $y = x$. This idea is generalized in the next definition.

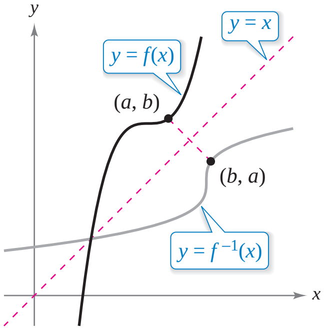
Reflective Property of Inverse Functions

The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).

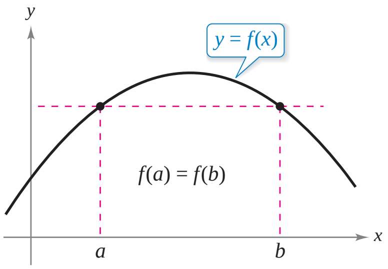
To see the validity of the Reflective Property of Inverse Functions, consider the point (a, b) on the graph of f. This implies $f(a) = b$ and you can write

$$f^{-1}(b) = f^{-1}(f(a)) = a.$$

So, (b, a) is on the graph of f^{-1}, as shown in Figure 1.37. A similar argument will verify this result in the other direction.



The graph of f^{-1} is a reflection of the graph of f in the line $y = x$.
Figure 1.37



If a horizontal line intersects the graph of f twice, then f is not one-to-one.
Figure 1.38

Existence of an Inverse Function

Not every function has an inverse, and the Reflective Property of Inverse Functions suggests a graphical test for those that do—the **Horizontal Line Test** for an inverse function. This test states that a function f has an inverse function if and only if every horizontal line intersects the graph of f at most once (see Figure 1.38). The next definition formally states why the Horizontal Line Test is valid.

> **The Existence of an Inverse Function**
> A function has an inverse function if and only if it is one-to-one.

EXAMPLE 2 The Existence of an Inverse Function

Which of the functions has an inverse function?

a. $f(x) = x^3 - 1$

b. $f(x) = x^3 - x + 1$

Solution

a. From the graph of f shown in Figure 1.39(a), it appears that f is one-to-one over its entire domain. To verify this, suppose that there exist x_1 and x_2 such that $f(x_1) = f(x_2)$. By showing that $x_1 = x_2$, it follows that f is one-to-one.

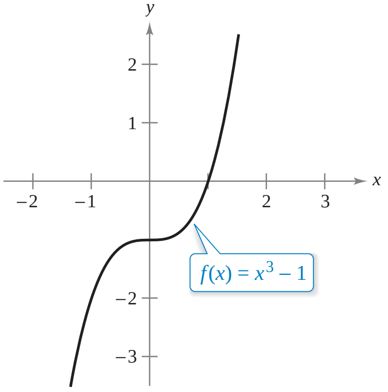
$$\begin{aligned} f(x_1) &= f(x_2) \\ x_1^3 - 1 &= x_2^3 - 1 \\ x_1^3 &= x_2^3 \\ \sqrt[3]{x_1^3} &= \sqrt[3]{x_2^3} \\ x_1 &= x_2 \end{aligned}$$

Because f is one-to-one, you can conclude that f must have an inverse function.

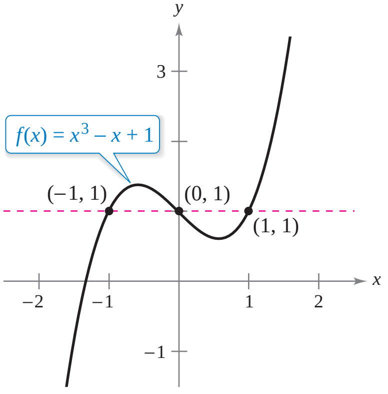
b. From the graph of f shown in Figure 1.39(b), you can see that the function does not pass the Horizontal Line Test. In other words, it is not one-to-one. For instance, f has the same value when $x = -1, 0,$ and 1.

$$f(-1) = f(1) = f(0) = 1 \qquad \text{Not one-to-one}$$

Therefore, f does not have an inverse function.



(a) Because f is one-to-one over its entire domain, it has an inverse function.



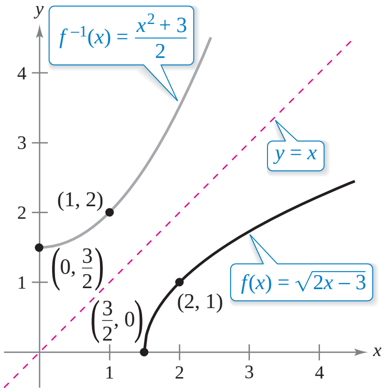
(b) Because f is not one-to-one, it does not have an inverse function.
Figure 1.39

Often it is easier to prove that a function has an inverse function than to find the inverse function. For instance, by sketching the graph of

$$f(x) = x^3 + x - 1$$

you can see that it is one-to-one. Yet it would be difficult to determine the inverse of this function algebraically.

> **Guidelines for Finding an Inverse of a Function**
> 1. Determine whether the function given by $y = f(x)$ has an inverse function.
> 2. Solve for x as a function of y: $x = g(y) = f^{-1}(y)$.
> 3. Interchange x and y. The resulting equation is $y = f^{-1}(x)$.
> 4. Define the domain of f^{-1} as the range of f.
> 5. Verify that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.



The domain of f^{-1}, $[0, \infty)$, is the range of f.
Figure 1.40

EXAMPLE 3 Finding an Inverse Function

Find the inverse function of

$$f(x) = \sqrt{2x - 3}.$$

Solution The function has an inverse function because it is one-to-one on its entire domain, $\left[\frac{3}{2}, \infty\right)$, as shown in Figure 1.40. To find an equation for the inverse function, let $y = f(x)$ and solve for x in terms of y.

$$\sqrt{2x-3} = y \qquad \text{Let } y = f(x).$$

$$2x - 3 = y^2 \qquad \text{Square each side.}$$

$$x = \frac{y^2+3}{2} \qquad \text{Solve for } x.$$

$$y = \frac{x^2+3}{2} \qquad \text{Interchange } x \text{ and } y.$$

$$f^{-1}(x) = \frac{x^2+3}{2} \qquad \text{Replace } y \text{ by } f^{-1}(x).$$

The domain of f^{-1} is the range of f, which is $[0, \infty)$. You can verify this result by showing that $f(f^{-1}(x)) = x$ and $f^{-1}(f(x)) = x$.

$$f(f^{-1}(x)) = \sqrt{2\left(\frac{x^2+3}{2}\right) - 3} = \sqrt{x^2} = x, \quad x \geq 0$$

$$f^{-1}(f(x)) = \frac{\left(\sqrt{2x-3}\right)^2 + 3}{2} = \frac{2x-3+3}{2} = x, \quad x \geq \frac{3}{2}$$

Consider a function that is *not* one-to-one on its entire domain. By restricting the domain to an interval on which the function is one-to-one, you can conclude that the new function has an inverse function on the restricted domain.

EXAMPLE 4 Testing Whether a Function Is One-to-One

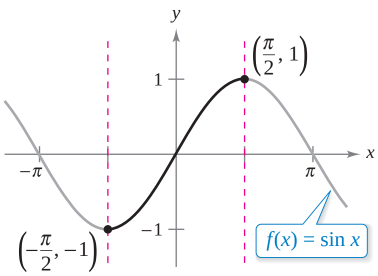
See LarsonCalculus.com for an interactive version of this type of example.

Show that the sine function $f(x) = \sin x$ is not one-to-one on the entire real line. Then show that f is one-to-one on the closed interval $[-\pi/2, \pi/2]$.

Solution It is clear that f is not one-to-one, because many different x-values yield the same y-value. For instance,

$$\sin(0) = 0 = \sin(\pi).$$

Moreover, from the graph of $f(x) = \sin x$ in Figure 1.41, you can see that when f is restricted to the interval $[-\pi/2, \pi/2]$, then the restricted function is one-to-one.



f is one-to-one on the interval $[-\pi/2, \pi/2]$.
Figure 1.41

Inverse Trigonometric Functions

From the graphs of the six basic trigonometric functions, you can see that they do not have inverse functions. (Graphs of the six basic trigonometric functions are shown in Appendix C.) The functions that are called "inverse trigonometric functions" are actually inverses of trigonometric functions whose domains have been restricted.

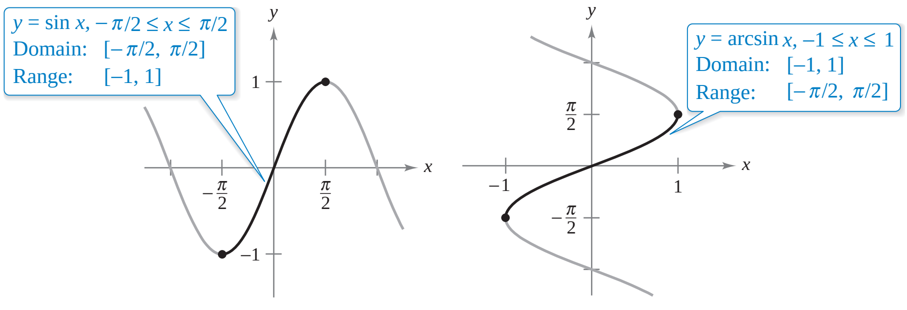
For instance, in Example 4, you saw that the sine function is one-to-one on the interval $[-\pi/2, \pi/2]$ (see Figure 1.42). On this interval, you can define the inverse of the *restricted* sine function as

$$y = \arcsin x \qquad \text{if and only if} \qquad \sin y = x$$

where

$$-1 \le x \le 1 \qquad \text{and} \qquad -\frac{\pi}{2} \le \arcsin x \le \frac{\pi}{2}.$$

From Figures 1.42 (a) and (b), you can see that you can obtain the graph of $y = \arcsin x$ by reflecting the graph of $y = \sin x$ in the line $y = x$ on the interval $[-\pi/2, \pi/2]$.



(a) **(b)**

Figure 1.42

Exploration

Inverse Secant Function
In the definition at the right, the inverse secant function is defined by restricting the domain of the secant function to the intervals

$$\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right].$$

Most other texts and reference books agree with this, but some disagree. What other domains might make sense? Explain your reasoning graphically. Most calculators do not have a key for the inverse secant function. How can you use a calculator to evaluate the inverse secant function?

Under suitable restrictions, each of the six trigonometric functions is one-to-one and so has an inverse function, as indicated in the next definition. (The term "iff" is used to represent the phrase "if and only if.")

Definition of Inverse Trigonometric Function

Function	**Domain**	**Range**
$y = \arcsin x$ iff $\sin y = x$	$-1 \le x \le 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}$
$y = \arccos x$ iff $\cos y = x$	$-1 \le x \le 1$	$0 \le y \le \pi$
$y = \arctan x$ iff $\tan y = x$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$y = \operatorname{arccot} x$ iff $\cot y = x$	$-\infty < x < \infty$	$0 < y < \pi$
$y = \operatorname{arcsec} x$ iff $\sec y = x$	$\lvert x \rvert \ge 1$	$0 \le y \le \pi, \quad y \ne \frac{\pi}{2}$
$y = \operatorname{arccsc} x$ iff $\csc y = x$	$\lvert x \rvert \ge 1$	$-\frac{\pi}{2} \le y \le \frac{\pi}{2}, \quad y \ne 0$

The term arcsin x is read as "the arcsine of x" or sometimes "the angle whose sine is x." An alternative notation for the inverse sine function is $\sin^{-1} x$.

The graphs of the six inverse trigonometric functions are shown in Figure 1.43.

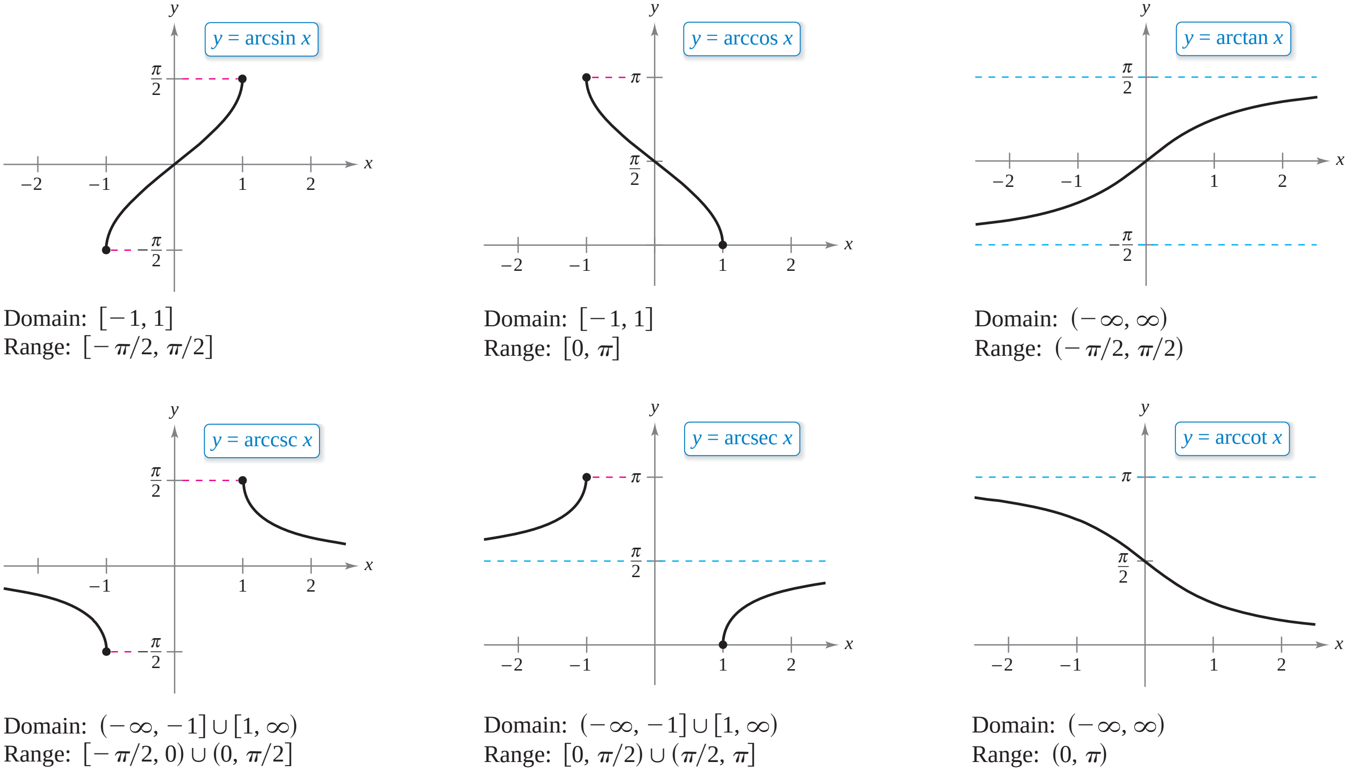


Figure 1.43

When evaluating inverse trigonometric functions, remember that they denote angles in *radian measure*.

EXAMPLE 5 Evaluating Inverse Trigonometric Functions

Evaluate each expression.

a. $\arcsin\left(-\dfrac{1}{2}\right)$ **b.** $\arccos 0$ **c.** $\arctan \sqrt{3}$ **d.** $\arcsin(0.3)$

Solution

a. By definition, $y = \arcsin\left(-\frac{1}{2}\right)$ implies that $\sin y = -\frac{1}{2}$. In the interval $[-\pi/2, \pi/2]$, the correct value of y is $-\pi/6$.

$$\arcsin\left(-\frac{1}{2}\right) = -\frac{\pi}{6}$$

b. By definition, $y = \arccos 0$ implies that $\cos y = 0$. In the interval $[0, \pi]$, you have $y = \pi/2$.

$$\arccos 0 = \frac{\pi}{2}$$

c. By definition, $y = \arctan \sqrt{3}$ implies that $\tan y = \sqrt{3}$. In the interval $(-\pi/2, \pi/2)$, you have $y = \pi/3$.

$$\arctan \sqrt{3} = \frac{\pi}{3}$$

d. Using a calculator set in *radian* mode produces

$$\arcsin(0.3) \approx 0.3047.$$

Inverse functions have the properties

$$f(f^{-1}(x)) = x \quad \text{and} \quad f^{-1}(f(x)) = x.$$

When applying these properties to inverse trigonometric functions, remember that the trigonometric functions have inverse functions only in restricted domains. For x-values outside these domains, these two properties do not hold. For example, $\arcsin(\sin \pi)$ is equal to 0, not π.

Properties of Inverse Trigonometric Functions

1. If $-1 \le x \le 1$ and $-\pi/2 \le y \le \pi/2$, then

$$\sin(\arcsin x) = x \quad \text{and} \quad \arcsin(\sin y) = y.$$

2. If $-\pi/2 < y < \pi/2$, then

$$\tan(\arctan x) = x \quad \text{and} \quad \arctan(\tan y) = y.$$

3. If $|x| \ge 1$ and $0 \le y < \pi/2$ or $\pi/2 < y \le \pi$, then

$$\sec(\text{arcsec } x) = x \quad \text{and} \quad \text{arcsec}(\sec y) = y.$$

Similar properties hold for the other inverse trigonometric functions.

EXAMPLE 6 Solving an Equation

Solve $\arctan(2x - 3) = \dfrac{\pi}{4}$ for x.

Solution

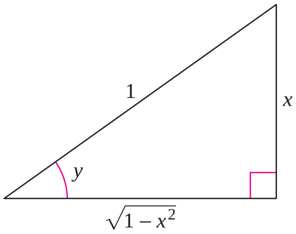
$$\arctan(2x - 3) = \frac{\pi}{4} \qquad \text{Write original equation.}$$

$$\tan[\arctan(2x - 3)] = \tan\frac{\pi}{4} \qquad \text{Take tangent of each side.}$$

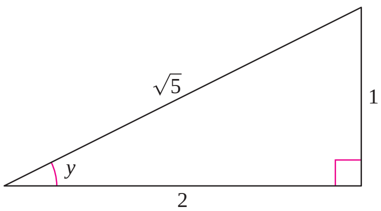
$$2x - 3 = 1 \qquad \tan(\arctan x) = x$$

$$x = 2 \qquad \text{Solve for } x.$$

Some problems in calculus require that you evaluate expressions such as $\cos(\arcsin x)$, as shown in Example 7.



$y = \arcsin x$
Figure 1.44



$y = \text{arcsec}\,\dfrac{\sqrt{5}}{2}$
Figure 1.45

EXAMPLE 7 Using Right Triangles

a. Given $y = \arcsin x$, where $0 < y < \pi/2$, find $\cos y$.

b. Given $y = \text{arcsec}(\sqrt{5}/2)$, find $\tan y$.

Solution

a. Because $y = \arcsin x$, you know that $\sin y = x$. This relationship between x and y can be represented by a right triangle, as shown in Figure 1.44.

$$\cos y = \cos(\arcsin x) = \frac{\text{adj.}}{\text{hyp.}} = \sqrt{1 - x^2}$$

(This result is also valid for $-\pi/2 < y < 0$.)

b. Use the right triangle shown in Figure 1.45.

$$\tan y = \tan\left[\text{arcsec}\left(\frac{\sqrt{5}}{2}\right)\right] = \frac{\text{opp.}}{\text{adj.}} = \frac{1}{2}$$

1.5 Exercises

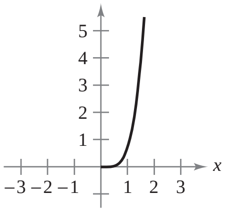
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Verifying Inverse Functions **In Exercises 1–8, show that f and g are inverse functions (a) analytically and (b) graphically.**

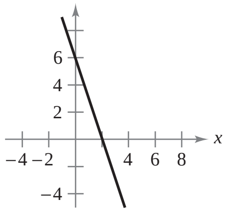
1. $f(x) = 5x + 1,\quad g(x) = \dfrac{x-1}{5}$
2. $f(x) = 3 - 4x,\quad g(x) = \dfrac{3-x}{4}$
3. $f(x) = x^3,\quad g(x) = \sqrt[3]{x}$
4. $f(x) = 1 - x^3,\quad g(x) = \sqrt[3]{1-x}$
5. $f(x) = \sqrt{x-4},\quad g(x) = x^2 + 4,\ \ x \ge 0$
6. $f(x) = 16 - x^2,\ \ x \ge 0,\quad g(x) = \sqrt{16-x}$
7. $f(x) = \dfrac{1}{x},\quad g(x) = \dfrac{1}{x}$
8. $f(x) = \dfrac{1}{1+x},\ \ x \ge 0,\quad g(x) = \dfrac{1-x}{x},\ \ 0 < x \le 1$

Matching **In Exercises 9–12, match the graph of the function with the graph of its inverse function. [The graphs of the inverse functions are labeled (a), (b), (c), and (d).]**

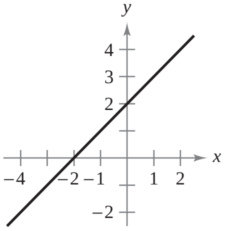
(a)



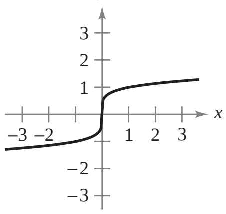
(b)



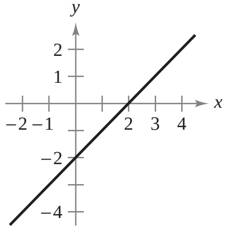
(c)



(d)



9.



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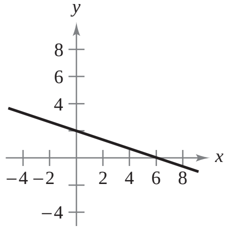

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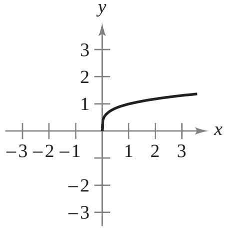

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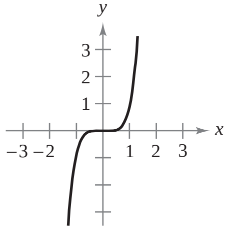
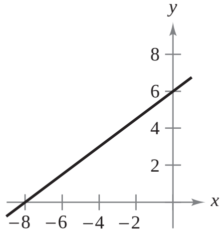
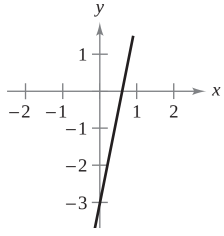


Using the Horizontal Line Test **In Exercises 13–16, use the Horizontal Line Test to determine whether the function is one-to-one on its entire domain and therefore has an inverse function. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

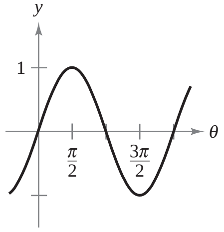
13. $f(x) = \frac{3}{4}x + 6$



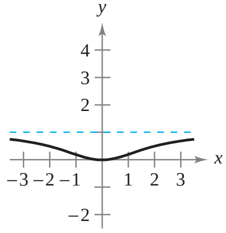
14. $f(x) = 5x - 3$



15. $f(\theta) = \sin\theta$



16. $f(x) = \dfrac{x^2}{x^2+4}$



The Existence of an Inverse Function **In Exercises 17–22, use a graphing utility to graph the function. Determine whether the function is one-to-one on its entire domain and therefore has an inverse function.**

17. $h(s) = \dfrac{1}{s-2} - 3$
18. $f(x) = \dfrac{6x}{x^2+4}$
19. $g(t) = \dfrac{1}{\sqrt{t^2+1}}$
20. $f(x) = 5x\sqrt{x-1}$
21. $g(x) = (x+5)^3$
22. $h(x) = |x+4| - |x-4|$

The Existence of an Inverse Function **In Exercises 23–26, determine whether the function is one-to-one on its entire domain and therefore has an inverse function.**

23. $f(x) = \dfrac{x^4}{4} - 2x^2$
24. $f(x) = \sin\dfrac{3x}{2}$
25. $f(x) = 2 - x - x^3$
26. $f(x) = \sqrt[3]{x+1}$

Finding an Inverse Function **In Exercises 27–34, (a) find the inverse function of f, (b) graph f and f^{-1} on the same set of coordinate axes, (c) describe the relationship between the graphs, and (d) state the domains and ranges of f and f^{-1}.**

27. $f(x) = 2x - 3$
28. $f(x) = 7 - 4x$
29. $f(x) = x^5$
30. $f(x) = x^3 - 1$
31. $f(x) = \sqrt{x}$
32. $f(x) = x^2,\ \ x \ge 0$
33. $f(x) = \sqrt{4 - x^2},\ \ 0 \le x \le 2$
34. $f(x) = \sqrt{x^2 - 4},\ \ x \ge 2$

Finding an Inverse Function **In Exercises 35–40, (a) find the inverse function of f, (b) use a graphing utility to graph f and f^{-1} in the same viewing window, (c) describe the relationship between the graphs, and (d) state the domains and ranges of f and f^{-1}.**

35. $f(x) = \sqrt[3]{x - 1}$ **36.** $f(x) = 3\sqrt[5]{2x - 1}$

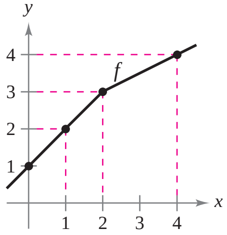
37. $f(x) = x^{2/3}, \quad x \geq 0$ **38.** $f(x) = x^{3/5}$

39. $f(x) = \dfrac{x}{\sqrt{x^2 + 7}}$

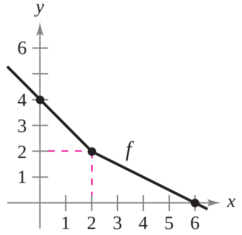
40. $f(x) = \dfrac{x + 2}{x}$

Finding an Inverse Function **In Exercises 41 and 42, use the graph of the function f to make a table of values for the given points. Then make a second table that can be used to find f^{-1}, and sketch the graph of f^{-1}. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

41.



42.



43. Cost You need 50 pounds of two commodities costing \$1.25 and \$1.60 per pound.

(a) Verify that the total cost is $y = 1.25x + 1.60(50 - x)$, where x is the number of pounds of the less expensive commodity.

(b) Find the inverse function of the cost function. What does each variable represent in the inverse function?

(c) What is the domain of the inverse function? Validate or explain your answer using the context of the problem.

(d) Determine the number of pounds of the less expensive commodity purchased when the total cost is \$73.

44. Temperature The formula $C = \frac{5}{9}(F - 32)$, where $F \geq -459.6$, represents the Celsius temperature C as a function of the Fahrenheit temperature F.

(a) Find the inverse function of C.

(b) What does the inverse function represent?

(c) What is the domain of the inverse function? Validate or explain your answer using the context of the problem.

(d) The temperature is 22°C. What is the corresponding temperature in degrees Fahrenheit?

Testing Whether a Function Is One-to-One **In Exercises 45–50, determine whether the function is one-to-one. If it is, find its inverse function.**

45. $f(x) = \sqrt{x - 2}$ **46.** $f(x) = \sqrt{9 - x^2}$

47. $f(x) = -3$ **48.** $f(x) = |x - 2|, \quad x \leq 2$

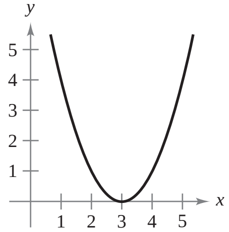
49. $f(x) = ax + b, \quad a \neq 0$ **50.** $f(x) = (x + a)^3 + b$

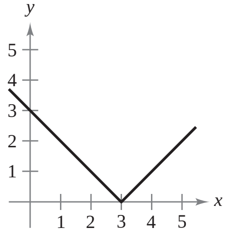
Showing a Function Is One-to-One **In Exercises 51–56, show that f is one-to-one on the given interval and therefore has an inverse function on that interval.**

Function	Interval
51. $f(x) = (x - 4)^2$	$[4, \infty)$
52. $f(x) = \lvert x + 2 \rvert$	$[-2, \infty)$
53. $f(x) = \dfrac{4}{x^2}$	$(0, \infty)$
54. $f(x) = \cot x$	$(0, \pi)$
55. $f(x) = \cos x$	$[0, \pi]$
56. $f(x) = \sec x$	$\left[0, \dfrac{\pi}{2}\right)$

Making a Function One-to-One **In Exercises 57 and 58, delete part of the domain so that the function that remains is one-to-one. Find the inverse function of the remaining function and give the domain of the inverse function. (*Note:* There is more than one correct answer.)**

57. $f(x) = (x - 3)^2$ **58.** $f(x) = |x - 3|$





Finding an Inverse Function **In Exercises 59–64, (a) sketch a graph of the function f, (b) determine an interval on which f is one-to-one, (c) find the inverse function of f on the interval found in part (b), and (d) give the domain of the inverse function. (*Note:* There is more than one correct answer.)**

59. $f(x) = (x + 5)^2$ **60.** $f(x) = (7 - x)^2$

61. $f(x) = \sqrt{x^2 - 4x}$ **62.** $f(x) = -\sqrt{25 - x^2}$

63. $f(x) = 3 \cos x$ **64.** $f(x) = 2 \sin x$

Finding Values **In Exercises 65–70, find $f^{-1}(a)$ for the function f and real number a.**

Function	Real Number
65. $f(x) = x^3 + 2x - 1$	$a = 2$
66. $f(x) = 2x^5 + x^3 + 1$	$a = -2$
67. $f(x) = \sin x, \quad -\dfrac{\pi}{2} \leq x \leq \dfrac{\pi}{2}$	$a = \dfrac{1}{2}$
68. $f(x) = \cos 2x, \quad 0 \leq x \leq \dfrac{\pi}{2}$	$a = 1$
69. $f(x) = x^3 - \dfrac{4}{x}, \quad x > 0$	$a = 6$
70. $f(x) = \sqrt{x - 4}$	$a = 2$

Using Composite and Inverse Functions In Exercises 71–74, use the functions $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3$ to find the indicated value.

71. $(f^{-1} \circ g^{-1})(1)$ **72.** $(g^{-1} \circ f^{-1})(-3)$

73. $(f^{-1} \circ f^{-1})(6)$ **74.** $(g^{-1} \circ g^{-1})(-4)$

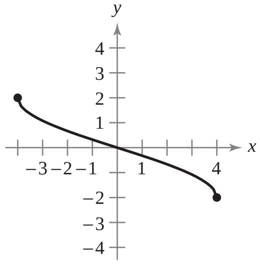
Using Composite and Inverse Functions In Exercises 75–78, use the functions $f(x) = x + 4$ and $g(x) = 2x - 5$ to find the indicated function.

75. $g^{-1} \circ f^{-1}$ **76.** $f^{-1} \circ g^{-1}$

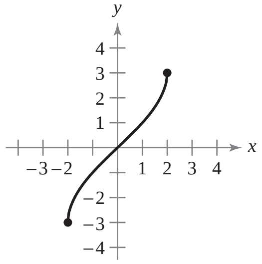
77. $(f \circ g)^{-1}$ **78.** $(g \circ f)^{-1}$

Graphical Reasoning In Exercises 79 and 80, (a) use the graph of the function f to determine whether f is one-to-one, (b) state the domain of f^{-1}, and (c) estimate the value of $f^{-1}(2)$.

79.

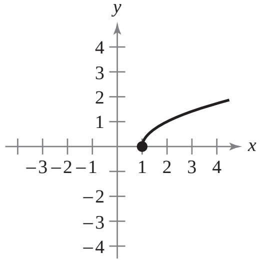


80.

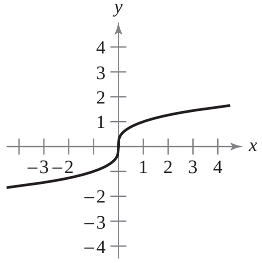


Graphical Reasoning In Exercises 81 and 82, use the graph of the function f to sketch the graph of f^{-1}. To print an enlarged copy of the graph, go to *MathGraphs.com*.

81.



82.

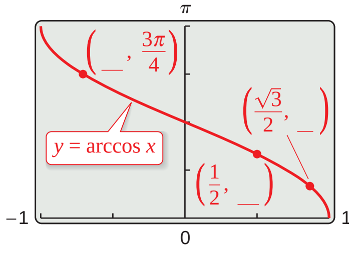


Numerical and Graphical Analysis In Exercises 83 and 84, (a) use a graphing utility to complete the table, (b) plot the points in the table and graph the function by hand, (c) use a graphing utility to graph the function and compare the result with your hand-drawn graph in part (b), and (d) determine any intercepts and symmetry of the graph.

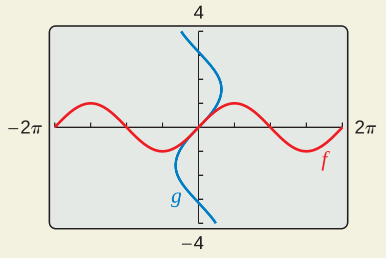
x	-1	-0.8	-0.6	-0.4	-0.2	0	0.2	0.4	0.6	0.8	1
y											

83. $y = \arcsin x$ **84.** $y = \arccos x$

85. Missing Coordinates Determine the missing coordinates of the points on the graph of the function.



86. HOW DO YOU SEE IT? You use a graphing utility to graph $f(x) = \sin x$ and then use the *draw inverse* feature to graph g (see figure). Is g the inverse function of f? Why or why not?



Evaluating Inverse Trigonometric Functions In Exercises 87–94, evaluate the expression without using a calculator.

87. $\arcsin \frac{1}{2}$ **88.** $\arcsin 0$

89. $\arccos \frac{1}{2}$ **90.** $\arccos 1$

91. $\arctan \frac{\sqrt{3}}{3}$ **92.** $\operatorname{arccot}(-\sqrt{3})$

93. $\operatorname{arccsc}(-\sqrt{2})$ **94.** $\operatorname{arcsec}(-\sqrt{2})$

Approximating Inverse Trigonometric Functions In Exercises 95–98, use a calculator to approximate the value. Round your answer to two decimal places.

95. $\arccos(-0.8)$ **96.** $\arcsin(-0.39)$

97. $\operatorname{arcsec} 1.269$ **98.** $\arctan(-5)$

Using Properties In Exercises 99 and 100, use the properties of inverse trigonometric functions to evaluate the expression.

99. $\cos[\arccos(-0.1)]$ **100.** $\arcsin(\sin 3\pi)$

Using a Right Triangle In Exercises 101–106, use the figure to write the expression in algebraic form given $y = \arccos x$, where $0 < y < \pi/2$.

101. $\cos y$

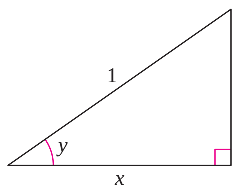
102. $\sin y$

103. $\tan y$

104. $\cot y$

105. $\sec y$

106. $\csc y$



Evaluating an Expression In Exercises 107–110, evaluate the expression without using a calculator. [*Hint:* Sketch a right triangle, as demonstrated in Example 7(b).]

107. (a) $\sin\left(\arctan \frac{3}{4}\right)$ (b) $\sec\left(\arcsin \frac{4}{5}\right)$

108. (a) $\tan\left(\arccos \frac{\sqrt{2}}{2}\right)$ (b) $\cos\left(\arcsin \frac{5}{13}\right)$

109. (a) $\cot\left[\arcsin\left(-\frac{1}{2}\right)\right]$ (b) $\csc\left[\arctan\left(-\frac{5}{12}\right)\right]$

110. (a) $\sec\left[\arctan\left(-\frac{3}{5}\right)\right]$ (b) $\tan\left[\arcsin\left(-\frac{5}{6}\right)\right]$

Simplifying an Expression In Exercises 111–116, write the expression in algebraic form. [*Hint:* Sketch a right triangle, as demonstrated in Example 7(a).]

111. $\cos(\arcsin 2x)$

112. $\sec(\arctan 4x)$

113. $\sin(\operatorname{arcsec} x)$

114. $\sec[\arcsin(x - 1)]$

115. $\tan\left(\operatorname{arcsec} \dfrac{x}{3}\right)$

116. $\csc\left(\arctan \dfrac{x}{\sqrt{2}}\right)$

Solving an Equation In Exercises 117–120, solve the equation for x.

117. $\arcsin(3x - \pi) = \frac{1}{2}$

118. $\arctan(2x - 5) = -1$

119. $\arcsin \sqrt{2x} = \arccos \sqrt{x}$

120. $\arccos x = \operatorname{arcsec} x$

Point of Intersection In Exercises 121 and 122, find the point of intersection of the graphs of the functions.

121. $y = \arccos x$
$y = \arctan x$

122. $y = \arcsin x$
$y = \arccos x$

WRITING ABOUT CONCEPTS

123. Inverse Functions Describe how to find the inverse function of a one-to-one function given by an equation in x and y. Give an example.

124. Describing Relationships Describe the relationship between the graph of a function and the graph of its inverse function.

125. Inverse Trigonometric Functions Explain why $\tan \pi = 0$ does not imply that $\arctan 0 = \pi$.

126. Inverse Trigonometric Functions and Technology Explain how to graph $y = \operatorname{arccot} x$ on a graphing utility that does not have the arccotangent function.

Fill in the Blank In Exercises 127 and 128, fill in the blank.

127. $\arctan \dfrac{9}{x} = \arcsin(\quad\;\;), \quad x > 0$

128. $\arcsin \dfrac{\sqrt{36 - x^2}}{6} = \arccos(\quad\;\;)$

Verifying an Identity In Exercises 129 and 130, verify each identity.

129. (a) $\operatorname{arccsc} x = \arcsin \dfrac{1}{x}, \quad |x| \geq 1$

(b) $\arctan x + \arctan \dfrac{1}{x} = \dfrac{\pi}{2}, \quad x > 0$

130. (a) $\arcsin(-x) = -\arcsin x, \quad |x| \leq 1$

(b) $\arccos(-x) = \pi - \arccos x, \quad |x| \leq 1$

Sketching a Graph In Exercises 131–134, sketch the graph of the function. Use a graphing utility to verify your graph.

131. $f(x) = \arcsin(x - 1)$

132. $f(x) = \operatorname{arcsec} 2x$

133. $f(x) = \arctan x + \dfrac{\pi}{2}$

134. $f(x) = \arccos \dfrac{x}{4}$

135. Think About It Given that f is a one-to-one function and $f(-3) = 8$, find $f^{-1}(8)$.

136. Think About It Given $f(x) = 5 + \arccos x$, find

$$f^{-1}\left(5 + \frac{\pi}{2}\right).$$

137. Proof Prove that if f and g are one-to-one functions, then $(f \circ g)^{-1}(x) = (g^{-1} \circ f^{-1})(x)$.

138. Proof Prove that if f has an inverse function, then $(f^{-1})^{-1} = f$.

139. Proof Prove that $\cos(\sin^{-1} x) = \sqrt{1 - x^2}$.

140. Proof Prove that if a function has an inverse function, then the inverse function is unique.

True or False? In Exercises 141–146, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

141. If f is an even function, then f^{-1} exists.

142. If the inverse function of f exists, then the y-intercept of f is an x-intercept of f^{-1}.

143. $\arcsin^2 x + \arccos^2 x = 1$

144. The range of $y = \arcsin x$ is $[0, \pi]$.

145. If $f(x) = x^n$ where n is odd, then f^{-1} exists.

146. There exists no function f such that $f = f^{-1}$.

147. Verifying an Identity Verify each identity.

(a) $\operatorname{arccot} x = \begin{cases} \pi + \arctan(1/x), & x < 0 \\ \pi/2, & x = 0 \\ \arctan(1/x), & x > 0 \end{cases}$

(b) $\operatorname{arcsec} x = \arccos(1/x), \quad |x| \geq 1$

(c) $\operatorname{arccsc} x = \arcsin(1/x), \quad |x| \geq 1$

148. Using an Identity Use the results of Exercise 147 and a graphing utility to evaluate each expression.

(a) $\operatorname{arccot} 0.5$

(b) $\operatorname{arcsec} 2.7$

(c) $\operatorname{arccsc}(-3.9)$

(d) $\operatorname{arccot}(-1.4)$

149. Proof Prove that

$$\arctan x + \arctan y = \arctan \frac{x + y}{1 - xy}, \quad xy \neq 1.$$

Use this formula to show that

$$\arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{\pi}{4}.$$

150. Think About It Use a graphing utility to graph $f(x) = \sin x$ and $g(x) = \arcsin(\sin x)$. Why isn't the graph of g the line $y = x$?

151. Determining Conditions Let $f(x) = ax^2 + bx + c$, where $a > 0$ and the domain is all real numbers such that $x \leq -\dfrac{b}{2a}$. Find f^{-1}.

152. Determining Conditions Determine conditions on the constants a, b, and c such that the graph of $f(x) = \dfrac{ax + b}{cx - a}$ is symmetric about the line $y = x$.

153. Determining Conditions Determine conditions on the constants a, b, c, and d such that $f(x) = \dfrac{ax + b}{cx + d}$ has an inverse function. Then find f^{-1}.

1.6 Exponential and Logarithmic Functions

- **Develop and use properties of exponential functions.**
- **Understand the definition of the number *e*.**
- **Understand the definition of the natural logarithmic function, and develop and use properties of the natural logarithmic function.**

Exponential Functions

An **exponential function** involves a constant raised to a power, such as $f(x) = 2^x$. You already know how to evaluate 2^x for *rational* values of x. For instance,

$$2^0 = 1, \quad 2^2 = 4, \quad 2^{-1} = \frac{1}{2}, \quad \text{and} \quad 2^{1/2} = \sqrt{2} \approx 1.4142136.$$

For *irrational* values of x, you can define 2^x by considering a sequence of rational numbers that approach x. A full discussion of this process would not be appropriate now, but here is the general idea. To define the number $2^{\sqrt{2}}$, note that

$$\sqrt{2} = 1.414213 \ldots$$

and consider the numbers below (which are of the form 2^r, where r is rational).

$$
\begin{aligned}
2^1 = 2 &< 2^{\sqrt{2}} < 4 = 2^2\\
2^{1.4} = 2.639015 \ldots &< 2^{\sqrt{2}} < 2.828427 \ldots = 2^{1.5}\\
2^{1.41} = 2.657371 \ldots &< 2^{\sqrt{2}} < 2.675855 \ldots = 2^{1.42}\\
2^{1.414} = 2.664749 \ldots &< 2^{\sqrt{2}} < 2.666597 \ldots = 2^{1.415}\\
2^{1.4142} = 2.665119 \ldots &< 2^{\sqrt{2}} < 2.665303 \ldots = 2^{1.4143}\\
2^{1.41421} = 2.665137 \ldots &< 2^{\sqrt{2}} < 2.665156 \ldots = 2^{1.41422}\\
2^{1.414213} = 2.665143 \ldots &< 2^{\sqrt{2}} < 2.665144 \ldots = 2^{1.414214}
\end{aligned}
$$

From these calculations, it seems reasonable to conclude that

$$2^{\sqrt{2}} \approx 2.66514.$$

In practice, you can use a calculator to approximate numbers such as $2^{\sqrt{2}}$.

In general, you can use any positive base a, $a \neq 1$, to define an exponential function. So, the exponential function with base a is written as $f(x) = a^x$. Exponential functions, even those with irrational values of x, obey the familiar properties of exponents.

Properties of Exponents

Let a and b be positive real numbers, and let x and y be any real numbers.

1. $a^0 = 1$ **2.** $a^x a^y = a^{x+y}$ **3.** $(a^x)^y = a^{xy}$ **4.** $(ab)^x = a^x b^x$

5. $\dfrac{a^x}{a^y} = a^{x-y}$ **6.** $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$ **7.** $a^{-x} = \dfrac{1}{a^x}$

EXAMPLE 1 **Using Properties of Exponents**

a. $(2^2)(2^3) = 2^{2+3} = 2^5$ **b.** $\dfrac{2^2}{2^3} = 2^{2-3} = 2^{-1} = \dfrac{1}{2}$

c. $(3^x)^3 = 3^{3x}$ **d.** $\left(\dfrac{1}{3}\right)^{-x} = (3^{-1})^{-x} = 3^x$

EXAMPLE 2 Sketching Graphs of Exponential Functions

See LarsonCalculus.com for an interactive version of this type of example.

Sketch the graphs of the functions

$$f(x) = 2^x, \quad g(x) = \left(\tfrac{1}{2}\right)^x = 2^{-x}, \quad \text{and} \quad h(x) = 3^x.$$

Solution To sketch the graphs of these functions by hand, you can complete a table of values, plot the corresponding points, and connect the points with smooth curves.

x	-3	-2	-1	0	1	2	3	4
2^x	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8	16
2^{-x}	8	4	2	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$
3^x	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9	27	81

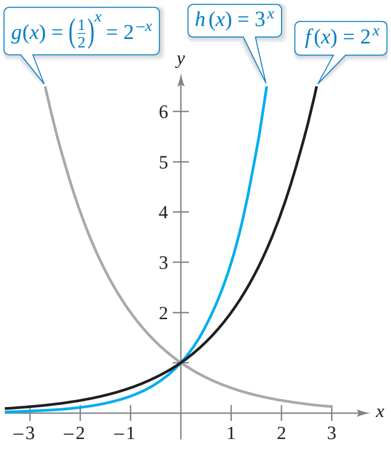


Figure 1.46

Another way to graph these functions is to use a graphing utility. In either case, you should obtain graphs similar to those shown in Figure 1.46. Note that the graphs of f and h are increasing, and the graph of g is decreasing. Also, the graph of h is increasing more rapidly than the graph of f.

The shapes of the graphs in Figure 1.46 are typical of the exponential functions $f(x) = a^x$ and $g(x) = a^{-x}$ where $a > 1$, as shown in Figure 1.47.

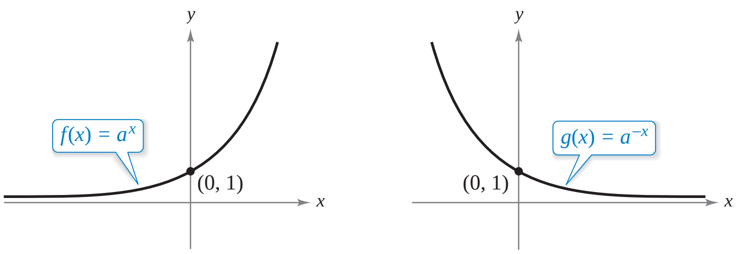


Figure 1.47

Properties of Exponential Functions

Let a be a real number that is greater than 1.

1. The domain of $f(x) = a^x$ and $g(x) = a^{-x}$ is $(-\infty, \infty)$.
2. The range of $f(x) = a^x$ and $g(x) = a^{-x}$ is $(0, \infty)$.
3. The y-intercept of $f(x) = a^x$ and $g(x) = a^{-x}$ is $(0, 1)$.
4. The functions $f(x) = a^x$ and $g(x) = a^{-x}$ are one-to-one.

TECHNOLOGY Functions of the form $h(x) = b^{cx}$ have the same types of properties and graphs as functions of the form $f(x) = a^x$ and $g(x) = a^{-x}$. To see why this is true, notice that

$$b^{cx} = (b^c)^x.$$

For instance, $f(x) = 2^{3x}$ can be written as

$$f(x) = (2^3)^x \quad \text{or} \quad f(x) = 8^x.$$

Try confirming this by graphing $f(x) = 2^{3x}$ and $g(x) = 8^x$ in the same viewing window.

The Number e

In calculus, the natural (or convenient) choice for a base of an exponential number is the irrational number e, whose decimal approximation is

$$e \approx 2.71828182846.$$

This choice may seem anything but natural. The convenience of this particular base, however, will become apparent as you continue in this course.

EXAMPLE 3 Investigating the Number e

Describe the behavior of the function $f(x) = (1 + x)^{1/x}$ at values of x that are close to 0.

Solution One way to examine the values of $f(x)$ near 0 is to construct a table.

x	-0.01	-0.001	-0.0001	0.0001	0.001	0.01
$(1 + x)^{1/x}$	2.7320	2.7196	2.7184	2.7181	2.7169	2.7048

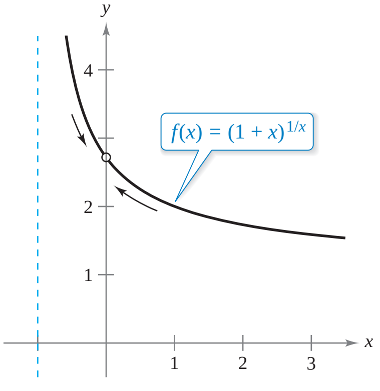


Figure 1.48

From the table, it appears that the closer x gets to 0, the closer $(1 + x)^{1/x}$ gets to e. The graph of f shown in Figure 1.48 supports this conclusion. Try using a graphing calculator to obtain this graph. Then zoom in closer and closer to $x = 0$. Although f is not defined when $x = 0$, it is defined for x-values that are arbitrarily close to zero. By zooming in, you can see that the value of $f(x)$ gets closer and closer to $e \approx 2.71828182846$ as x gets closer and closer to 0. Later, when you study limits, you will learn that this result can be written as

$$\lim_{x \to 0} (1 + x)^{1/x} = e$$

which is read as "the limit of $(1 + x)^{1/x}$ as x approaches 0 is e."

EXAMPLE 4 The Graph of the Natural Exponential Function

Sketch the graph of $f(x) = e^x$.

Solution To sketch the graph of f by hand, you can complete a table of values, plot the corresponding points, and connect the points with a smooth curve (see Figure 1.49).

x	-2	-1	0	1	2
e^x	$\frac{1}{e^2} \approx 0.135$	$\frac{1}{e} \approx 0.368$	1	$e \approx 2.718$	$e^2 \approx 7.389$

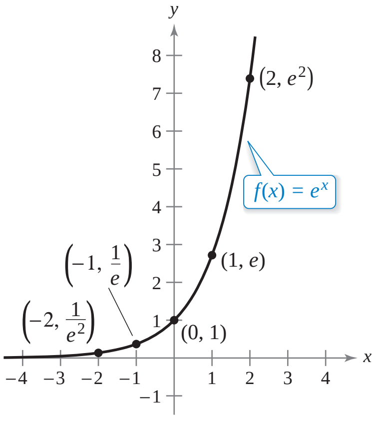


Figure 1.49

The Natural Logarithmic Function

Because the natural exponential function

$$f(x) = e^x$$

is one-to-one, it must have an inverse function. Its inverse is called the **natural logarithmic function.** The domain of the natural logarithmic function is the set of positive real numbers.

REMARK The notation $\ln x$ is read as "el en of x" or "the natural log of x."

Definition of the Natural Logarithmic Function

Let x be a positive real number. The **natural logarithmic function,** denoted by $\ln x$, is defined as

$$\ln x = b \quad \text{if and only if} \quad e^b = x.$$

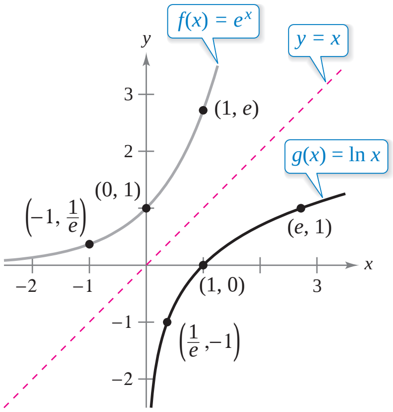


Figure 1.50

This definition tells you that a logarithmic equation can be written in an equivalent exponential form, and vice versa. Here are some examples.

Logarithmic Form	*Exponential Form*
$\ln 1 = 0$	$e^0 = 1$
$\ln e = 1$	$e^1 = e$
$\ln e^{-1} = -1$	$e^{-1} = \dfrac{1}{e}$

Because the function $g(x) = \ln x$ is defined to be the inverse of $f(x) = e^x$, it follows that the graph of the natural logarithmic function is a reflection of the graph of the natural exponential function in the line $y = x$, as shown in Figure 1.50. Several other properties of the natural logarithmic function also follow directly from its definition as the inverse of the natural exponential function.

Properties of the Natural Logarithmic Function

1. The domain of $g(x) = \ln x$ is $(0, \infty)$.
2. The range of $g(x) = \ln x$ is $(-\infty, \infty)$.
3. The x-intercept of $g(x) = \ln x$ is $(1, 0)$.
4. The function $g(x) = \ln x$ is one-to-one.

Because $f(x) = e^x$ and $g(x) = \ln x$ are inverses of each other, you can conclude that

$$\ln e^x = x \quad \text{and} \quad e^{\ln x} = x.$$

One of the properties of exponents states that when you multiply two exponential functions (having the same base), you add their exponents. For instance,

$$e^x e^y = e^{x+y}.$$

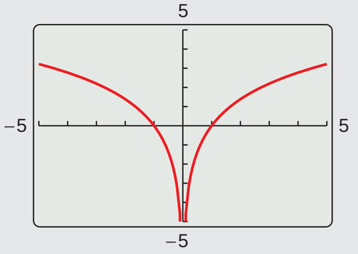
The logarithmic version of this property states that the natural logarithm of the product of two numbers is equal to the sum of the natural logs of the numbers. That is,

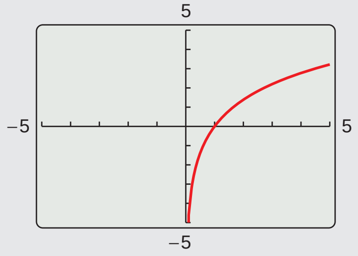
$$\ln xy = \ln x + \ln y.$$

This property and the properties dealing with the natural log of a quotient and the natural log of a power are listed on the next page.

Exploration

A graphing utility is used to graph $f(x) = \ln x^2$ and $g(x) = 2 \ln x$. Which of the graphs below is the graph of f? Which is the graph of g?





Properties of Logarithms

Let x, y, and z be real numbers such that $x > 0$ and $y > 0$.

1. $\ln xy = \ln x + \ln y$
2. $\ln \frac{x}{y} = \ln x - \ln y$
3. $\ln x^z = z \ln x$

EXAMPLE 5 Expanding Logarithmic Expressions

a. $\ln \frac{10}{9} = \ln 10 - \ln 9$ Property 2

b. $\ln \sqrt{3x + 2} = \ln(3x + 2)^{1/2}$ Rewrite with rational exponent.

$= \frac{1}{2} \ln(3x + 2)$ Property 3

c. $\ln \frac{6x}{5} = \ln(6x) - \ln 5$ Property 2

$= \ln 6 + \ln x - \ln 5$ Property 1

d. $$\ln \frac{(x^2 + 3)^2}{x\sqrt[3]{x^2 + 1}} = \ln(x^2 + 3)^2 - \ln\left(x\sqrt[3]{x^2 + 1}\right)$$
$$= 2 \ln(x^2 + 3) - [\ln x + \ln(x^2 + 1)^{1/3}]$$
$$= 2 \ln(x^2 + 3) - \ln x - \ln(x^2 + 1)^{1/3}$$
$$= 2 \ln(x^2 + 3) - \ln x - \frac{1}{3} \ln(x^2 + 1)$$

When using the properties of logarithms to rewrite logarithmic functions, you must check to see whether the domain of the rewritten function is the same as the domain of the original function. For instance, the domain of $f(x) = \ln x^2$ is all real numbers except $x = 0$, and the domain of $g(x) = 2 \ln x$ is all positive real numbers.

EXAMPLE 6 Solving Exponential and Logarithmic Equations

Solve for x.

a. $7 = e^{x+1}$ **b.** $\ln(2x - 3) = 5$

Solution

a.

$7 = e^{x+1}$ Write original equation.

$\ln 7 = \ln(e^{x+1})$ Take natural log of each side.

$\ln 7 = x + 1$ Apply inverse property.

$-1 + \ln 7 = x$ Solve for x.

$0.946 \approx x$ Use a calculator.

b. $\ln(2x - 3) = 5$ Write original equation.

$e^{\ln(2x-3)} = e^5$ Exponentiate each side.

$2x - 3 = e^5$ Apply inverse property.

$x = \frac{1}{2}(e^5 + 3)$ Solve for x.

$x \approx 75.707$ Use a calculator.

1.6 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Evaluating an Expression In Exercises 1 and 2, evaluate the expressions.

1. (a) $25^{3/2}$ (b) $81^{1/2}$ (c) 3^{-2} (d) $27^{-1/3}$

2. (a) $64^{1/3}$ (b) 5^{-4} (c) $\left(\frac{1}{8}\right)^{1/3}$ (d) $\left(\frac{1}{4}\right)^{3}$

Using Properties of Exponents In Exercises 3–6, use the properties of exponents to simplify the expressions.

3. (a) $(5^2)(5^3)$ (b) $(5^2)(5^{-3})$ (c) $\dfrac{5^3}{25^2}$ (d) $\left(\dfrac{1}{4}\right)^2 2^6$

4. (a) $(2^2)^3$ (b) $(5^4)^{1/2}$ (c) $[(27^{-1})(27^{2/3})]^3$ (d) $(25^{3/2})(3^2)$

5. (a) $e^2(e^4)$ (b) $(e^3)^4$ (c) $(e^3)^{-2}$ (d) $\dfrac{e^5}{e^3}$

6. (a) $\left(\dfrac{1}{e}\right)^{-2}$ (b) $\left(\dfrac{e^5}{e^2}\right)^{-1}$ (c) e^0 (d) $\dfrac{1}{e^{-3}}$

Solving an Equation In Exercises 7–22, solve for x.

7. $3^x = 81$
8. $4^x = 64$
9. $6^{x-2} = 36$
10. $5^{x+1} = 125$
11. $\left(\frac{1}{2}\right)^x = 32$
12. $\left(\frac{1}{4}\right)^x = 16$
13. $\left(\frac{1}{3}\right)^{x-1} = 27$
14. $\left(\frac{1}{5}\right)^{2x} = 625$
15. $4^3 = (x+2)^3$
16. $18^2 = (5x-7)^2$
17. $x^{3/4} = 8$
18. $(x+3)^{4/3} = 16$
19. $e^x = 5$
20. $e^x = 1$
21. $e^{-2x} = e^5$
22. $e^{3x} = e^{-4}$

Comparing Numbers In Exercises 23 and 24, compare the given number with the number e. Is the number less than or greater than e?

23. $\left(1 + \dfrac{1}{1{,}000{,}000}\right)^{1{,}000{,}000}$

24. $1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040}$

Sketching the Graph of a Function In Exercises 25–38, sketch the graph of the function.

25. $y = 3^x$
26. $y = 3^{x-1}$
27. $y = \left(\frac{1}{3}\right)^x$
28. $y = 2^{-x^2}$
29. $f(x) = 3^{-x^2}$
30. $f(x) = 3^{|x|}$
31. $y = e^{-x}$
32. $y = \frac{1}{2}e^x$
33. $y = e^x + 2$
34. $y = e^{x-1}$
35. $h(x) = e^{x-2}$
36. $g(x) = -e^{x/2}$
37. $y = e^{-x^2}$
38. $y = e^{-x/4}$

Finding the Domain In Exercises 39–44, find the domain of the function.

39. $f(x) = \dfrac{1}{3 + e^x}$
40. $f(x) = \dfrac{1}{2 - e^x}$
41. $f(x) = \sqrt{1 - 4^x}$
42. $f(x) = \sqrt{1 + 3^{-x}}$
43. $f(x) = \sin e^{-x}$
44. $f(x) = \cos e^{-x}$

45. Identifying a Relationship Use a graphing utility to graph $f(x) = e^x$ and the given function in the same viewing window. How are the two graphs related?

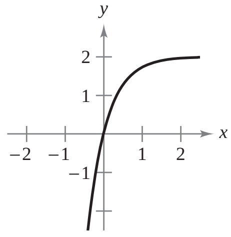
(a) $g(x) = e^{x-2}$ (b) $h(x) = -\frac{1}{2}e^x$ (c) $q(x) = e^{-x} + 3$

46. Describing the Shape of a Graph Use a graphing utility to graph the function. Describe the shape of the graph for very large and very small values of x.

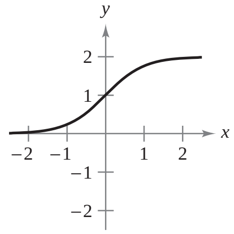
(a) $f(x) = \dfrac{8}{1 + e^{-0.5x}}$ (b) $g(x) = \dfrac{8}{1 + e^{-0.5/x}}$

Matching In Exercises 47–50, match the equation with the correct graph. Assume that a and C are positive real numbers. [The graphs are labeled (a), (b), (c), and (d).]

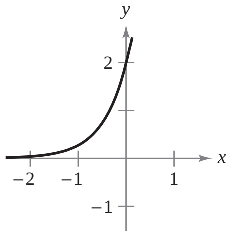
(a)



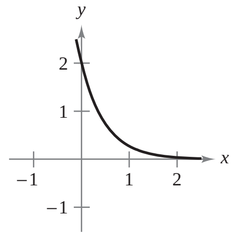
(b)



(c)



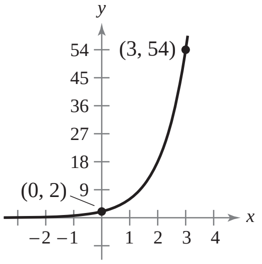
(d)



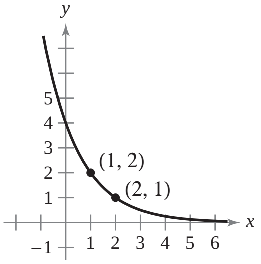
47. $y = Ce^{ax}$
48. $y = Ce^{-ax}$
49. $y = C(1 - e^{-ax})$
50. $y = \dfrac{C}{1 + e^{-ax}}$

Finding an Exponential Function In Exercises 51 and 52, find the exponential function $y = Ca^x$ that fits the graph.

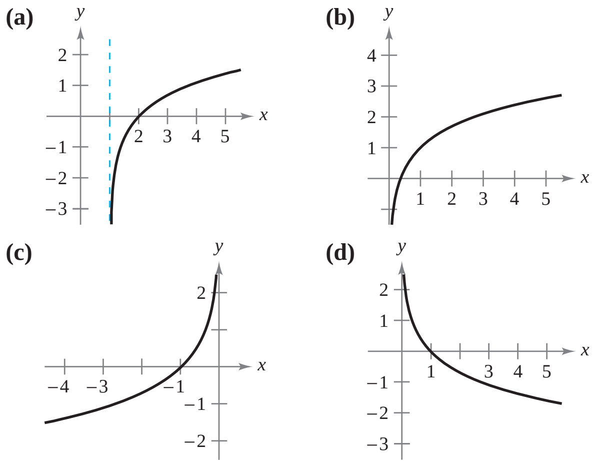
51.



52.



Matching In Exercises 53–56, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]



53. $f(x) = \ln x + 1$

54. $f(x) = -\ln x$

55. $f(x) = \ln(x - 1)$

56. $f(x) = -\ln(-x)$

Writing Exponential or Logarithmic Equations In Exercises 57–60, write the exponential equation as a logarithmic equation, or vice versa.

57. $e^0 = 1$

58. $e^{-2} = 0.1353 \ldots$

59. $\ln 2 = 0.6931 \ldots$

60. $\ln 0.5 = -0.6931 \ldots$

Sketching a Graph In Exercises 61–68, sketch the graph of the function and state its domain.

61. $f(x) = 3 \ln x$

62. $f(x) = -2 \ln x$

63. $f(x) = \ln 2x$

64. $f(x) = \ln|x|$

65. $f(x) = \ln(x - 3)$

66. $f(x) = \ln x - 4$

67. $f(x) = \ln(x + 2)$

68. $f(x) = \ln(x - 2) + 1$

Writing an Equation In Exercises 69–72, write an equation for the function having the given characteristics.

69. The shape of $f(x) = e^x$, but shifted eight units upward and reflected in the x-axis

70. The shape of $f(x) = e^x$, but shifted two units to the left and six units downward

71. The shape of $f(x) = \ln x$, but shifted five units to the right and one unit downward

72. The shape of $f(x) = \ln x$, but shifted three units upward and reflected in the y-axis

Inverse Functions In Exercises 73–76, illustrate that the functions f and g are inverses of each other by using a graphing utility to graph them in the same viewing window.

73. $f(x) = e^{2x}$, $g(x) = \ln\sqrt{x}$

74. $f(x) = e^{x/3}$, $g(x) = \ln x^3$

75. $f(x) = e^x - 1$, $g(x) = \ln(x + 1)$

76. $f(x) = e^{x-1}$, $g(x) = 1 + \ln x$

Finding Inverse Functions In Exercises 77–80, (a) find the inverse of the function, (b) use a graphing utility to graph f and f^{-1} in the same viewing window, and (c) verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.

77. $f(x) = e^{4x-1}$

78. $f(x) = 3e^{-x}$

79. $f(x) = 2\ln(x - 1)$

80. $f(x) = 3 + \ln(2x)$

Applying Inverse Properties In Exercises 81–86, apply the inverse properties of $\ln x$ and e^x to simplify the given expression.

81. $\ln e^{x^2}$

82. $\ln e^{2x-1}$

83. $e^{\ln(5x+2)}$

84. $e^{\ln\sqrt{x}}$

85. $-1 + \ln e^{2x}$

86. $-8 + e^{\ln x^3}$

Using Properties of Logarithms In Exercises 87 and 88, use the properties of logarithms to approximate the indicated logarithms, given that $\ln 2 \approx 0.6931$ and $\ln 3 \approx 1.0986$.

87. (a) $\ln 6$ (b) $\ln \frac{2}{3}$ (c) $\ln 81$ (d) $\ln\sqrt{3}$

88. (a) $\ln 0.25$ (b) $\ln 24$ (c) $\ln \sqrt[3]{12}$ (d) $\ln \frac{1}{72}$

Expanding a Logarithmic Expression In Exercises 89–98, use the properties of logarithms to expand the logarithmic expression.

89. $\ln \dfrac{x}{4}$

90. $\ln\sqrt{x^5}$

91. $\ln \dfrac{xy}{z}$

92. $\ln(xyz)$

93. $\ln\left(x\sqrt{x^2 + 5}\right)$

94. $\ln \sqrt[3]{z + 1}$

95. $\ln\sqrt{\dfrac{x - 1}{x}}$

96. $\ln z(z - 1)^2$

97. $\ln(3e^2)$

98. $\ln \dfrac{1}{e}$

Condensing a Logarithmic Expression In Exercises 99–106, write the expression as the logarithm of a single quantity.

99. $\ln x + \ln 7$

100. $\ln y + \ln x^2$

101. $\ln(x - 2) - \ln(x + 2)$

102. $3\ln x + 2\ln y - 4\ln z$

103. $\frac{1}{3}[2\ln(x + 3) + \ln x - \ln(x^2 - 1)]$

104. $2[\ln x - \ln(x + 1) - \ln(x - 1)]$

105. $2\ln 3 - \frac{1}{2}\ln(x^2 + 1)$

106. $\frac{3}{2}[\ln(x^2 + 1) - \ln(x + 1) - \ln(x - 1)]$

Solving an Exponential or Logarithmic Equation In Exercises 107–110, solve for x accurate to three decimal places.

107. (a) $e^{\ln x} = 4$
(b) $\ln e^{2x} = 3$

108. (a) $e^{\ln 2x} = 12$
(b) $\ln e^{-x} = 0$

109. (a) $\ln x = 2$
(b) $e^x = 4$

110. (a) $\ln x^2 = 8$
(b) $e^{-2x} = 5$

Solving an Inequality **In Exercises 111–114, solve the inequality for x.**

111. $e^x > 5$

112. $e^{1-x} < 6$

113. $-2 < \ln x < 0$

114. $1 < \ln x < 100$

Solving an Inequality **In Exercises 115 and 116, show that $f = g$ by using a graphing utility to graph f and g in the same viewing window. (Assume $x > 0$.)**

115. $f(x) = \ln \dfrac{x^2}{4}$

$g(x) = 2 \ln x - \ln 4$

116. $f(x) = \ln \sqrt{x(x^2 + 1)}$

$g(x) = \frac{1}{2}[\ln x + \ln(x^2 + 1)]$

WRITING ABOUT CONCEPTS

117. Stating Properties In your own words, state the properties of the natural logarithmic function.

118. Think About It Explain why $\ln e^x = x$.

119. Stating Properties In your own words, state the properties of the natural exponential function.

120. Describe the Relationship Describe the relationship between the graphs of $f(x) = \ln x$ and $g(x) = e^x$.

121. Analyze a Statement The table of values below was obtained by evaluating a function. Determine which of the statements may be true and which must be false, and explain why.

(a) y is an exponential function of x.

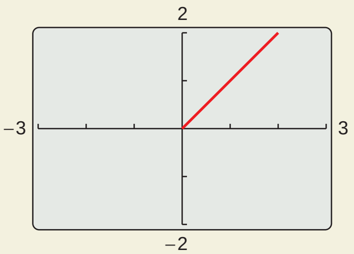
(b) y is a logarithmic function of x.

(c) x is an exponential function of y.

(d) y is a linear function of x.

x	1	2	8
y	0	1	3

122. HOW DO YOU SEE IT? The figure below shows the graph of $y_1 = \ln e^x$ or $y_2 = e^{\ln x}$. Which graph is it? What are the domains of y_1 and y_2? Does $\ln e^x = e^{\ln x}$ for all real values of x? Explain.



Sound Intensity **In Exercises 123 and 124, use the following information. The relationship between the number of decibels β and the intensity of a sound in watts per centimeter squared is**

$$\beta = \frac{10}{\ln 10} \ln\left(\frac{I}{10^{-16}}\right).$$

123. Use the properties of logarithms to write the formula in simpler form.

124. Determine the number of decibels of a sound with an intensity of 10^{-5} watt per square centimeter.

True or False? **In Exercises 125 and 126, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

125. $\ln(x + 25) = \ln x + \ln 25$

126. $\ln xy = \ln x \ln y$

127. Comparing Functions Use a graphing utility to graph the functions

$f(x) = 6^x$ and $g(x) = x^6$

in the same viewing window. Where do these graphs intersect? As x increases, which function grows more rapidly?

128. Comparing Functions Use a graphing utility to graph the functions

$f(x) = \ln x$ and $g(x) = x^{1/4}$

in the same viewing window. Where do these graphs intersect? As x increases, which function grows more rapidly?

129. Analyzing a Function Let $f(x) = \ln\left(x + \sqrt{x^2 + 1}\right)$.

(a) Use a graphing utility to graph f and determine its domain.

(b) Show that f is an odd function.

(c) Find the inverse function of f.

130. Prime Number Theorem There are 25 prime numbers less than 100. The **Prime Number Theorem** states that the number of primes less than x approaches

$$p(x) \approx \frac{x}{\ln x}.$$

Use this approximation to estimate the rate (in primes per 100 integers) at which the prime numbers occur when

(a) $x = 1000$.

(b) $x = 1{,}000{,}000$.

(c) $x = 1{,}000{,}000{,}000$.

Stirling's Formula **For large values of n,**

$$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n - 1) \cdot n$$

can be approximated by Stirling's Formula,

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2\pi n}.$$

In Exercises 131 and 132, find the exact value of $n!$, and then approximate $n!$ using Stirling's Formula.

131. $n = 12$

132. $n = 15$

133. Proof Prove that $\ln(x/y) = \ln x - \ln y$, $x > 0, y > 0$.

134. Proof Prove that $\ln x^y = y \ln x$.

Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding Intercepts **In Exercises 1–4, find any intercepts.**

1. $y = 5x - 8$

2. $y = x^2 - 8x + 12$

3. $y = \dfrac{x - 3}{x - 4}$

4. $y = (x - 3)\sqrt{x + 4}$

Testing for Symmetry **In Exercises 5–8, test for symmetry with respect to each axis and to the origin.**

5. $y = x^2 + 4x$

6. $y = x^4 - x^2 + 3$

7. $y^2 = x^2 - 5$

8. $xy = -2$

Using Intercepts and Symmetry to Sketch a Graph **In Exercises 9–14, sketch the graph of the equation. Identify any intercepts and test for symmetry.**

9. $y = -\frac{1}{2}x + 3$

10. $y = -x^2 + 4$

11. $y = x^3 - 4x$

12. $y^2 = 9 - x$

13. $y = 2\sqrt{4 - x}$

14. $y = |x - 4| - 4$

Finding Points of Intersection **In Exercises 15–18, find the points of intersection of the graphs of the equations.**

15. $5x + 3y = -1$
$x - y = -5$

16. $2x + 4y = 9$
$6x - 4y = 7$

17. $x - y = -5$
$x^2 - y = 1$

18. $x^2 + y^2 = 1$
$-x + y = 1$

Finding the Slope of a Line **In Exercises 19 and 20, plot the points and find the slope of the line passing through them.**

19. $\left(\frac{3}{2}, 1\right), \left(5, \frac{5}{2}\right)$

20. $(-7, 8), (-1, 8)$

Finding an Equation of a Line **In Exercises 21–24, find an equation of the line that passes through the point and has the indicated slope. Then sketch the line.**

	Point	Slope
21.	$(3, -5)$	$m = \frac{7}{4}$
22.	$(-8, 1)$	m is undefined.
23.	$(-3, 0)$	$m = -\frac{2}{3}$
24.	$(5, 4)$	$m = 0$

Sketching Lines in the Plane **In Exercises 25–28, use the slope and *y*-intercept to sketch a graph of the equation.**

25. $y = 6$

26. $x = -3$

27. $y = 4x - 2$

28. $3x + 2y = 12$

Finding an Equation of a Line **In Exercises 29 and 30, find an equation of the line that passes through the points. Then sketch the line.**

29. $(0, 0), (8, 2)$

30. $(-5, 5), (10, -1)$

31. Finding Equations of Lines Find equations of the lines passing through $(-3, 5)$ and having the following characteristics.

(a) Slope of $\frac{7}{16}$

(b) Parallel to the line $5x - 3y = 3$

(c) Perpendicular to the line $3x + 4y = 8$

(d) Parallel to the y-axis

32. Finding Equations of Lines Find equations of the lines passing through $(2, 4)$ and having the following characteristics.

(a) Slope of $-\frac{2}{3}$

(b) Perpendicular to the line $x + y = 0$

(c) Passing through the point $(6, 1)$

(d) Parallel to the x-axis

33. Rate of Change The purchase price of a new machine is \$12,500, and its value will decrease by \$850 per year. Use this information to write a linear equation that gives the value V of the machine t years after it is purchased. Find its value at the end of 3 years.

34. Break-Even Analysis A contractor purchases a piece of equipment for \$36,500 that costs an average of \$9.25 per hour for fuel and maintenance. The equipment operator is paid \$13.50 per hour, and customers are charged \$30 per hour.

(a) Write an equation for the cost C of operating this equipment for t hours.

(b) Write an equation for the revenue R derived from t hours of use.

(c) Find the break-even point for this equipment by finding the time at which $R = C$.

Evaluating a Function **In Exercises 35–38, evaluate the function at the given value(s) of the independent variable. Simplify the result.**

35. $f(x) = 5x + 4$

(a) $f(0)$

(b) $f(5)$

(c) $f(-3)$

(d) $f(t + 1)$

36. $f(x) = x^3 - 2x$

(a) $f(-3)$

(b) $f(2)$

(c) $f(-1)$

(d) $f(c - 1)$

37. $f(x) = 4x^2$

$\dfrac{f(x + \Delta x) - f(x)}{\Delta x}$

38. $f(x) = 2x - 6$

$\dfrac{f(x) - f(-1)}{x - 1}$

Finding the Domain and Range of a Function **In Exercises 39–42, find the domain and range of the function.**

39. $f(x) = x^2 + 3$

40. $g(x) = \sqrt{6 - x}$

41. $f(x) = -|x + 1|$

42. $h(x) = \dfrac{2}{x + 1}$

Using the Vertical Line Test **In Exercises 43–46, sketch the graph of the equation and use the Vertical Line Test to determine whether y is a function of x.**

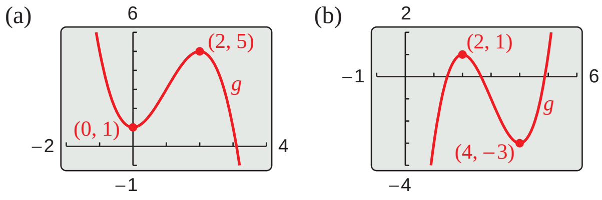
43. $x - y^2 = 6$

44. $x^2 - y = 0$

45. $y = \dfrac{|x - 2|}{x - 2}$

46. $x = 9 - y^2$

47. **Transformations of Functions** Use a graphing utility to graph $f(x) = x^3 - 3x^2$. Use the graph to write a formula for the function g shown in the figure. To print an enlarged copy of the graph, go to *MathGraphs.com.*



48. **Conjecture**

(a) Use a graphing utility to graph the functions f, g, and h in the same viewing window. Write a description of any similarities and differences you observe among the graphs.

Odd powers: $f(x) = x$, $g(x) = x^3$, $h(x) = x^5$

Even powers: $f(x) = x^2$, $g(x) = x^4$, $h(x) = x^6$

(b) Use the result in part (a) to make a conjecture about the graphs of the functions $y = x^7$ and $y = x^8$. Use a graphing utility to verify your conjecture.

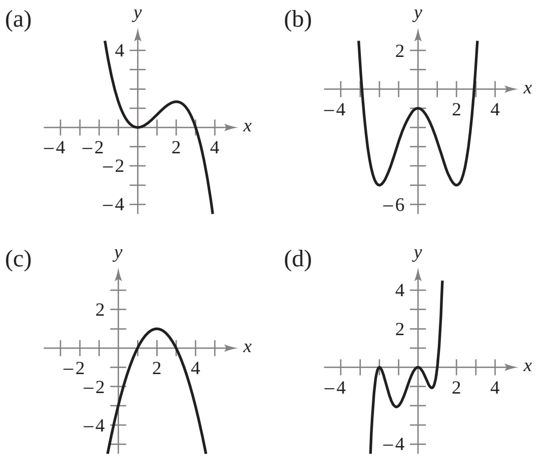
49. **Think About It** Use the results of Exercise 48 to guess the shapes of the graphs of the functions f, g, and h. Then use a graphing utility to graph each function and compare the result with your guess.

(a) $f(x) = x^2(x - 6)^2$

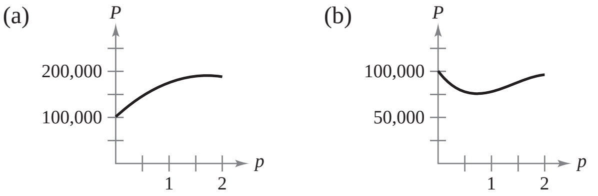
(b) $g(x) = x^3(x - 6)^2$

(c) $h(x) = x^3(x - 6)^3$

50. **Think About It** What is the minimum degree of the polynomial function whose graph approximates the given graph? What sign must the leading coefficient have?



51. **Writing** The following graphs give the profits P for two small companies over a period p of 2 years. Create a story to describe the behavior of each profit function for some hypothetical product the company produces.



52. **Area** A wire 24 inches long is to be cut into four pieces to form a rectangle whose shortest side has a length of x.

(a) Write the area A of the rectangle as a function of x.

(b) Determine the domain of the function and use a graphing utility to graph the function over that domain.

(c) Use the graph of the function to approximate the maximum area of the rectangle. Make a conjecture about the dimensions that yield a maximum area.

53. **Stress Test** A machine part was tested by bending it x centimeters 10 times per minute until the time y (in hours) of failure. The results are recorded in the table.

x	3	6	9	12	15
y	61	56	53	55	48

x	18	21	24	27	30
y	35	36	33	44	23

(a) Use the regression capabilities of a graphing utility to find a linear model for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the graph to determine whether there may have been an error made in conducting one of the tests or in recording the results. If so, eliminate the erroneous point and find the model for the remaining data.

54. **Median Income** The data in the table show the median income y (in thousands of dollars) for males of various ages x in the United States in 2009. *(Source: U.S. Census Bureau)*

x	20	30	40	50	60	70
y	10.0	31.9	42.2	44.7	41.3	25.9

(a) Use the regression capabilities of a graphing utility to find a quadratic model for the data.

(b) Use a graphing utility to plot the data and graph the model.

(c) Use the model to approximate the median income for a male who is 26 years old.

(d) Use the model to approximate the median income for a male who is 34 years old.

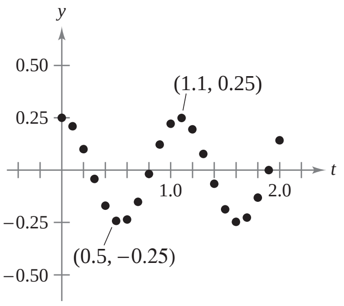
55. Summer Olympics The table lists the U.S. media rights fees y (in millions of dollars) for the Summer Olympics in year t, where $t = 4$ corresponds to 1984. *(Source: 2012 Olympics Media Guide, NBC Sports Group)*

t	4	8	12	16	20	24	28	32
y	225	300	401	456	705	793	894	1180

(a) Use the regression capabilities of a graphing utility to find a quadratic model for the data.

(b) Use a graphing utility to plot the data and graph the model. How well does the model fit the data? Explain your reasoning.

56. Harmonic Motion The motion of an oscillating weight suspended by a spring was measured by a motion detector. The data collected and the approximate maximum (positive and negative) displacements from equilibrium are shown in the figure. The displacement y is measured in feet, and the time t is measured in seconds.



(a) Is y a function of t? Explain.

(b) Approximate the amplitude and period of the oscillations.

(c) Find a model for the data.

(d) Use a graphing utility to graph the model in part (c). Compare the result with the data in the figure.

Finding an Inverse Function **In Exercises 57–62, (a) find the inverse of the function, (b) use a graphing utility to graph f and f^{-1} in the same viewing window, and (c) verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.**

57. $f(x) = \frac{1}{2}x - 3$

58. $f(x) = 5x - 7$

59. $f(x) = \sqrt{x + 1}$

60. $f(x) = x^3 + 2$

61. $f(x) = \sqrt[3]{x + 1}$

62. $f(x) = x^2 - 5, \quad x \geq 0$

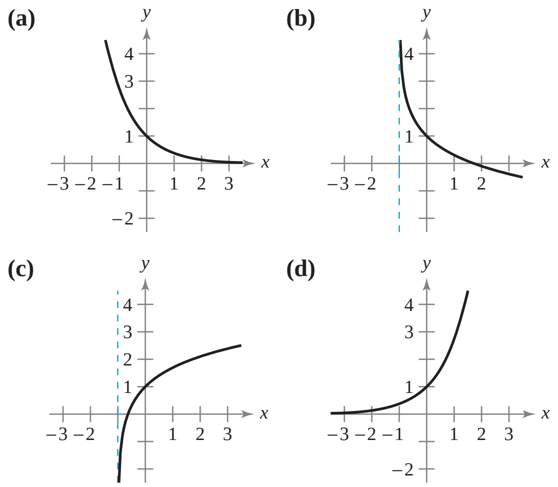
Sketching a Graph **In Exercises 63 and 64, sketch the graph of the function by hand.**

63. $f(x) = 2 \arctan(x + 3)$ **64.** $h(x) = -3 \arcsin 2x$

Evaluating an Expression **In Exercises 65 and 66, evaluate the expression without using a calculator. (*Hint:* Make a sketch of a right triangle.)**

65. $\sin\left(\arcsin \frac{1}{2}\right)$ **66.** $\tan(\operatorname{arccot} 2)$

Matching **In Exercises 67–70, match the function with its graph. [The graphs are labeled (a), (b), (c), and (d).]**



67. $f(x) = e^x$ **68.** $f(x) = e^{-x}$

69. $f(x) = \ln(x + 1) + 1$ **70.** $f(x) = -\ln(x + 1) + 1$

Sketching a Graph **In Exercises 71 and 72, sketch the graph of the function by hand.**

71. $f(x) = \ln x + 3$ **72.** $f(x) = \ln(x - 1)$

Expanding a Logarithmic Expression **In Exercises 73 and 74, use the properties of logarithms to expand the logarithmic function.**

73. $\ln \sqrt[5]{\dfrac{4x^2 - 1}{4x^2 + 1}}$ **74.** $\ln[(x^2 + 1)(x - 1)]$

Condensing a Logarithmic Expression **In Exercises 75 and 76, write the expression as the logarithm of a single quantity.**

75. $\ln 3 + \frac{1}{3}\ln(4 - x^2) - \ln x$

76. $3[\ln x - 2\ln(x^2 + 1)] + 2\ln 5$

Solving an Equation **In Exercises 77 and 78, solve the equation for x.**

77. $\ln \sqrt{x + 1} = 2$ **78.** $\ln x + \ln(x - 3) = 0$

Finding Inverse Functions **In Exercises 79 and 80, (a) find the inverse function of f, (b) use a graphing utility to graph f and f^{-1} in the same viewing window, and (c) verify that $f^{-1}(f(x)) = x$ and $f(f^{-1}(x)) = x$.**

79. $f(x) = \ln \sqrt{x}$ **80.** $f(x) = e^{1-x}$

Sketching a Graph **In Exercises 81 and 82, sketch the graph of the function by hand.**

81. $y = e^{-x/2}$ **82.** $y = 4e^{-x^2}$

P.S. Problem Solving

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

1. Finding Tangent Lines Consider the circle

$$x^2 + y^2 - 6x - 8y = 0,$$

as shown in the figure.

(a) Find the center and radius of the circle.

(b) Find an equation of the tangent line to the circle at the point $(0, 0)$.

(c) Find an equation of the tangent line to the circle at the point $(6, 0)$.

(d) Where do the two tangent lines intersect?

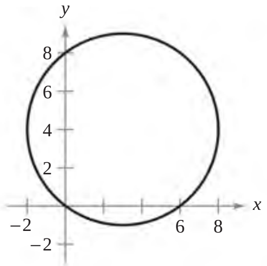


Figure for 1

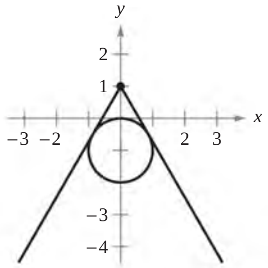


Figure for 2

2. Finding Tangent Lines There are two tangent lines from the point $(0, 1)$ to the circle $x^2 + (y + 1)^2 = 1$ (see figure). Find equations of these two lines by using the fact that each tangent line intersects the circle at *exactly* one point.

3. Heaviside Function The Heaviside function $H(x)$ is widely used in engineering applications.

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

Sketch the graph of the Heaviside function and the graphs of the following functions by hand.

(a) $H(x) - 2$ (b) $H(x - 2)$ (c) $-H(x)$

(d) $H(-x)$ (e) $\frac{1}{2}H(x)$ (f) $-H(x - 2) + 2$

OLIVER HEAVISIDE (1850–1925)

Heaviside was a British mathematician and physicist who contributed to the field of applied mathematics, especially applications of mathematics to electrical engineering. The *Heaviside function* is a classic type of "on-off" function that has applications to electricity and computer science.

Science and Society/SuperStock

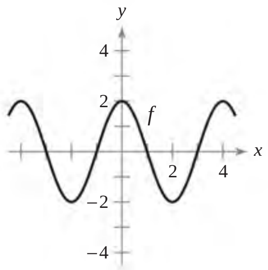
4. Sketching Transformations Consider the graph of the function f shown below. Use this graph to sketch the graphs of the following functions. To print an enlarged copy of the graph, go to *MathGraphs.com.*

(a) $f(x + 1)$ (b) $f(x) + 1$

(c) $2f(x)$ (d) $f(-x)$

(e) $-f(x)$ (f) $|f(x)|$

(g) $f(|x|)$



5. Maximum Area A rancher plans to fence a rectangular pasture adjacent to a river. The rancher has 100 meters of fencing, and no fencing is needed along the river (see figure).

(a) Write the area A of the pasture as a function of x, the length of the side parallel to the river. What is the domain of A?

(b) Graph the area function and estimate the dimensions that yield the maximum amount of area for the pasture.

(c) Find the dimensions that yield the maximum amount of area for the pasture by completing the square.

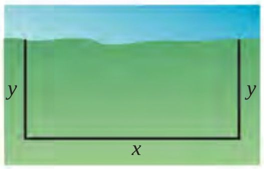


Figure for 5

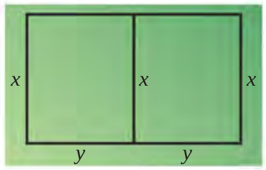


Figure for 6

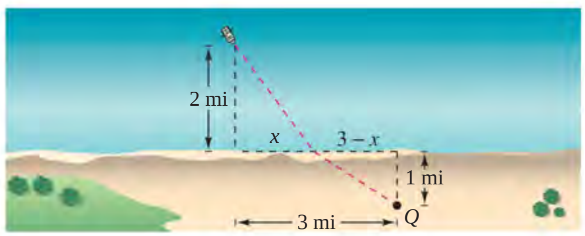
6. Maximum Area A rancher has 300 feet of fencing to enclose two adjacent pastures (see figure).

(a) Write the total area A of the two pastures as a function of x. What is the domain of A?

(b) Graph the area function and estimate the dimensions that yield the maximum amount of area for the pastures.

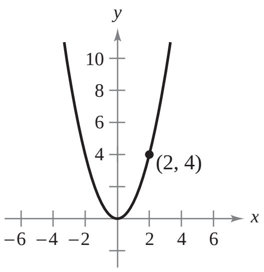
(c) Find the dimensions that yield the maximum amount of area for the pastures by completing the square.

7. Writing a Function You are in a boat 2 miles from the nearest point on the coast. You are to go to a point Q located 3 miles down the coast and 1 mile inland (see figure). You can row at 2 miles per hour and walk at 4 miles per hour. Write the total time T of the trip as a function of x.



8. Analyzing a Function Graph the function $f(x) = e^x - e^{-x}$. From the graph, the function appears to be one-to-one. Assuming that the function has an inverse, find $f^{-1}(x)$.

9. Slope of a Tangent Line One of the fundamental themes of calculus is to find the slope of the tangent line to a curve at a point. To see how this can be done, consider the point $(2, 4)$ on the graph of $f(x) = x^2$ (see figure).



(a) Find the slope of the line joining $(2, 4)$ and $(3, 9)$. Is the slope of the tangent line at $(2, 4)$ greater than or less than this number?

(b) Find the slope of the line joining $(2, 4)$ and $(1, 1)$. Is the slope of the tangent line at $(2, 4)$ greater than or less than this number?

(c) Find the slope of the line joining $(2, 4)$ and $(2.1, 4.41)$. Is the slope of the tangent line at $(2, 4)$ greater than or less than this number?

(d) Find the slope of the line joining $(2, 4)$ and $(2 + h, f(2 + h))$ in terms of the nonzero number h. Verify that $h = 1, -1$, and 0.1 yield the solutions to parts (a)–(c) above.

(e) What is the slope of the tangent line at $(2, 4)$? Explain how you arrived at your answer.

10. Slope of a Tangent Line Sketch the graph of the function $f(x) = \sqrt{x}$ and label the point $(4, 2)$ on the graph.

(a) Find the slope of the line joining $(4, 2)$ and $(9, 3)$. Is the slope of the tangent line at $(4, 2)$ greater than or less than this number?

(b) Find the slope of the line joining $(4, 2)$ and $(1, 1)$. Is the slope of the tangent line at $(4, 2)$ greater than or less than this number?

(c) Find the slope of the line joining $(4, 2)$ and $(4.41, 2.1)$. Is the slope of the tangent line at $(4, 2)$ greater than or less than this number?

(d) Find the slope of the line joining $(4, 2)$ and $(4 + h, f(4 + h))$ in terms of the nonzero number h.

(e) What is the slope of the tangent line at $(4, 2)$? Explain how you arrived at your answer.

11. Composite Functions Let $f(x) = \dfrac{1}{1 - x}$.

(a) What are the domain and range of f?

(b) Find the composition $f(f(x))$. What is the domain of this function?

(c) Find $f(f(f(x)))$. What is the domain of this function?

(d) Graph $f(f(f(x)))$. Is the graph a line? Why or why not?

12. Graphing an Equation Explain how you would graph the equation

$$y + |y| = x + |x|.$$

Then sketch the graph.

13. Sound Intensity A large room contains two speakers that are 3 meters apart. The sound intensity I of one speaker is twice that of the other, as shown in the figure. (To print an enlarged copy of the graph, go to *MathGraphs.com.*) Suppose the listener is free to move about the room to find those positions that receive equal amounts of sound from both speakers. Such a location satisfies two conditions: (1) the sound intensity at the listener's position is directly proportional to the sound level of a source, and (2) the sound intensity is inversely proportional to the square of the distance from the source.

(a) Find the points on the x-axis that receive equal amounts of sound from both speakers.

(b) Find and graph the equation of all locations (x, y) where one could stand and receive equal amounts of sound from both speakers.

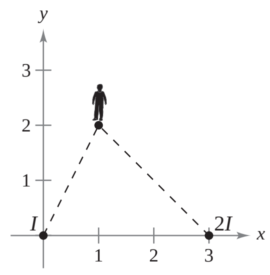


Figure for 13

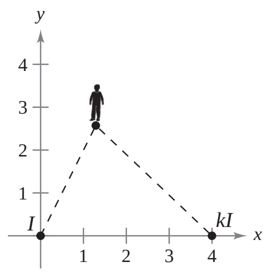


Figure for 14

14. Sound Intensity Suppose the speakers in Exercise 13 are 4 meters apart and the sound intensity of one speaker is k times that of the other, as shown in the figure. To print an enlarged copy of the graph, go to *MathGraphs.com.*

(a) Find the equation of all locations (x, y) where one could stand and receive equal amounts of sound from both speakers.

(b) Graph the equation for the case $k = 3$.

(c) Describe the set of locations of equal sound as k becomes very large.

15. Lemniscate Let d_1 and d_2 be the distances from the point (x, y) to the points $(-1, 0)$ and $(1, 0)$, respectively, as shown in the figure. Show that the equation of the graph of all points (x, y) satisfying $d_1 d_2 = 1$ is

$$(x^2 + y^2)^2 = 2(x^2 - y^2).$$

This curve is called a **lemniscate.** Graph the lemniscate and identify three points on the graph.