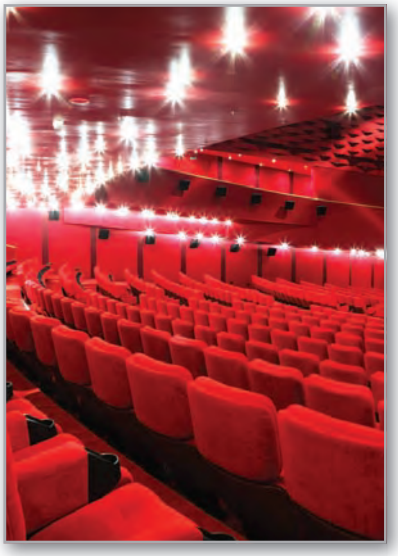
11 Vectors and the Geometry of Space

Geography *(Exercise 45, p. 803)*

Torque *(Exercise 29, p. 781)*

Work *(Exercise 64, p. 774)*



Auditorium Lights
(Exercise 101, p. 765)

Navigation *(Exercise 84, p. 757)*

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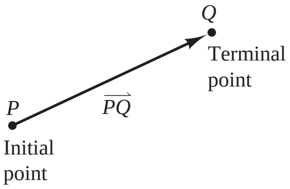
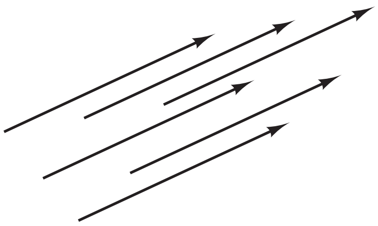
11.1 Vectors in the Plane

- **Write the component form of a vector.**
- **Perform vector operations and interpret the results geometrically.**
- **Write a vector as a linear combination of standard unit vectors.**

Component Form of a Vector



A directed line segment
Figure 11.1


Equivalent directed line segments
Figure 11.2

Many quantities in geometry and physics, such as area, volume, temperature, mass, and time, can be characterized by a single real number that is scaled to appropriate units of measure. These are called **scalar quantities,** and the real number associated with each is called a **scalar.**

Other quantities, such as force, velocity, and acceleration, involve both magnitude and direction and cannot be characterized completely by a single real number. A **directed line segment** is used to represent such a quantity, as shown in Figure 11.1. The directed line segment $\overrightarrow{PQ}$ has **initial point** P and **terminal point** Q, and its **length** (or **magnitude**) is denoted by $\|\overrightarrow{PQ}\|$. Directed line segments that have the same length and direction are **equivalent,** as shown in Figure 11.2. The set of all directed line segments that are equivalent to a given directed line segment $\overrightarrow{PQ}$ is a **vector in the plane** and is denoted by

$$\mathbf{v} = \overrightarrow{PQ}.$$

In typeset material, vectors are usually denoted by lowercase, boldface letters such as **u**, **v**, and **w**. When written by hand, however, vectors are often denoted by letters with arrows above them, such as $\vec{u}$, $\vec{v}$, and $\vec{w}$.

Be sure you understand that a vector represents a *set* of directed line segments (each having the same length and direction). In practice, however, it is common not to distinguish between a vector and one of its representatives.

EXAMPLE 1 **Vector Representation: Directed Line Segments**

Let **v** be represented by the directed line segment from (0, 0) to (3, 2), and let **u** be represented by the directed line segment from (1, 2) to (4, 4). Show that **v** and **u** are equivalent.

Solution Let $P(0, 0)$ and $Q(3, 2)$ be the initial and terminal points of **v**, and let $R(1, 2)$ and $S(4, 4)$ be the initial and terminal points of **u**, as shown in Figure 11.3. You can use the Distance Formula to show that $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the *same length.*

$$\|\overrightarrow{PQ}\| = \sqrt{(3-0)^2 + (2-0)^2} = \sqrt{13}$$
$$\|\overrightarrow{RS}\| = \sqrt{(4-1)^2 + (4-2)^2} = \sqrt{13}$$

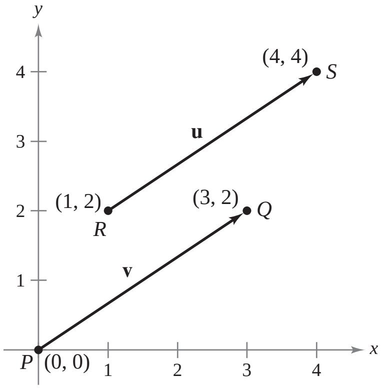
Both line segments have the *same direction,* because they both are directed toward the upper right on lines having the same slope.

$$\text{Slope of } \overrightarrow{PQ} = \frac{2-0}{3-0} = \frac{2}{3}$$

and

$$\text{Slope of } \overrightarrow{RS} = \frac{4-2}{4-1} = \frac{2}{3}$$

Because $\overrightarrow{PQ}$ and $\overrightarrow{RS}$ have the same length and direction, you can conclude that the two vectors are equivalent. That is, **v** and **u** are equivalent.



The vectors **u** and **v** are equivalent.
Figure 11.3

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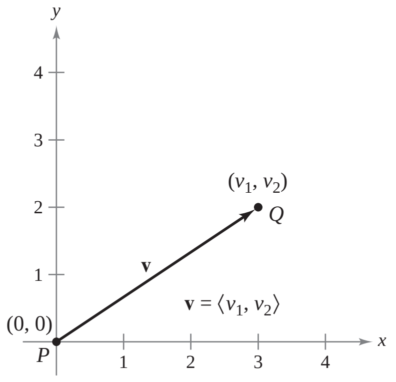


A vector in standard position
Figure 11.4

The directed line segment whose initial point is the origin is often the most convenient representative of a set of equivalent directed line segments such as those shown in Figure 11.3. This representation of **v** is said to be in **standard position.** A directed line segment whose initial point is the origin can be uniquely represented by the coordinates of its terminal point $Q(v_1, v_2)$, as shown in Figure 11.4.

Definition of Component Form of a Vector in the Plane

If **v** is a vector in the plane whose initial point is the origin and whose terminal point is (v_1, v_2), then the **component form of v** is

$$\mathbf{v} = \langle v_1, v_2 \rangle.$$

The coordinates v_1 and v_2 are called the **components of v.** If both the initial point and the terminal point lie at the origin, then **v** is called the **zero vector** and is denoted by $\mathbf{0} = \langle 0, 0 \rangle$.

This definition implies that two vectors $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ are **equal** if and only if $u_1 = v_1$ and $u_2 = v_2$.

The procedures listed below can be used to convert directed line segments to component form or vice versa.

1. If $P(p_1, p_2)$ and $Q(q_1, q_2)$ are the initial and terminal points of a directed line segment, then the component form of the vector **v** represented by $\overrightarrow{PQ}$ is

$$\langle v_1, v_2 \rangle = \langle q_1 - p_1, q_2 - p_2 \rangle.$$

Moreover, from the Distance Formula, you can see that the **length** (or **magnitude**) **of v** is

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{(q_1 - p_1)^2 + (q_2 - p_2)^2}\\ &= \sqrt{v_1^2 + v_2^2}.\end{aligned}$$

Length of a vector

2. If $\mathbf{v} = \langle v_1, v_2 \rangle$, then **v** can be represented by the directed line segment, in standard position, from $P(0, 0)$ to $Q(v_1, v_2)$.

The length of **v** is also called the **norm of v.** If $\|\mathbf{v}\| = 1$, then **v** is a **unit vector.** Moreover, $\|\mathbf{v}\| = 0$ if and only if **v** is the zero vector **0**.

EXAMPLE 2 Component Form and Length of a Vector

Find the component form and length of the vector **v** that has initial point $(3, -7)$ and terminal point $(-2, 5)$.

Solution Let $P(3, -7) = (p_1, p_2)$ and $Q(-2, 5) = (q_1, q_2)$. Then the components of $\mathbf{v} = \langle v_1, v_2 \rangle$ are

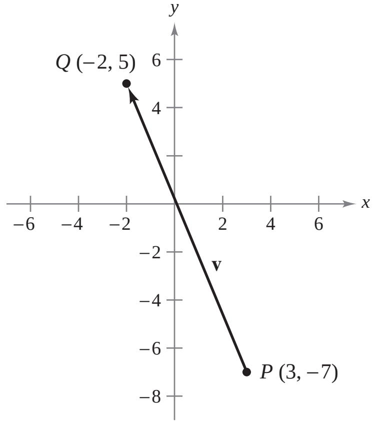
$$v_1 = q_1 - p_1 = -2 - 3 = -5$$

and

$$v_2 = q_2 - p_2 = 5 - (-7) = 12.$$

So, as shown in Figure 11.5, $\mathbf{v} = \langle -5, 12 \rangle$, and the length of **v** is

$$\begin{aligned}\|\mathbf{v}\| &= \sqrt{(-5)^2 + 12^2}\\ &= \sqrt{169}\\ &= 13.\end{aligned}$$



Component form of **v**: $\mathbf{v} = \langle -5, 12 \rangle$
Figure 11.5

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Vector Operations

Definitions of Vector Addition and Scalar Multiplication

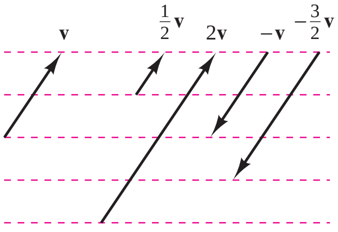
Let $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ be vectors and let c be a scalar.

1. The **vector sum** of $\mathbf{u}$ and $\mathbf{v}$ is the vector $\mathbf{u} + \mathbf{v} = \langle u_1 + v_1, u_2 + v_2 \rangle$.
2. The **scalar multiple** of c and $\mathbf{u}$ is the vector

   $$c\mathbf{u} = \langle cu_1, cu_2 \rangle.$$
3. The **negative** of $\mathbf{v}$ is the vector

   $$-\mathbf{v} = (-1)\mathbf{v} = \langle -v_1, -v_2 \rangle.$$
4. The **difference** of $\mathbf{u}$ and $\mathbf{v}$ is

   $$\mathbf{u} - \mathbf{v} = \mathbf{u} + (-\mathbf{v}) = \langle u_1 - v_1, u_2 - v_2 \rangle.$$



The scalar multiplication of **v**
Figure 11.6

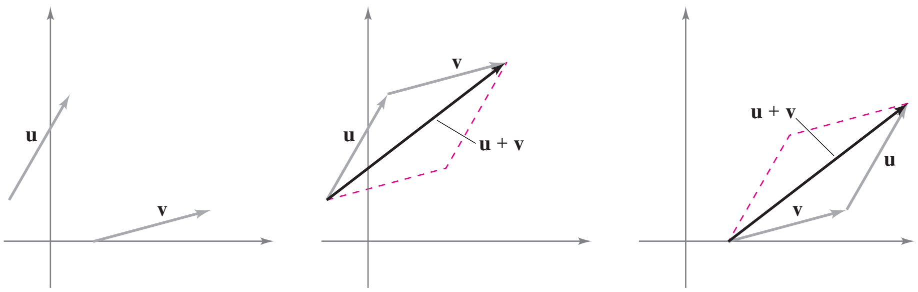
Geometrically, the scalar multiple of a vector $\mathbf{v}$ and a scalar c is the vector that is $|c|$ times as long as $\mathbf{v}$, as shown in Figure 11.6. If c is positive, then $c\mathbf{v}$ has the same direction as $\mathbf{v}$. If c is negative, then $c\mathbf{v}$ has the opposite direction.

The sum of two vectors can be represented geometrically by positioning the vectors (without changing their magnitudes or directions) so that the initial point of one coincides with the terminal point of the other, as shown in Figure 11.7. The vector $\mathbf{u} + \mathbf{v}$, called the **resultant vector,** is the diagonal of a parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as its adjacent sides.

WILLIAM ROWAN HAMILTON (1805–1865)

Some of the earliest work with vectors was done by the Irish mathematician William Rowan Hamilton. Hamilton spent many years developing a system of vector-like quantities called *quaternions*. It wasn't until the latter half of the nineteenth century that the Scottish physicist James Maxwell (1831–1879) restructured Hamilton's quaternions in a form useful for representing physical quantities such as force, velocity, and acceleration.

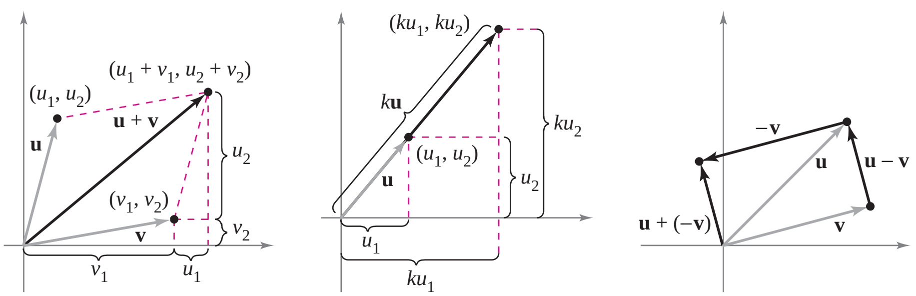
See LarsonCalculus.com to read more of this biography.



To find $\mathbf{u} + \mathbf{v}$, (1) move the initial point of **v** to the terminal point of **u**, or (2) move the initial point of **u** to the terminal point of **v**.

Figure 11.7

Figure 11.8 shows the equivalence of the geometric and algebraic definitions of vector addition and scalar multiplication, and presents (at far right) a geometric interpretation of $\mathbf{u} - \mathbf{v}$.



Vector addition — Scalar multiplication — Vector subtraction
Figure 11.8

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EXAMPLE 3 Vector Operations

For $\mathbf{v} = \langle -2, 5\rangle$ and $\mathbf{w} = \langle 3, 4\rangle$, find each of the vectors.

a. $\frac{1}{2}\mathbf{v}$ **b.** $\mathbf{w} - \mathbf{v}$ **c.** $\mathbf{v} + 2\mathbf{w}$

Solution

a. $\frac{1}{2}\mathbf{v} = \left\langle \frac{1}{2}(-2), \frac{1}{2}(5)\right\rangle = \left\langle -1, \frac{5}{2}\right\rangle$

b. $\mathbf{w} - \mathbf{v} = \langle w_1 - v_1, w_2 - v_2\rangle = \langle 3 - (-2), 4 - 5\rangle = \langle 5, -1\rangle$

c. Using $2\mathbf{w} = \langle 6, 8\rangle$, you have

$$\begin{aligned}\mathbf{v} + 2\mathbf{w} &= \langle -2, 5\rangle + \langle 6, 8\rangle \\ &= \langle -2 + 6, 5 + 8\rangle \\ &= \langle 4, 13\rangle.\end{aligned}$$

Vector addition and scalar multiplication share many properties of ordinary arithmetic, as shown in the next theorem.

EMMY NOETHER (1882–1935)

One person who contributed to our knowledge of axiomatic systems was the German mathematician Emmy Noether. Noether is generally recognized as the leading woman mathematician in recent history.

THEOREM 11.1 Properties of Vector Operations

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in the plane, and let c and d be scalars.

1. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ — Commutative Property
2. $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ — Associative Property
3. $\mathbf{u} + \mathbf{0} = \mathbf{u}$ — Additive Identity Property
4. $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ — Additive Inverse Property
5. $c(d\mathbf{u}) = (cd)\mathbf{u}$
6. $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ — Distributive Property
7. $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ — Distributive Property
8. $1(\mathbf{u}) = \mathbf{u}$, $0(\mathbf{u}) = \mathbf{0}$

Proof The proof of the *Associative Property* of vector addition uses the Associative Property of addition of real numbers.

$$\begin{aligned}(\mathbf{u} + \mathbf{v}) + \mathbf{w} &= [\langle u_1, u_2\rangle + \langle v_1, v_2\rangle] + \langle w_1, w_2\rangle \\ &= \langle u_1 + v_1, u_2 + v_2\rangle + \langle w_1, w_2\rangle \\ &= \langle (u_1 + v_1) + w_1, (u_2 + v_2) + w_2\rangle \\ &= \langle u_1 + (v_1 + w_1), u_2 + (v_2 + w_2)\rangle \\ &= \langle u_1, u_2\rangle + \langle v_1 + w_1, v_2 + w_2\rangle \\ &= \mathbf{u} + (\mathbf{v} + \mathbf{w})\end{aligned}$$

The other properties can be proved in a similar manner.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

FOR FURTHER INFORMATION For more information on Emmy Noether, see the article "Emmy Noether, Greatest Woman Mathematician" by Clark Kimberling in *Mathematics Teacher.* To view this article, go to *MathArticles.com.*

Any set of vectors (with an accompanying set of scalars) that satisfies the eight properties listed in Theorem 11.1 is a **vector space.*** The eight properties are the *vector space axioms.* So, this theorem states that the set of vectors in the plane (with the set of real numbers) forms a vector space.

* For more information about vector spaces, see *Elementary Linear Algebra*, Seventh Edition, by Ron Larson (Boston, Massachusetts: Brooks/Cole, Cengage Learning, 2013).

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THEOREM 11.2 Length of a Scalar Multiple

Let **v** be a vector and let c be a scalar. Then

$$\|c\mathbf{v}\| = |c|\,\|\mathbf{v}\|. \qquad |c| \text{ is the absolute value of } c.$$

Proof Because $c\mathbf{v} = \langle cv_1, cv_2\rangle$, it follows that

$$\begin{aligned}\|c\mathbf{v}\| &= \|\langle cv_1, cv_2\rangle\|\\ &= \sqrt{(cv_1)^2 + (cv_2)^2}\\ &= \sqrt{c^2v_1^2 + c^2v_2^2}\\ &= \sqrt{c^2(v_1^2 + v_2^2)}\\ &= |c|\sqrt{v_1^2 + v_2^2}\\ &= |c|\,\|\mathbf{v}\|.\end{aligned}$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

In many applications of vectors, it is useful to find a unit vector that has the same direction as a given vector. The next theorem gives a procedure for doing this.

THEOREM 11.3 Unit Vector in the Direction of v

If **v** is a nonzero vector in the plane, then the vector

$$\mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{1}{\|\mathbf{v}\|}\mathbf{v}$$

has length 1 and the same direction as **v**.

Proof Because $1/\|\mathbf{v}\|$ is positive and $\mathbf{u} = (1/\|\mathbf{v}\|)\mathbf{v}$, you can conclude that **u** has the same direction as **v**. To see that $\|\mathbf{u}\| = 1$, note that

$$\|\mathbf{u}\| = \left\|\left(\frac{1}{\|\mathbf{v}\|}\right)\mathbf{v}\right\| = \left|\frac{1}{\|\mathbf{v}\|}\right|\,\|\mathbf{v}\| = \frac{1}{\|\mathbf{v}\|}\|\mathbf{v}\| = 1.$$

So, **u** has length 1 and the same direction as **v**.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

In Theorem 11.3, **u** is called a **unit vector in the direction of v.** The process of multiplying **v** by $1/\|\mathbf{v}\|$ to get a unit vector is called **normalization of v.**

EXAMPLE 4 Finding a Unit Vector

Find a unit vector in the direction of $\mathbf{v} = \langle -2, 5\rangle$ and verify that it has length 1.

Solution From Theorem 11.3, the unit vector in the direction of **v** is

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{\langle -2, 5\rangle}{\sqrt{(-2)^2 + (5)^2}} = \frac{1}{\sqrt{29}}\langle -2, 5\rangle = \left\langle \frac{-2}{\sqrt{29}}, \frac{5}{\sqrt{29}}\right\rangle.$$

This vector has length 1, because

$$\sqrt{\left(\frac{-2}{\sqrt{29}}\right)^2 + \left(\frac{5}{\sqrt{29}}\right)^2} = \sqrt{\frac{4}{29} + \frac{25}{29}} = \sqrt{\frac{29}{29}} = 1.$$

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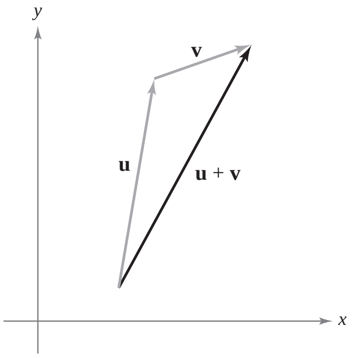


Triangle inequality
Figure 11.9

Generally, the length of the sum of two vectors is not equal to the sum of their lengths. To see this, consider the vectors **u** and **v** as shown in Figure 11.9. With **u** and **v** as two sides of a triangle, the length of the third side is $\|\mathbf{u} + \mathbf{v}\|$, and

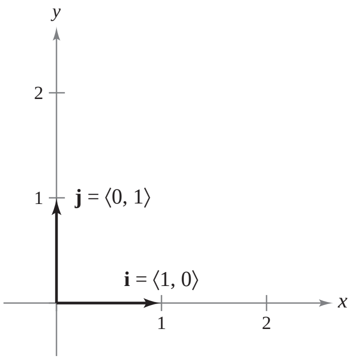
$$\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|.$$

Equality occurs only when the vectors **u** and **v** have the *same direction*. This result is called the **triangle inequality** for vectors. (You are asked to prove this in Exercise 77, Section 11.3.)

Standard Unit Vectors

The unit vectors $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$ are called the **standard unit vectors** in the plane and are denoted by

$$\mathbf{i} = \langle 1, 0\rangle \quad \text{and} \quad \mathbf{j} = \langle 0, 1\rangle$$ Standard unit vectors



Standard unit vectors **i** and **j**
Figure 11.10

as shown in Figure 11.10. These vectors can be used to represent any vector uniquely, as follows.

$$\mathbf{v} = \langle v_1, v_2\rangle = \langle v_1, 0\rangle + \langle 0, v_2\rangle = v_1\langle 1, 0\rangle + v_2\langle 0, 1\rangle = v_1\mathbf{i} + v_2\mathbf{j}$$

The vector $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$ is called a **linear combination** of **i** and **j**. The scalars v_1 and v_2 are called the **horizontal** and **vertical components of v.**

EXAMPLE 5 Writing a Linear Combination of Unit Vectors

Let **u** be the vector with initial point $(2, -5)$ and terminal point $(-1, 3)$, and let $\mathbf{v} = 2\mathbf{i} - \mathbf{j}$. Write each vector as a linear combination of **i** and **j**.

a. **u** **b.** $\mathbf{w} = 2\mathbf{u} - 3\mathbf{v}$

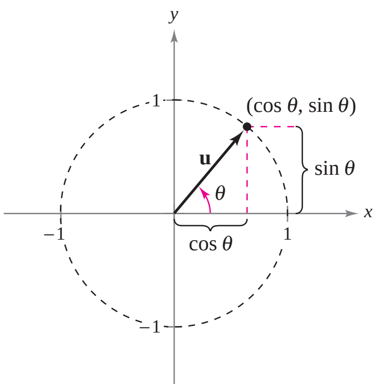
Solution

a. $\mathbf{u} = \langle q_1 - p_1, q_2 - p_2\rangle = \langle -1 - 2, 3 - (-5)\rangle = \langle -3, 8\rangle = -3\mathbf{i} + 8\mathbf{j}$

b. $\mathbf{w} = 2\mathbf{u} - 3\mathbf{v} = 2(-3\mathbf{i} + 8\mathbf{j}) - 3(2\mathbf{i} - \mathbf{j}) = -6\mathbf{i} + 16\mathbf{j} - 6\mathbf{i} + 3\mathbf{j} = -12\mathbf{i} + 19\mathbf{j}$

If **u** is a unit vector and θ is the angle (measured counterclockwise) from the positive x-axis to **u**, then the terminal point of **u** lies on the unit circle, and you have

$$\mathbf{u} = \langle \cos\theta, \sin\theta\rangle = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$$ Unit vector



The angle θ from the positive x-axis to the vector **u**
Figure 11.11

as shown in Figure 11.11. Moreover, it follows that any other nonzero vector **v** making an angle θ with the positive x-axis has the same direction as **u**, and you can write

$$\mathbf{v} = \|\mathbf{v}\|\langle \cos\theta, \sin\theta\rangle = \|\mathbf{v}\|\cos\theta\mathbf{i} + \|\mathbf{v}\|\sin\theta\mathbf{j}.$$

EXAMPLE 6 Writing a Vector of Given Magnitude and Direction

The vector **v** has a magnitude of 3 and makes an angle of $30° = \pi/6$ with the positive x-axis. Write **v** as a linear combination of the unit vectors **i** and **j**.

Solution Because the angle between **v** and the positive x-axis is $\theta = \pi/6$, you can write

$$\mathbf{v} = \|\mathbf{v}\|\cos\theta\mathbf{i} + \|\mathbf{v}\|\sin\theta\mathbf{j} = 3\cos\frac{\pi}{6}\mathbf{i} + 3\sin\frac{\pi}{6}\mathbf{j} = \frac{3\sqrt{3}}{2}\mathbf{i} + \frac{3}{2}\mathbf{j}.$$

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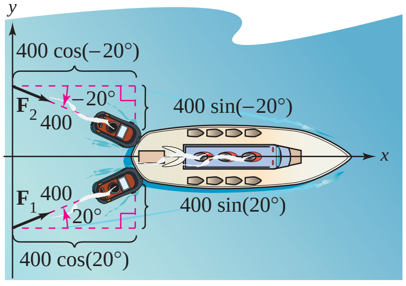
Vectors have many applications in physics and engineering. One example is force. A vector can be used to represent force, because force has both magnitude and direction. If two or more forces are acting on an object, then the **resultant force** on the object is the vector sum of the vector forces.

EXAMPLE 7 Finding the Resultant Force

Two tugboats are pushing an ocean liner, as shown in Figure 11.12. Each boat is exerting a force of 400 pounds. What is the resultant force on the ocean liner?



The resultant force on the ocean liner that is exerted by the two tugboats
Figure 11.12

Solution Using Figure 11.12, you can represent the forces exerted by the first and second tugboats as

$$\mathbf{F}_1 = 400\langle \cos 20°, \sin 20° \rangle = 400\cos(20°)\mathbf{i} + 400\sin(20°)\mathbf{j}$$
$$\mathbf{F}_2 = 400\langle \cos(-20°), \sin(-20°) \rangle = 400\cos(20°)\mathbf{i} - 400\sin(20°)\mathbf{j}.$$

The resultant force on the ocean liner is

$$\begin{aligned}\mathbf{F} &= \mathbf{F}_1 + \mathbf{F}_2\\ &= [400\cos(20°)\mathbf{i} + 400\sin(20°)\mathbf{j}] + [400\cos(20°)\mathbf{i} - 400\sin(20°)\mathbf{j}]\\ &= 800\cos(20°)\mathbf{i}\\ &\approx 752\mathbf{i}.\end{aligned}$$

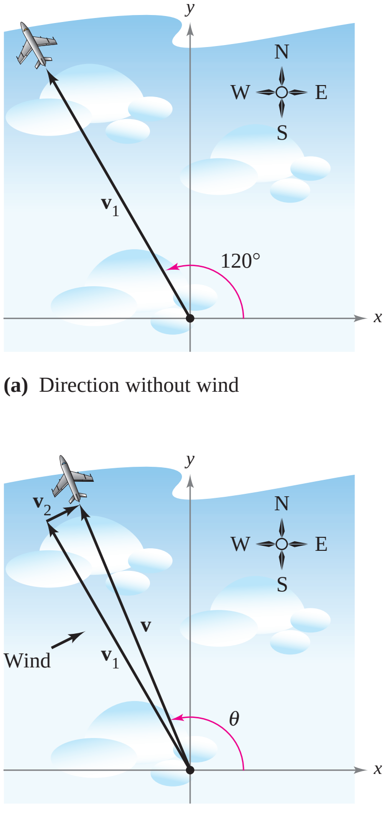
So, the resultant force on the ocean liner is approximately 752 pounds in the direction of the positive *x*-axis.

In surveying and navigation, a **bearing** is a direction that measures the acute angle that a path or line of sight makes with a fixed north-south line. In air navigation, bearings are measured in degrees clockwise from north.

EXAMPLE 8 Finding a Velocity

See LarsonCalculus.com for an interactive version of this type of example.

An airplane is traveling at a fixed altitude with a negligible wind factor. The airplane is traveling at a speed of 500 miles per hour with a bearing of 330°, as shown in Figure 11.13(a). As the airplane reaches a certain point, it encounters wind with a velocity of 70 miles per hour in the direction N 45° E (45° east of north), as shown in Figure 11.13(b). What are the resultant speed and direction of the airplane?



(a) Direction without wind

(b) Direction with wind
Figure 11.13

Solution Using Figure 11.13(a), represent the velocity of the airplane (alone) as

$$\mathbf{v}_1 = 500\cos(120°)\mathbf{i} + 500\sin(120°)\mathbf{j}.$$

The velocity of the wind is represented by the vector

$$\mathbf{v}_2 = 70\cos(45°)\mathbf{i} + 70\sin(45°)\mathbf{j}.$$

The resultant velocity of the airplane (in the wind) is

$$\begin{aligned}\mathbf{v} &= \mathbf{v}_1 + \mathbf{v}_2\\ &= 500\cos(120°)\mathbf{i} + 500\sin(120°)\mathbf{j} + 70\cos(45°)\mathbf{i} + 70\sin(45°)\mathbf{j}\\ &\approx -200.5\mathbf{i} + 482.5\mathbf{j}.\end{aligned}$$

To find the resultant speed and direction, write $\mathbf{v} = \|\mathbf{v}\|(\cos\theta\,\mathbf{i} + \sin\theta\,\mathbf{j})$. Because $\|\mathbf{v}\| \approx \sqrt{(-200.5)^2 + (482.5)^2} \approx 522.5$, you can write

$$\mathbf{v} \approx 522.5\left(\frac{-200.5}{522.5}\mathbf{i} + \frac{482.5}{522.5}\mathbf{j}\right) \approx 522.5[\cos(112.6°)\mathbf{i} + \sin(112.6°)\mathbf{j}].$$

The new speed of the airplane, as altered by the wind, is approximately 522.5 miles per hour in a path that makes an angle of 112.6° with the positive *x*-axis.

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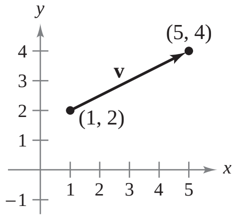
11.1 Exercises

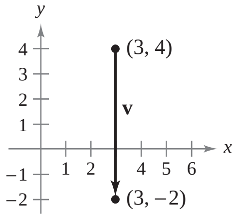
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Sketching a Vector In Exercises 1–4, (a) find the component form of the vector **v** and (b) sketch the vector with its initial point at the origin.

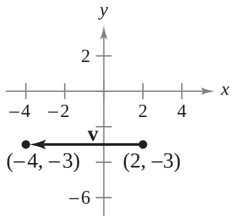
1.



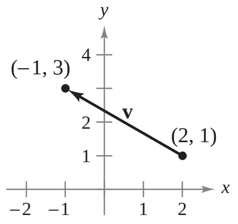
2.



3.



4.



Equivalent Vectors In Exercises 5–8, find the vectors **u** and **v** whose initial and terminal points are given. Show that **u** and **v** are equivalent.

5. **u**: $(3, 2)$, $(5, 6)$
v: $(1, 4)$, $(3, 8)$

6. **u**: $(-4, 0)$, $(1, 8)$
v: $(2, -1)$, $(7, 7)$

7. **u**: $(0, 3)$, $(6, -2)$
v: $(3, 10)$, $(9, 5)$

8. **u**: $(-4, -1)$, $(11, -4)$
v: $(10, 13)$, $(25, 10)$

Writing a Vector in Different Forms In Exercises 9–16, the initial and terminal points of a vector **v** are given. (a) Sketch the given directed line segment, (b) write the vector in component form, (c) write the vector as the linear combination of the standard unit vectors **i** and **j**, and (d) sketch the vector with its initial point at the origin.

	Initial Point	Terminal Point		Initial Point	Terminal Point
9.	$(2, 0)$	$(5, 5)$	**10.**	$(4, -6)$	$(3, 6)$
11.	$(8, 3)$	$(6, -1)$	**12.**	$(0, -4)$	$(-5, -1)$
13.	$(6, 2)$	$(6, 6)$	**14.**	$(7, -1)$	$(-3, -1)$
15.	$\left(\frac{3}{2}, \frac{4}{3}\right)$	$\left(\frac{1}{2}, 3\right)$	**16.**	$(0.12, 0.60)$	$(0.84, 1.25)$

Sketching Scalar Multiples In Exercises 17 and 18, sketch each scalar multiple of **v**.

17. $\mathbf{v} = \langle 3, 5 \rangle$

(a) $2\mathbf{v}$ (b) $-3\mathbf{v}$ (c) $\frac{7}{2}\mathbf{v}$ (d) $\frac{2}{3}\mathbf{v}$

18. $\mathbf{v} = \langle -2, 3 \rangle$

(a) $4\mathbf{v}$ (b) $-\frac{1}{2}\mathbf{v}$ (c) $0\mathbf{v}$ (d) $-6\mathbf{v}$

Using Vector Operations In Exercises 19 and 20, find (a) $\frac{2}{3}\mathbf{u}$, (b) $3\mathbf{v}$, (c) $\mathbf{v} - \mathbf{u}$, and (d) $2\mathbf{u} + 5\mathbf{v}$.

19. $\mathbf{u} = \langle 4, 9 \rangle, \mathbf{v} = \langle 2, -5 \rangle$

20. $\mathbf{u} = \langle -3, -8 \rangle, \mathbf{v} = \langle 8, 25 \rangle$

Sketching a Vector In Exercises 21–26, use the figure to sketch a graph of the vector. To print an enlarged copy of the graph, go to *MathGraphs.com*.

21. $-\mathbf{u}$

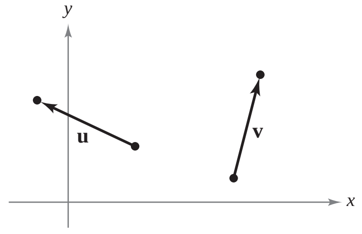
22. $2\mathbf{u}$

23. $-\mathbf{v}$

24. $\frac{1}{2}\mathbf{v}$

25. $\mathbf{u} - \mathbf{v}$

26. $\mathbf{u} + 2\mathbf{v}$



Finding a Terminal Point In Exercises 27 and 28, the vector **v** and its initial point are given. Find the terminal point.

27. $\mathbf{v} = \langle -1, 3 \rangle$; Initial point: $(4, 2)$

28. $\mathbf{v} = \langle 4, -9 \rangle$; Initial point: $(5, 3)$

Finding a Magnitude of a Vector In Exercises 29–34, find the magnitude of **v**.

29. $\mathbf{v} = 7\mathbf{i}$

30. $\mathbf{v} = -3\mathbf{i}$

31. $\mathbf{v} = \langle 4, 3 \rangle$

32. $\mathbf{v} = \langle 12, -5 \rangle$

33. $\mathbf{v} = 6\mathbf{i} - 5\mathbf{j}$

34. $\mathbf{v} = -10\mathbf{i} + 3\mathbf{j}$

Finding a Unit Vector In Exercises 35–38, find the unit vector in the direction of **v** and verify that it has length 1.

35. $\mathbf{v} = \langle 3, 12 \rangle$

36. $\mathbf{v} = \langle -5, 15 \rangle$

37. $\mathbf{v} = \left\langle \frac{3}{2}, \frac{5}{2} \right\rangle$

38. $\mathbf{v} = \langle -6.2, 3.4 \rangle$

Finding Magnitudes In Exercises 39–42, find the following.

(a) $\|\mathbf{u}\|$ (b) $\|\mathbf{v}\|$ (c) $\|\mathbf{u} + \mathbf{v}\|$

(d) $\left\| \dfrac{\mathbf{u}}{\|\mathbf{u}\|} \right\|$ (e) $\left\| \dfrac{\mathbf{v}}{\|\mathbf{v}\|} \right\|$ (f) $\left\| \dfrac{\mathbf{u} + \mathbf{v}}{\|\mathbf{u} + \mathbf{v}\|} \right\|$

39. $\mathbf{u} = \langle 1, -1 \rangle, \mathbf{v} = \langle -1, 2 \rangle$

40. $\mathbf{u} = \langle 0, 1 \rangle, \mathbf{v} = \langle 3, -3 \rangle$

41. $\mathbf{u} = \left\langle 1, \frac{1}{2} \right\rangle, \mathbf{v} = \langle 2, 3 \rangle$

42. $\mathbf{u} = \langle 2, -4 \rangle, \mathbf{v} = \langle 5, 5 \rangle$

Using the Triangle Inequality In Exercises 43 and 44, sketch a graph of **u**, **v**, and **u** + **v**. Then demonstrate the triangle inequality using the vectors **u** and **v**.

43. $\mathbf{u} = \langle 2, 1 \rangle, \mathbf{v} = \langle 5, 4 \rangle$

44. $\mathbf{u} = \langle -3, 2 \rangle, \mathbf{v} = \langle 1, -2 \rangle$

Finding a Vector In Exercises 45–48, find the vector **v** with the given magnitude and the same direction as **u**.

	Magnitude	Direction
45.	$\|\mathbf{v}\| = 6$	$\mathbf{u} = \langle 0, 3 \rangle$
46.	$\|\mathbf{v}\| = 4$	$\mathbf{u} = \langle 1, 1 \rangle$
47.	$\|\mathbf{v}\| = 5$	$\mathbf{u} = \langle -1, 2 \rangle$
48.	$\|\mathbf{v}\| = 2$	$\mathbf{u} = \left\langle \sqrt{3}, 3 \right\rangle$

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Finding a Vector In Exercises 49–52, find the component form of **v** given its magnitude and the angle it makes with the positive x-axis.

49. $\|\mathbf{v}\| = 3, \quad \theta = 0°$

50. $\|\mathbf{v}\| = 5, \quad \theta = 120°$

51. $\|\mathbf{v}\| = 2, \quad \theta = 150°$

52. $\|\mathbf{v}\| = 4, \quad \theta = 3.5°$

Finding a Vector In Exercises 53–56, find the component form of **u** + **v** given the lengths of **u** and **v** and the angles that **u** and **v** make with the positive x-axis.

53. $\|\mathbf{u}\| = 1, \quad \theta_{\mathbf{u}} = 0°$
$\|\mathbf{v}\| = 3, \quad \theta_{\mathbf{v}} = 45°$

54. $\|\mathbf{u}\| = 4, \quad \theta_{\mathbf{u}} = 0°$
$\|\mathbf{v}\| = 2, \quad \theta_{\mathbf{v}} = 60°$

55. $\|\mathbf{u}\| = 2, \quad \theta_{\mathbf{u}} = 4$
$\|\mathbf{v}\| = 1, \quad \theta_{\mathbf{v}} = 2$

56. $\|\mathbf{u}\| = 5, \quad \theta_{\mathbf{u}} = -0.5$
$\|\mathbf{v}\| = 5, \quad \theta_{\mathbf{v}} = 0.5$

WRITING ABOUT CONCEPTS

57. Scalar and Vector In your own words, state the difference between a scalar and a vector. Give examples of each.

58. Scalar or Vector Identify the quantity as a scalar or as a vector. Explain your reasoning.

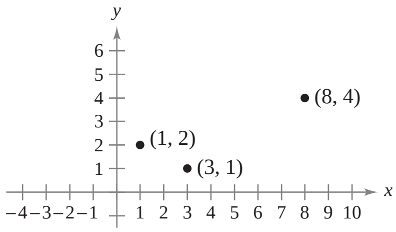
(a) The muzzle velocity of a gun

(b) The price of a company's stock

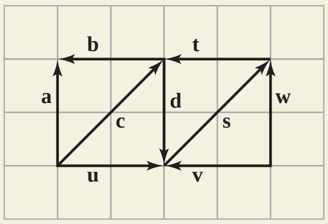
(c) The air temperature in a room

(d) The weight of a car

59. Using a Parallelogram Three vertices of a parallelogram are (1, 2), (3, 1), and (8, 4). Find the three possible fourth vertices (see figure).



60. HOW DO YOU SEE IT? Use the figure to determine whether each statement is true or false. Justify your answer.



(a) $\mathbf{a} = -\mathbf{d}$

(b) $\mathbf{c} = \mathbf{s}$

(c) $\mathbf{a} + \mathbf{u} = \mathbf{c}$

(d) $\mathbf{v} + \mathbf{w} = -\mathbf{s}$

(e) $\mathbf{a} + \mathbf{d} = \mathbf{0}$

(f) $\mathbf{u} - \mathbf{v} = -2(\mathbf{b} + \mathbf{t})$

Finding Values In Exercises 61–66, find a and b such that $\mathbf{v} = a\mathbf{u} + b\mathbf{w}$, where $\mathbf{u} = \langle 1, 2\rangle$ and $\mathbf{w} = \langle 1, -1\rangle$.

61. $\mathbf{v} = \langle 2, 1\rangle$

62. $\mathbf{v} = \langle 0, 3\rangle$

63. $\mathbf{v} = \langle 3, 0\rangle$

64. $\mathbf{v} = \langle 3, 3\rangle$

65. $\mathbf{v} = \langle 1, 1\rangle$

66. $\mathbf{v} = \langle -1, 7\rangle$

Finding Unit Vectors In Exercises 67–72, find a unit vector (a) parallel to and (b) perpendicular to the graph of f at the given point. Then sketch the graph of f and sketch the vectors at the given point.

67. $f(x) = x^2, \quad (3, 9)$

68. $f(x) = -x^2 + 5, \quad (1, 4)$

69. $f(x) = x^3, \quad (1, 1)$

70. $f(x) = x^3, \quad (-2, -8)$

71. $f(x) = \sqrt{25 - x^2}, \quad (3, 4)$

72. $f(x) = \tan x, \quad \left(\frac{\pi}{4}, 1\right)$

Finding a Vector In Exercises 73 and 74, find the component form of **v** given the magnitudes of **u** and **u** + **v** and the angles that **u** and **u** + **v** make with the positive x-axis.

73. $\|\mathbf{u}\| = 1, \theta = 45°$
$\|\mathbf{u} + \mathbf{v}\| = \sqrt{2}, \theta = 90°$

74. $\|\mathbf{u}\| = 4, \theta = 30°$
$\|\mathbf{u} + \mathbf{v}\| = 6, \theta = 120°$

75. Resultant Force Forces with magnitudes of 500 pounds and 200 pounds act on a machine part at angles of 30° and −45°, respectively, with the x-axis (see figure). Find the direction and magnitude of the resultant force.

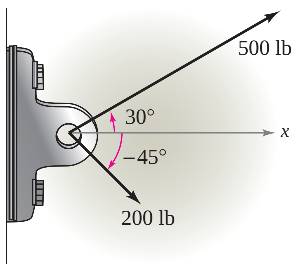


Figure for 75

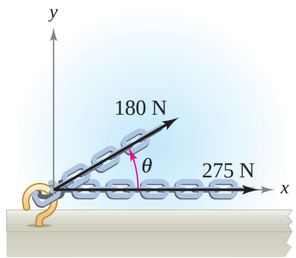


Figure for 76

76. Numerical and Graphical Analysis Forces with magnitudes of 180 newtons and 275 newtons act on a hook (see figure). The angle between the two forces is θ degrees.

(a) When $\theta = 30°$, find the direction and magnitude of the resultant force.

(b) Write the magnitude M and direction α of the resultant force as functions of θ, where $0° \le \theta \le 180°$.

(c) Use a graphing utility to complete the table.

θ	0°	30°	60°	90°	120°	150°	180°
M							
α							

(d) Use a graphing utility to graph the two functions M and α.

(e) Explain why one of the functions decreases for increasing values of θ, whereas the other does not.

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77. Resultant Force Three forces with magnitudes of 75 pounds, 100 pounds, and 125 pounds act on an object at angles of 30°, 45°, and 120°, respectively, with the positive x-axis. Find the direction and magnitude of the resultant force.

78. Resultant Force Three forces with magnitudes of 400 newtons, 280 newtons, and 350 newtons act on an object at angles of $-30°$, 45°, and 135°, respectively, with the positive x-axis. Find the direction and magnitude of the resultant force.

79. Think About It Consider two forces of equal magnitude acting on a point.

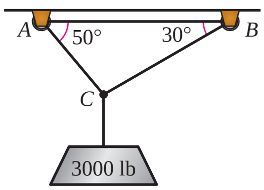
(a) When the magnitude of the resultant is the sum of the magnitudes of the two forces, make a conjecture about the angle between the forces.

(b) When the resultant of the forces is 0, make a conjecture about the angle between the forces.

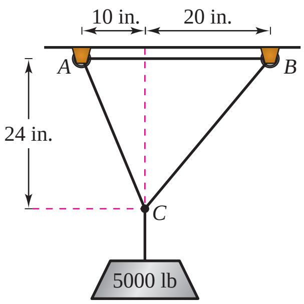
(c) Can the magnitude of the resultant be greater than the sum of the magnitudes of the two forces? Explain.

80. Cable Tension Determine the tension in each cable supporting the given load for each figure.

(a)



(b)



81. Projectile Motion A gun with a muzzle velocity of 1200 feet per second is fired at an angle of 6° above the horizontal. Find the vertical and horizontal components of the velocity.

82. Shared Load To carry a 100-pound cylindrical weight, two workers lift on the ends of short ropes tied to an eyelet on the top center of the cylinder. One rope makes a 20° angle away from the vertical and the other makes a 30° angle (see figure).

(a) Find each rope's tension when the resultant force is vertical.

(b) Find the vertical component of each worker's force.

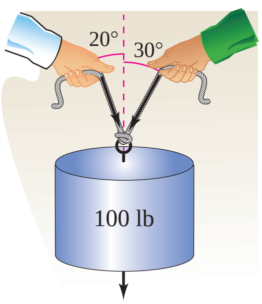


Figure for 82

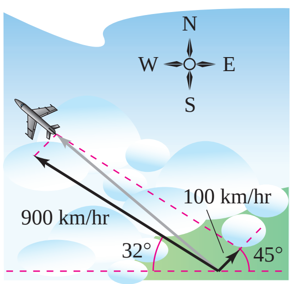


Figure for 83

Mikael Damkier/Shutterstock.com

83. Navigation A plane is flying with a bearing of 302°. Its speed with respect to the air is 900 kilometers per hour. The wind at the plane's altitude is from the southwest at 100 kilometers per hour (see figure). What is the true direction of the plane, and what is its speed with respect to the ground?

84. Navigation

A plane flies at a constant groundspeed of 400 miles per hour due east and encounters a 50-mile-per-hour wind from the northwest. Find the airspeed and compass direction that will allow the plane to maintain its groundspeed and eastward direction.

True or False? **In Exercises 85–90, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

85. If **u** and **v** have the same magnitude and direction, then **u** and **v** are equivalent.

86. If **u** is a unit vector in the direction of **v**, then $\mathbf{v} = \|\mathbf{v}\|\,\mathbf{u}$.

87. If $\mathbf{u} = a\mathbf{i} + b\mathbf{j}$ is a unit vector, then $a^2 + b^2 = 1$.

88. If $\mathbf{v} = a\mathbf{i} + b\mathbf{j} = \mathbf{0}$, then $a = -b$.

89. If $a = b$, then $\|a\mathbf{i} + b\mathbf{j}\| = \sqrt{2}a$.

90. If **u** and **v** have the same magnitude but opposite directions, then $\mathbf{u} + \mathbf{v} = \mathbf{0}$.

91. Proof Prove that

$$\mathbf{u} = (\cos\theta)\mathbf{i} - (\sin\theta)\mathbf{j} \quad \text{and} \quad \mathbf{v} = (\sin\theta)\mathbf{i} + (\cos\theta)\mathbf{j}$$

are unit vectors for any angle θ.

92. Geometry Using vectors, prove that the line segment joining the midpoints of two sides of a triangle is parallel to, and one-half the length of, the third side.

93. Geometry Using vectors, prove that the diagonals of a parallelogram bisect each other.

94. Proof Prove that the vector $\mathbf{w} = \|\mathbf{u}\|\mathbf{v} + \|\mathbf{v}\|\mathbf{u}$ bisects the angle between **u** and **v**.

95. Using a Vector Consider the vector $\mathbf{u} = \langle x, y\rangle$. Describe the set of all points (x, y) such that $\|\mathbf{u}\| = 5$.

PUTNAM EXAM CHALLENGE

96. A coast artillery gun can fire at any angle of elevation between 0° and 90° in a fixed vertical plane. If air resistance is neglected and the muzzle velocity is constant $(= v_0)$, determine the set H of points in the plane and above the horizontal which can be hit.

This problem was composed by the Committee on the Putnam Prize Competition.
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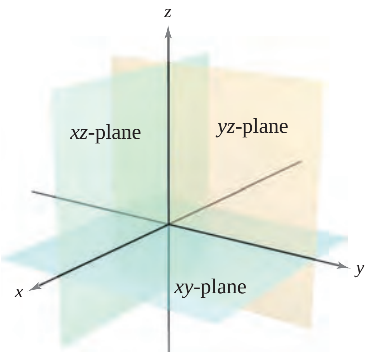
11.2 Space Coordinates and Vectors in Space

- **Understand the three-dimensional rectangular coordinate system.**
- **Analyze vectors in space.**

Coordinates in Space

Up to this point in the text, you have been primarily concerned with the two-dimensional coordinate system. Much of the remaining part of your study of calculus will involve the three-dimensional coordinate system.

Before extending the concept of a vector to three dimensions, you must be able to identify points in the **three-dimensional coordinate system.** You can construct this system by passing a z-axis perpendicular to both the x- and y-axes at the origin, as shown in Figure 11.14. Taken as pairs, the axes determine three **coordinate planes:** the ***xy*-plane,** the ***xz*-plane,** and the ***yz*-plane.** These three coordinate planes separate three-space into eight **octants.** The first octant is the one for which all three coordinates are positive. In this three-dimensional system, a point P in space is determined by an ordered triple (x, y, z), where x, y, and z are as follows.



The three-dimensional coordinate system
Figure 11.14

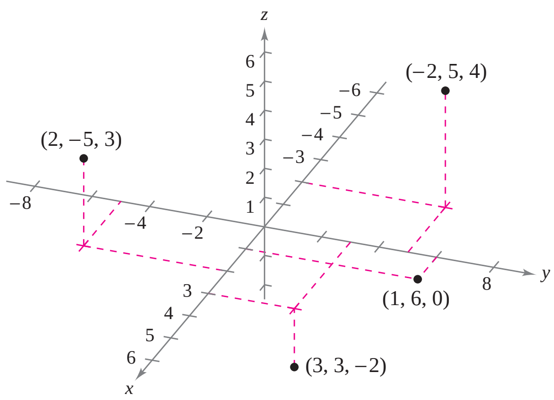
$x =$ directed distance from yz-plane to P

$y =$ directed distance from xz-plane to P

$z =$ directed distance from xy-plane to P

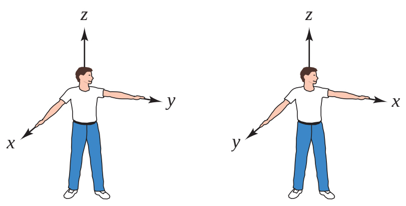
Several points are shown in Figure 11.15.

REMARK The three-dimensional rotatable graphs that are available at *LarsonCalculus.com* can help you visualize points or objects in a three-dimensional coordinate system.



Points in the three-dimensional coordinate system are represented by ordered triples.
Figure 11.15

A three-dimensional coordinate system can have either a **right-handed** or a **left-handed** orientation. To determine the orientation of a system, imagine that you are standing at the origin, with your arms pointing in the direction of the positive x- and y-axes, and with the z-axis pointing up, as shown in Figure 11.16. The system is right-handed or left-handed depending on which hand points along the x-axis. In this text, you will work exclusively with the right-handed system.



Right-handed system
Left-handed system
Figure 11.16

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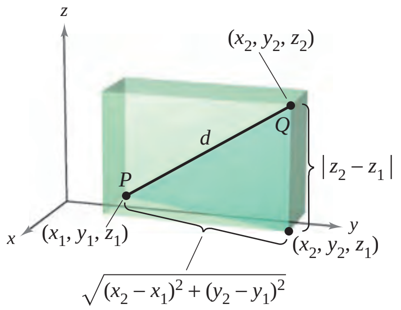

The distance between two points in space
Figure 11.17

Many of the formulas established for the two-dimensional coordinate system can be extended to three dimensions. For example, to find the distance between two points in space, you can use the Pythagorean Theorem twice, as shown in Figure 11.17. By doing this, you will obtain the formula for the distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$ Distance Formula

EXAMPLE 1 Finding the Distance Between Two Points in Space

Find the distance between the points $(2, -1, 3)$ and $(1, 0, -2)$.

Solution

$$\begin{aligned} d &= \sqrt{(1-2)^2 + (0+1)^2 + (-2-3)^2} && \text{Distance Formula} \\ &= \sqrt{1 + 1 + 25} \\ &= \sqrt{27} \\ &= 3\sqrt{3} \end{aligned}$$

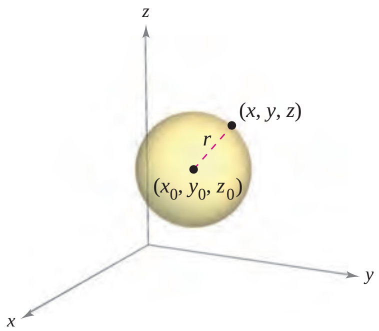

Figure 11.18

A **sphere** with center at (x_0, y_0, z_0) and radius r is defined to be the set of all points (x, y, z) such that the distance between (x, y, z) and (x_0, y_0, z_0) is r. You can use the Distance Formula to find the **standard equation of a sphere** of radius r, centered at (x_0, y_0, z_0). If (x, y, z) is an arbitrary point on the sphere, then the equation of the sphere is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$ Equation of sphere

as shown in Figure 11.18. Moreover, the midpoint of the line segment joining the points (x_1, y_1, z_1) and (x_2, y_2, z_2) has coordinates

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right).$$ Midpoint Formula

EXAMPLE 2 Finding the Equation of a Sphere

Find the standard equation of the sphere that has the points

$(5, -2, 3)$ and $(0, 4, -3)$

as endpoints of a diameter.

Solution Using the Midpoint Formula, the center of the sphere is

$$\left(\frac{5+0}{2}, \frac{-2+4}{2}, \frac{3-3}{2}\right) = \left(\frac{5}{2}, 1, 0\right).$$ Midpoint Formula

By the Distance Formula, the radius is

$$r = \sqrt{\left(0 - \frac{5}{2}\right)^2 + (4-1)^2 + (-3-0)^2} = \sqrt{\frac{97}{4}} = \frac{\sqrt{97}}{2}.$$

Therefore, the standard equation of the sphere is

$$\left(x - \frac{5}{2}\right)^2 + (y-1)^2 + z^2 = \frac{97}{4}.$$ Equation of sphere

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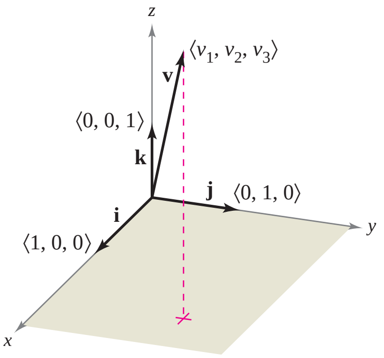

The standard unit vectors in space
Figure 11.19

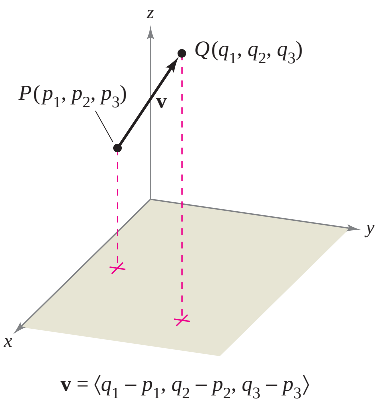

Figure 11.20

Vectors in Space

In space, vectors are denoted by ordered triples $\mathbf{v} = \langle v_1, v_2, v_3\rangle$. The **zero vector** is denoted by $\mathbf{0} = \langle 0, 0, 0\rangle$. Using the unit vectors

$$\mathbf{i} = \langle 1, 0, 0\rangle, \quad \mathbf{j} = \langle 0, 1, 0\rangle, \quad \text{and} \quad \mathbf{k} = \langle 0, 0, 1\rangle$$

the **standard unit vector notation** for **v** is

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

as shown in Figure 11.19. If **v** is represented by the directed line segment from $P(p_1, p_2, p_3)$ to $Q(q_1, q_2, q_3)$, as shown in Figure 11.20, then the component form of **v** is written by subtracting the coordinates of the initial point from the coordinates of the terminal point, as follows.

$$\mathbf{v} = \langle v_1, v_2, v_3\rangle = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3\rangle$$

Vectors in Space

Let $\mathbf{u} = \langle u_1, u_2, u_3\rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ be vectors in space and let c be a scalar.

1. *Equality of Vectors:* $\mathbf{u} = \mathbf{v}$ if and only if $u_1 = v_1$, $u_2 = v_2$, and $u_3 = v_3$.
2. *Component Form:* If **v** is represented by the directed line segment from $P(p_1, p_2, p_3)$ to $Q(q_1, q_2, q_3)$, then
$$\mathbf{v} = \langle v_1, v_2, v_3\rangle = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3\rangle.$$
3. *Length:* $\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$
4. *Unit Vector in the Direction of* **v**: $\dfrac{\mathbf{v}}{\|\mathbf{v}\|} = \left(\dfrac{1}{\|\mathbf{v}\|}\right)\langle v_1, v_2, v_3\rangle, \quad \mathbf{v} \neq \mathbf{0}$
5. *Vector Addition:* $\mathbf{v} + \mathbf{u} = \langle v_1 + u_1, v_2 + u_2, v_3 + u_3\rangle$
6. *Scalar Multiplication:* $c\mathbf{v} = \langle cv_1, cv_2, cv_3\rangle$

Note that the properties of vector operations listed in Theorem 11.1 (see Section 11.1) are also valid for vectors in space.

EXAMPLE 3 Finding the Component Form of a Vector in Space

See LarsonCalculus.com for an interactive version of this type of example.

Find the component form and magnitude of the vector **v** having initial point $(-2, 3, 1)$ and terminal point $(0, -4, 4)$. Then find a unit vector in the direction of **v**.

Solution The component form of **v** is

$$\mathbf{v} = \langle q_1 - p_1, q_2 - p_2, q_3 - p_3\rangle = \langle 0 - (-2), -4 - 3, 4 - 1\rangle = \langle 2, -7, 3\rangle$$

which implies that its magnitude is

$$\|\mathbf{v}\| = \sqrt{2^2 + (-7)^2 + 3^2} = \sqrt{62}.$$

The unit vector in the direction of **v** is

$$\begin{aligned}\mathbf{u} &= \frac{\mathbf{v}}{\|\mathbf{v}\|}\\ &= \frac{1}{\sqrt{62}}\langle 2, -7, 3\rangle\\ &= \left\langle \frac{2}{\sqrt{62}}, \frac{-7}{\sqrt{62}}, \frac{3}{\sqrt{62}}\right\rangle.\end{aligned}$$

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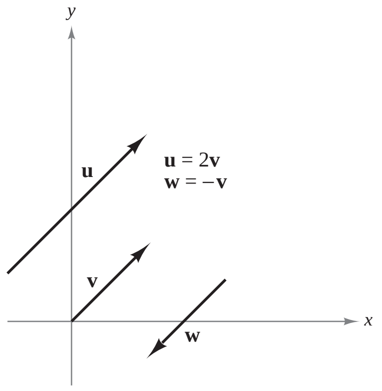


Parallel vectors
Figure 11.21

Recall from the definition of scalar multiplication that positive scalar multiples of a nonzero vector **v** have the same direction as **v**, whereas negative multiples have the direction opposite of **v**. In general, two nonzero vectors **u** and **v** are **parallel** when there is some scalar c such that $\mathbf{u} = c\mathbf{v}$. For example, in Figure11.21, the vectors **u**, **v**, and **w** are parallel because

$$\mathbf{u} = 2\mathbf{v} \quad \text{and} \quad \mathbf{w} = -\mathbf{v}.$$

Definition of Parallel Vectors

Two nonzero vectors **u** and **v** are **parallel** when there is some scalar c such that $\mathbf{u} = c\mathbf{v}$.

EXAMPLE 4 Parallel Vectors

Vector **w** has initial point $(2, -1, 3)$ and terminal point $(-4, 7, 5)$. Which of the following vectors is parallel to **w**?

a. $\mathbf{u} = \langle 3, -4, -1 \rangle$

b. $\mathbf{v} = \langle 12, -16, 4 \rangle$

Solution Begin by writing **w** in component form.

$$\mathbf{w} = \langle -4 - 2, 7 - (-1), 5 - 3 \rangle = \langle -6, 8, 2 \rangle$$

a. Because $\mathbf{u} = \langle 3, -4, -1 \rangle = -\frac{1}{2}\langle -6, 8, 2 \rangle = -\frac{1}{2}\mathbf{w}$, you can conclude that **u** is parallel to **w**.

b. In this case, you want to find a scalar c such that

$$\langle 12, -16, 4 \rangle = c\langle -6, 8, 2 \rangle.$$

To find c, equate the corresponding components and solve as shown.

$$12 = -6c \quad \Rightarrow \quad c = -2$$
$$-16 = 8c \quad \Rightarrow \quad c = -2$$
$$4 = 2c \quad \Rightarrow \quad c = 2$$

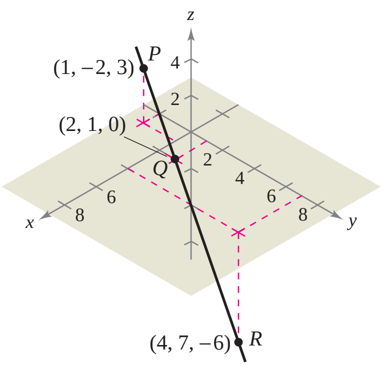
Note that $c = -2$ for the first two components and $c = 2$ for the third component. This means that the equation $\langle 12, -16, 4 \rangle = c\langle -6, 8, 2 \rangle$ has no solution, and the vectors are not parallel.

EXAMPLE 5 Using Vectors to Determine Collinear Points

Determine whether the points

$$P(1, -2, 3), \quad Q(2, 1, 0), \quad \text{and} \quad R(4, 7, -6)$$

are collinear.



The points P, Q, and R lie on the same line.
Figure 11.22

Solution The component forms of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are

$$\overrightarrow{PQ} = \langle 2 - 1, 1 - (-2), 0 - 3 \rangle = \langle 1, 3, -3 \rangle$$

and

$$\overrightarrow{PR} = \langle 4 - 1, 7 - (-2), -6 - 3 \rangle = \langle 3, 9, -9 \rangle.$$

These two vectors have a common initial point. So, P, Q, and R lie on the same line if and only if $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel—which they are because $\overrightarrow{PR} = 3\,\overrightarrow{PQ}$, as shown in Figure 11.22.

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EXAMPLE 6 Standard Unit Vector Notation

a. Write the vector $\mathbf{v} = 4\mathbf{i} - 5\mathbf{k}$ in component form.

b. Find the terminal point of the vector $\mathbf{v} = 7\mathbf{i} - \mathbf{j} + 3\mathbf{k}$, given that the initial point is $P(-2, 3, 5)$.

c. Find the magnitude of the vector $\mathbf{v} = -6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$. Then find a unit vector in the direction of $\mathbf{v}$.

Solution

a. Because $\mathbf{j}$ is missing, its component is 0 and

$$\mathbf{v} = 4\mathbf{i} - 5\mathbf{k} = \langle 4, 0, -5 \rangle.$$

b. You need to find $Q(q_1, q_2, q_3)$ such that

$$\mathbf{v} = \overrightarrow{PQ} = 7\mathbf{i} - \mathbf{j} + 3\mathbf{k}.$$

This implies that $q_1 - (-2) = 7$, $q_2 - 3 = -1$, and $q_3 - 5 = 3$. The solution of these three equations is $q_1 = 5$, $q_2 = 2$, and $q_3 = 8$. Therefore, Q is $(5, 2, 8)$.

c. Note that $v_1 = -6$, $v_2 = 2$, and $v_3 = -3$. So, the magnitude of $\mathbf{v}$ is

$$\|\mathbf{v}\| = \sqrt{(-6)^2 + 2^2 + (-3)^2} = \sqrt{49} = 7.$$

The unit vector in the direction of $\mathbf{v}$ is

$$\tfrac{1}{7}(-6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}) = -\tfrac{6}{7}\mathbf{i} + \tfrac{2}{7}\mathbf{j} - \tfrac{3}{7}\mathbf{k}.$$

EXAMPLE 7 Measuring Force

A television camera weighing 120 pounds is supported by a tripod, as shown in Figure 11.23. Represent the force exerted on each leg of the tripod as a vector.

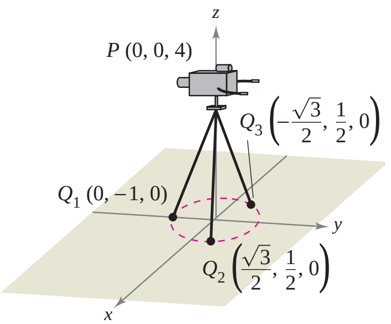

Figure 11.23

Solution Let the vectors $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ represent the forces exerted on the three legs. From Figure 11.23, you can determine the directions of $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ to be as follows.

$$\overrightarrow{PQ_1} = \langle 0 - 0, -1 - 0, 0 - 4 \rangle = \langle 0, -1, -4 \rangle$$

$$\overrightarrow{PQ_2} = \left\langle \frac{\sqrt{3}}{2} - 0, \frac{1}{2} - 0, 0 - 4 \right\rangle = \left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}, -4 \right\rangle$$

$$\overrightarrow{PQ_3} = \left\langle -\frac{\sqrt{3}}{2} - 0, \frac{1}{2} - 0, 0 - 4 \right\rangle = \left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}, -4 \right\rangle$$

Because each leg has the same length, and the total force is distributed equally among the three legs, you know that $\|\mathbf{F}_1\| = \|\mathbf{F}_2\| = \|\mathbf{F}_3\|$. So, there exists a constant c such that

$$\mathbf{F}_1 = c\langle 0, -1, -4 \rangle, \quad \mathbf{F}_2 = c\left\langle \frac{\sqrt{3}}{2}, \frac{1}{2}, -4 \right\rangle, \quad \text{and} \quad \mathbf{F}_3 = c\left\langle -\frac{\sqrt{3}}{2}, \frac{1}{2}, -4 \right\rangle.$$

Let the total force exerted by the object be given by $\mathbf{F} = \langle 0, 0, -120 \rangle$. Then, using the fact that

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3$$

you can conclude that $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}_3$ all have a vertical component of -40. This implies that $c(-4) = -40$ and $c = 10$. Therefore, the forces exerted on the legs can be represented by

$$\mathbf{F}_1 = \langle 0, -10, -40 \rangle,$$
$$\mathbf{F}_2 = \left\langle 5\sqrt{3}, 5, -40 \right\rangle,$$

and

$$\mathbf{F}_3 = \left\langle -5\sqrt{3}, 5, -40 \right\rangle.$$

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11.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Plotting Points **In Exercises 1–4, plot the points in the same three-dimensional coordinate system.**

1. (a) $(2, 1, 3)$ (b) $(-1, 2, 1)$

2. (a) $(3, -2, 5)$ (b) $\left(\frac{3}{2}, 4, -2\right)$

3. (a) $(5, -2, 2)$ (b) $(5, -2, -2)$

4. (a) $(0, 4, -5)$ (b) $(4, 0, 5)$

Finding Coordinates of a Point **In Exercises 5–8, find the coordinates of the point.**

5. The point is located three units behind the *yz*-plane, four units to the right of the *xz*-plane, and five units above the *xy*-plane.

6. The point is located seven units in front of the *yz*-plane, two units to the left of the *xz*-plane, and one unit below the *xy*-plane.

7. The point is located on the *x*-axis, 12 units in front of the *yz*-plane.

8. The point is located in the *yz*-plane, three units to the right of the *xz*-plane, and two units above the *xy*-plane.

9. Think About It What is the *z*-coordinate of any point in the *xy*-plane?

10. Think About It What is the *x*-coordinate of any point in the *yz*-plane?

Using the Three-Dimensional Coordinate System **In Exercises 11–22, determine the location of a point (x, y, z) that satisfies the condition(s).**

11. $z = 6$ **12.** $y = 2$

13. $x = -3$ **14.** $z = -\frac{5}{2}$

15. $y < 0$ **16.** $x > 0$

17. $|y| \le 3$ **18.** $|x| > 4$

19. $xy > 0, \quad z = -3$ **20.** $xy < 0, \quad z = 4$

21. $xyz < 0$ **22.** $xyz > 0$

Finding the Distance Between Two Points in Space **In Exercises 23–26, find the distance between the points.**

23. $(0, 0, 0), (-4, 2, 7)$ **24.** $(-2, 3, 2), (2, -5, -2)$

25. $(1, -2, 4), (6, -2, -2)$ **26.** $(2, 2, 3), (4, -5, 6)$

Classifying a Triangle **In Exercises 27–30, find the lengths of the sides of the triangle with the indicated vertices, and determine whether the triangle is a right triangle, an isosceles triangle, or neither.**

27. $(0, 0, 4), (2, 6, 7), (6, 4, -8)$

28. $(3, 4, 1), (0, 6, 2), (3, 5, 6)$

29. $(-1, 0, -2), (-1, 5, 2), (-3, -1, 1)$

30. $(4, -1, -1), (2, 0, -4), (3, 5, -1)$

31. Think About It The triangle in Exercise 27 is translated five units upward along the *z*-axis. Determine the coordinates of the translated triangle.

32. Think About It The triangle in Exercise 28 is translated three units to the right along the *y*-axis. Determine the coordinates of the translated triangle.

Finding the Midpoint **In Exercises 33–36, find the coordinates of the midpoint of the line segment joining the points.**

33. $(3, 4, 6), (1, 8, 0)$ **34.** $(7, 2, 2), (-5, -2, -3)$

35. $(5, -9, 7), (-2, 3, 3)$ **36.** $(4, 0, -6), (8, 8, 20)$

Finding the Equation of a Sphere **In Exercises 37–40, find the standard equation of the sphere.**

37. Center: $(0, 2, 5)$
Radius: 2

38. Center: $(4, -1, 1)$
Radius: 5

39. Endpoints of a diameter: $(2, 0, 0), (0, 6, 0)$

40. Center: $(-3, 2, 4)$, tangent to the *yz*-plane

Finding the Equation of a Sphere **In Exercises 41–44, complete the square to write the equation of the sphere in standard form. Find the center and radius.**

41. $x^2 + y^2 + z^2 - 2x + 6y + 8z + 1 = 0$

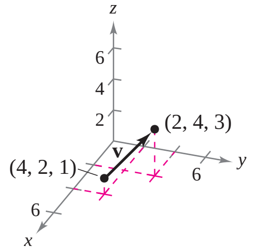
42. $x^2 + y^2 + z^2 + 9x - 2y + 10z + 19 = 0$

43. $9x^2 + 9y^2 + 9z^2 - 6x + 18y + 1 = 0$

44. $4x^2 + 4y^2 + 4z^2 - 24x - 4y + 8z - 23 = 0$

Finding the Component Form of a Vector in Space **In Exercises 45–48, (a) find the component form of the vector v, (b) write the vector using standard unit vector notation, and (c) sketch the vector with its initial point at the origin.**

45.



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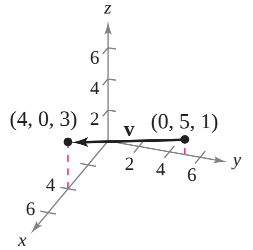

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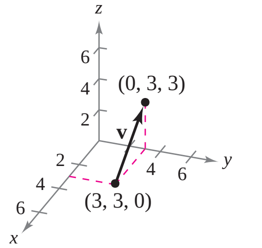

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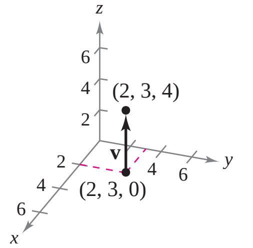

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Finding the Component Form of a Vector in Space **In Exercises 49 and 50, find the component form and magnitude of the vector v with the given initial and terminal points. Then find a unit vector in the direction of v.**

49. Initial point: $(3, 2, 0)$
Terminal point: $(4, 1, 6)$

50. Initial point: $(1, -2, 4)$
Terminal point: $(2, 4, -2)$

Writing a Vector in Different Forms **In Exercises 51 and 52, the initial and terminal points of a vector v are given. (a) Sketch the directed line segment, (b) find the component form of the vector, (c) write the vector using standard unit vector notation, and (d) sketch the vector with its initial point at the origin.**

51. Initial point: $(-1, 2, 3)$
Terminal point: $(3, 3, 4)$

52. Initial point: $(2, -1, -2)$
Terminal point: $(-4, 3, 7)$

Finding a Terminal Point **In Exercises 53 and 54, the vector v and its initial point are given. Find the terminal point.**

53. $\mathbf{v} = \langle 3, -5, 6 \rangle$
Initial point: $(0, 6, 2)$

54. $\mathbf{v} = \left\langle 1, -\frac{2}{3}, \frac{1}{2} \right\rangle$
Initial point: $\left(0, 2, \frac{5}{2}\right)$

Finding Scalar Multiples **In Exercises 55 and 56, find each scalar multiple of v and sketch its graph.**

55. $\mathbf{v} = \langle 1, 2, 2 \rangle$
(a) $2\mathbf{v}$ (b) $-\mathbf{v}$
(c) $\frac{3}{2}\mathbf{v}$ (d) $0\mathbf{v}$

56. $\mathbf{v} = \langle 2, -2, 1 \rangle$
(a) $-\mathbf{v}$ (b) $2\mathbf{v}$
(c) $\frac{1}{2}\mathbf{v}$ (d) $\frac{5}{2}\mathbf{v}$

Finding a Vector **In Exercises 57–60, find the vector z, given that $\mathbf{u} = \langle 1, 2, 3 \rangle$, $\mathbf{v} = \langle 2, 2, -1 \rangle$, and $\mathbf{w} = \langle 4, 0, -4 \rangle$.**

57. $\mathbf{z} = \mathbf{u} - \mathbf{v} + 2\mathbf{w}$

58. $\mathbf{z} = 5\mathbf{u} - 3\mathbf{v} - \frac{1}{2}\mathbf{w}$

59. $2\mathbf{z} - 3\mathbf{u} = \mathbf{w}$

60. $2\mathbf{u} + \mathbf{v} - \mathbf{w} + 3\mathbf{z} = \mathbf{0}$

Parallel Vectors **In Exercises 61–64, determine which of the vectors is (are) parallel to z. Use a graphing utility to confirm your results.**

61. $\mathbf{z} = \langle 3, 2, -5 \rangle$
(a) $\langle -6, -4, 10 \rangle$
(b) $\left\langle 2, \frac{4}{3}, -\frac{10}{3} \right\rangle$
(c) $\langle 6, 4, 10 \rangle$
(d) $\langle 1, -4, 2 \rangle$

62. $\mathbf{z} = \frac{1}{2}\mathbf{i} - \frac{2}{3}\mathbf{j} + \frac{3}{4}\mathbf{k}$
(a) $6\mathbf{i} - 4\mathbf{j} + 9\mathbf{k}$
(b) $-\mathbf{i} + \frac{4}{3}\mathbf{j} - \frac{3}{2}\mathbf{k}$
(c) $12\mathbf{i} + 9\mathbf{k}$
(d) $\frac{3}{4}\mathbf{i} - \mathbf{j} + \frac{9}{8}\mathbf{k}$

63. **z** has initial point $(1, -1, 3)$ and terminal point $(-2, 3, 5)$.
(a) $-6\mathbf{i} + 8\mathbf{j} + 4\mathbf{k}$ (b) $4\mathbf{j} + 2\mathbf{k}$

64. **z** has initial point $(5, 4, 1)$ and terminal point $(-2, -4, 4)$.
(a) $\langle 7, 6, 2 \rangle$ (b) $\langle 14, 16, -6 \rangle$

Using Vectors to Determine Collinear Points **In Exercises 65–68, use vectors to determine whether the points are collinear.**

65. $(0, -2, -5), (3, 4, 4), (2, 2, 1)$

66. $(4, -2, 7), (-2, 0, 3), (7, -3, 9)$

67. $(1, 2, 4), (2, 5, 0), (0, 1, 5)$

68. $(0, 0, 0), (1, 3, -2), (2, -6, 4)$

Verifying a Parallelogram **In Exercises 69 and 70, use vectors to show that the points form the vertices of a parallelogram.**

69. $(2, 9, 1), (3, 11, 4), (0, 10, 2), (1, 12, 5)$

70. $(1, 1, 3), (9, -1, -2), (11, 2, -9), (3, 4, -4)$

Finding the Magnitude **In Exercises 71–76, find the magnitude of v.**

71. $\mathbf{v} = \langle 0, 0, 0 \rangle$

72. $\mathbf{v} = \langle 1, 0, 3 \rangle$

73. $\mathbf{v} = 3\mathbf{j} - 5\mathbf{k}$

74. $\mathbf{v} = 2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$

75. $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$

76. $\mathbf{v} = -4\mathbf{i} + 3\mathbf{j} + 7\mathbf{k}$

Finding Unit Vectors **In Exercises 77–80, find a unit vector (a) in the direction of v and (b) in the direction opposite of v.**

77. $\mathbf{v} = \langle 2, -1, 2 \rangle$

78. $\mathbf{v} = \langle 6, 0, 8 \rangle$

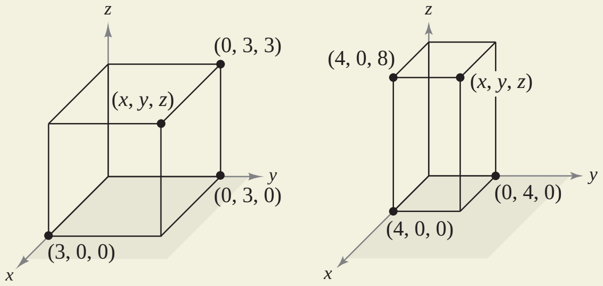
79. $\mathbf{v} = 4\mathbf{i} - 5\mathbf{j} + 3\mathbf{k}$

80. $\mathbf{v} = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$

81. Using Vectors Consider the two nonzero vectors **u** and **v**, and let s and t be real numbers. Describe the geometric figure generated by the terminal points of the three vectors $t\mathbf{v}$, $\mathbf{u} + t\mathbf{v}$, and $s\mathbf{u} + t\mathbf{v}$.

82. HOW DO YOU SEE IT? Determine (x, y, z) for each figure. Then find the component form of the vector from the point on the x-axis to the point (x, y, z).

(a) (b)



Finding a Vector **In Exercises 83–86, find the vector v with the given magnitude and the same direction as u.**

	Magnitude	Direction
83.	$\|\mathbf{v}\| = 10$	$\mathbf{u} = \langle 0, 3, 3 \rangle$
84.	$\|\mathbf{v}\| = 3$	$\mathbf{u} = \langle 1, 1, 1 \rangle$
85.	$\|\mathbf{v}\| = \frac{3}{2}$	$\mathbf{u} = \langle 2, -2, 1 \rangle$
86.	$\|\mathbf{v}\| = 7$	$\mathbf{u} = \langle -4, 6, 2 \rangle$

Sketching a Vector **In Exercises 87 and 88, sketch the vector v and write its component form.**

87. **v** lies in the yz-plane, has magnitude 2, and makes an angle of 30° with the positive y-axis.

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88. $\mathbf{v}$ lies in the xz-plane, has magnitude 5, and makes an angle of 45° with the positive z-axis.

Finding a Point Using Vectors **In Exercises 89 and 90, use vectors to find the point that lies two-thirds of the way from P to Q.**

89. $P(4, 3, 0)$, $Q(1, -3, 3)$

90. $P(1, 2, 5)$, $Q(6, 8, 2)$

91. Using Vectors Let $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$, and $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$.

(a) Sketch $\mathbf{u}$ and $\mathbf{v}$.

(b) If $\mathbf{w} = \mathbf{0}$, show that a and b must both be zero.

(c) Find a and b such that $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + \mathbf{k}$.

(d) Show that no choice of a and b yields $\mathbf{w} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$.

92. Writing The initial and terminal points of the vector $\mathbf{v}$ are (x_1, y_1, z_1) and (x, y, z). Describe the set of all points (x, y, z) such that $\|\mathbf{v}\| = 4$.

WRITING ABOUT CONCEPTS

93. Describing Coordinates A point in the three-dimensional coordinate system has coordinates (x_0, y_0, z_0). Describe what each coordinate measures.

94. Distance Formula Give the formula for the distance between the points (x_1, y_1, z_1) and (x_2, y_2, z_2).

95. Standard Equation of a Sphere Give the standard equation of a sphere of radius r, centered at (x_0, y_0, z_0).

96. Parallel Vectors State the definition of parallel vectors.

97. Using a Triangle and Vectors Let A, B, and C be vertices of a triangle. Find $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}$.

98. Using Vectors Let $\mathbf{r} = \langle x, y, z\rangle$ and $\mathbf{r}_0 = \langle 1, 1, 1\rangle$. Describe the set of all points (x, y, z) such that $\|\mathbf{r} - \mathbf{r}_0\| = 2$.

99. Diagonal of a Cube Find the component form of the unit vector $\mathbf{v}$ in the direction of the diagonal of the cube shown in the figure.

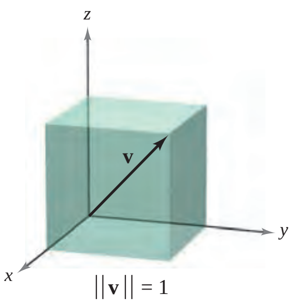


Figure for 99

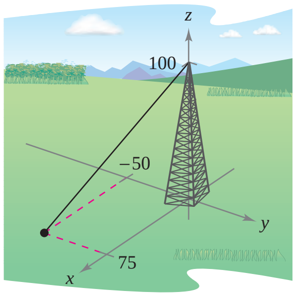


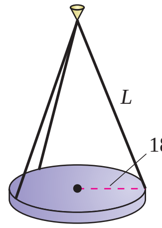
Figure for 100

100. Tower Guy Wire The guy wire supporting a 100-foot tower has a tension of 550 pounds. Using the distances shown in the figure, write the component form of the vector $\mathbf{F}$ representing the tension in the wire.

Losevsky Photo and Video/Shutterstock.com

101. Auditorium Lights

The lights in an auditorium are 24-pound discs of radius 18 inches. Each disc is supported by three equally spaced cables that are L inches long (see figure).



(a) Write the tension T in each cable as a function of L. Determine the domain of the function.

(b) Use a graphing utility and the function in part (a) to complete the table.

L	20	25	30	35	40	45	50
T							

(c) Use a graphing utility to graph the function in part (a). Determine the asymptotes of the graph.

(d) Confirm the asymptotes of the graph in part (c) analytically.

(e) Determine the minimum length of each cable when a cable is designed to carry a maximum load of 10 pounds.

102. Think About It Suppose the length of each cable in Exercise 101 has a fixed length $L = a$, and the radius of each disc is r_0 inches. Make a conjecture about the limit $\lim_{r_0 \to a^-} T$ and give a reason for your answer.

103. Load Supports Find the tension in each of the supporting cables in the figure when the weight of the crate is 500 newtons.

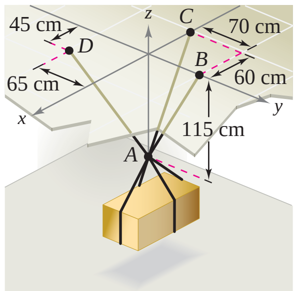


Figure for 103

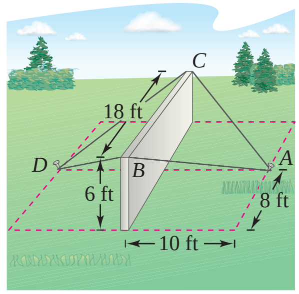


Figure for 104

104. Construction A precast concrete wall is temporarily kept in its vertical position by ropes (see figure). Find the total force exerted on the pin at position A. The tensions in AB and AC are 420 pounds and 650 pounds.

105. Geometry Write an equation whose graph consists of the set of points $P(x, y, z)$ that are twice as far from $A(0, -1, 1)$ as from $B(1, 2, 0)$. Describe the geometric figure represented by the equation.

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11.3 The Dot Product of Two Vectors

- **Use properties of the dot product of two vectors.**
- **Find the angle between two vectors using the dot product.**
- **Find the direction cosines of a vector in space.**
- **Find the projection of a vector onto another vector.**
- **Use vectors to find the work done by a constant force.**

The Dot Product

So far, you have studied two operations with vectors—vector addition and multiplication by a scalar—each of which yields another vector. In this section, you will study a third vector operation, the **dot product.** This product yields a scalar, rather than a vector.

REMARK Because the dot product of two vectors yields a scalar, it is also called the *scalar product* (or *inner product*) of the two vectors.

Definition of Dot Product

The **dot product** of $\mathbf{u} = \langle u_1, u_2 \rangle$ and $\mathbf{v} = \langle v_1, v_2 \rangle$ is

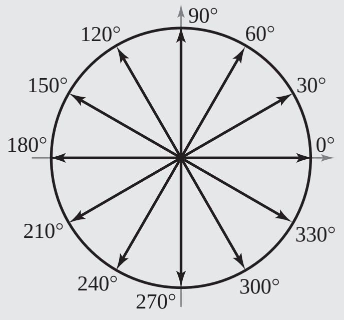
$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2.$$

The **dot product** of $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ is

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3.$$

Exploration

Interpreting a Dot Product Several vectors are shown below on the unit circle. Find the dot products of several pairs of vectors. Then find the angle between each pair that you used. Make a conjecture about the relationship between the dot product of two vectors and the angle between the vectors.



THEOREM 11.4 Properties of the Dot Product

Let **u**, **v**, and **w** be vectors in the plane or in space and let c be a scalar.

1. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$ — Commutative Property
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = \mathbf{u} \cdot \mathbf{v} + \mathbf{u} \cdot \mathbf{w}$ — Distributive Property
3. $c(\mathbf{u} \cdot \mathbf{v}) = c\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot c\mathbf{v}$
4. $\mathbf{0} \cdot \mathbf{v} = 0$
5. $\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2$

Proof To prove the first property, let $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ and $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then

$$\mathbf{u} \cdot \mathbf{v} = u_1v_1 + u_2v_2 + u_3v_3 = v_1u_1 + v_2u_2 + v_3u_3 = \mathbf{v} \cdot \mathbf{u}.$$

For the fifth property, let $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$. Then

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + v_2^2 + v_3^2 = \left(\sqrt{v_1^2 + v_2^2 + v_3^2}\right)^2 = \|\mathbf{v}\|^2.$$

Proofs of the other properties are left to you.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 1 Finding Dot Products

Let $\mathbf{u} = \langle 2, -2 \rangle$, $\mathbf{v} = \langle 5, 8 \rangle$, and $\mathbf{w} = \langle -4, 3 \rangle$.

a. $\mathbf{u} \cdot \mathbf{v} = \langle 2, -2 \rangle \cdot \langle 5, 8 \rangle = 2(5) + (-2)(8) = -6$

b. $(\mathbf{u} \cdot \mathbf{v})\mathbf{w} = -6\langle -4, 3 \rangle = \langle 24, -18 \rangle$

c. $\mathbf{u} \cdot (2\mathbf{v}) = 2(\mathbf{u} \cdot \mathbf{v}) = 2(-6) = -12$

d. $\|\mathbf{w}\|^2 = \mathbf{w} \cdot \mathbf{w} = \langle -4, 3 \rangle \cdot \langle -4, 3 \rangle = (-4)(-4) + (3)(3) = 25$

Notice that the result of part (b) is a *vector* quantity, whereas the results of the other three parts are *scalar* quantities.

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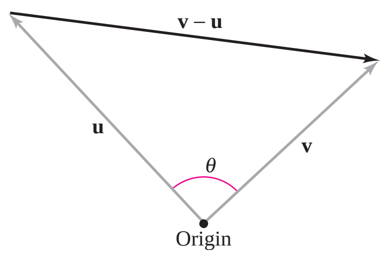
Angle Between Two Vectors

The **angle between two nonzero vectors** is the angle θ, $0 \le \theta \le \pi$, between their respective standard position vectors, as shown in Figure 11.24. The next theorem shows how to find this angle using the dot product. (Note that the angle between the zero vector and another vector is not defined here.)



The angle between two vectors
Figure 11.24

THEOREM 11.5 Angle Between Two Vectors

If θ is the angle between two nonzero vectors **u** and **v**, where $0 \le \theta \le \pi$, then

$$\cos\theta = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.$$

Proof Consider the triangle determined by vectors **u**, **v**, and $\mathbf{v}-\mathbf{u}$, as shown in Figure 11.24. By the Law of Cosines, you can write

$$\|\mathbf{v}-\mathbf{u}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta.$$

Using the properties of the dot product, the left side can be rewritten as

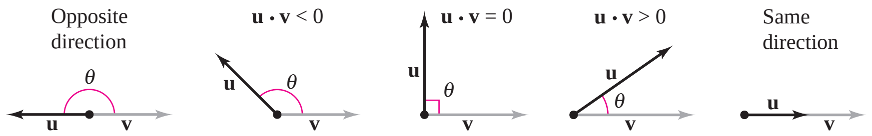
$$\begin{aligned}\|\mathbf{v}-\mathbf{u}\|^2 &= (\mathbf{v}-\mathbf{u})\cdot(\mathbf{v}-\mathbf{u})\\ &= (\mathbf{v}-\mathbf{u})\cdot\mathbf{v} - (\mathbf{v}-\mathbf{u})\cdot\mathbf{u}\\ &= \mathbf{v}\cdot\mathbf{v} - \mathbf{u}\cdot\mathbf{v} - \mathbf{v}\cdot\mathbf{u} + \mathbf{u}\cdot\mathbf{u}\\ &= \|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{u}\|^2\end{aligned}$$

and substitution back into the Law of Cosines yields

$$\begin{aligned}\|\mathbf{v}\|^2 - 2\mathbf{u}\cdot\mathbf{v} + \|\mathbf{u}\|^2 &= \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta\\ -2\mathbf{u}\cdot\mathbf{v} &= -2\|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta\\ \cos\theta &= \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|}.\end{aligned}$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Note in Theorem 11.5 that because $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ are always positive, $\mathbf{u}\cdot\mathbf{v}$ and $\cos\theta$ will always have the same sign. Figure 11.25 shows the possible orientations of two vectors.



$\theta = \pi$	$\pi/2 < \theta < \pi$	$\theta = \pi/2$	$0 < \theta < \pi/2$	$\theta = 0$
$\cos\theta = -1$	$-1 < \cos\theta < 0$	$\cos\theta = 0$	$0 < \cos\theta < 1$	$\cos\theta = 1$

Figure 11.25

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From Theorem 11.5, you can see that two nonzero vectors meet at a right angle if and only if their dot product is zero. Two such vectors are said to be **orthogonal.**

Definition of Orthogonal Vectors

The vectors $\mathbf{u}$ and $\mathbf{v}$ are orthogonal when $\mathbf{u} \cdot \mathbf{v} = 0$.

REMARK The terms "perpendicular," "orthogonal," and "normal" all mean essentially the same thing—meeting at right angles. It is common, however, to say that two vectors are *orthogonal*, two lines or planes are *perpendicular*, and a vector is *normal* to a line or plane.

From this definition, it follows that the zero vector is orthogonal to every vector $\mathbf{u}$, because $\mathbf{0} \cdot \mathbf{u} = 0$. Moreover, for $0 \le \theta \le \pi$, you know that $\cos\theta = 0$ if and only if $\theta = \pi/2$. So, you can use Theorem 11.5 to conclude that two *nonzero* vectors are orthogonal if and only if the angle between them is $\pi/2$.

EXAMPLE 2 Finding the Angle Between Two Vectors

See LarsonCalculus.com for an interactive version of this type of example.

For $\mathbf{u} = \langle 3, -1, 2\rangle$, $\mathbf{v} = \langle -4, 0, 2\rangle$, $\mathbf{w} = \langle 1, -1, -2\rangle$, and $\mathbf{z} = \langle 2, 0, -1\rangle$, find the angle between each pair of vectors.

a. $\mathbf{u}$ and $\mathbf{v}$ **b.** $\mathbf{u}$ and $\mathbf{w}$ **c.** $\mathbf{v}$ and $\mathbf{z}$

Solution

a. $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{u}\|\,\|\mathbf{v}\|} = \dfrac{-12+0+4}{\sqrt{14}\sqrt{20}} = \dfrac{-8}{2\sqrt{14}\sqrt{5}} = \dfrac{-4}{\sqrt{70}}$

Because $\mathbf{u}\cdot\mathbf{v} < 0$, $\theta = \arccos\dfrac{-4}{\sqrt{70}} \approx 2.069$ radians.

REMARK The angle between $\mathbf{u}$ and $\mathbf{v}$ in Example 3(a) can also be written as approximately 118.561°.

b. $\cos\theta = \dfrac{\mathbf{u}\cdot\mathbf{w}}{\|\mathbf{u}\|\,\|\mathbf{w}\|} = \dfrac{3+1-4}{\sqrt{14}\sqrt{6}} = \dfrac{0}{\sqrt{84}} = 0$

Because $\mathbf{u}\cdot\mathbf{w} = 0$, $\mathbf{u}$ and $\mathbf{w}$ are *orthogonal*. So, $\theta = \pi/2$.

c. $\cos\theta = \dfrac{\mathbf{v}\cdot\mathbf{z}}{\|\mathbf{v}\|\,\|\mathbf{z}\|} = \dfrac{-8+0-2}{\sqrt{20}\sqrt{5}} = \dfrac{-10}{\sqrt{100}} = -1$

Consequently, $\theta = \pi$. Note that $\mathbf{v}$ and $\mathbf{z}$ are parallel, with $\mathbf{v} = -2\mathbf{z}$.

When the angle between two vectors is known, rewriting Theorem 11.5 in the form

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta$$

Alternative form of dot product

produces an alternative way to calculate the dot product.

EXAMPLE 3 Alternative Form of the Dot Product

Given that $\|\mathbf{u}\| = 10$, $\|\mathbf{v}\| = 7$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\pi/4$, find $\mathbf{u}\cdot\mathbf{v}$.

Solution Use the alternative form of the dot product as shown.

$$\mathbf{u}\cdot\mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\cos\theta = (10)(7)\cos\frac{\pi}{4} = 35\sqrt{2}$$

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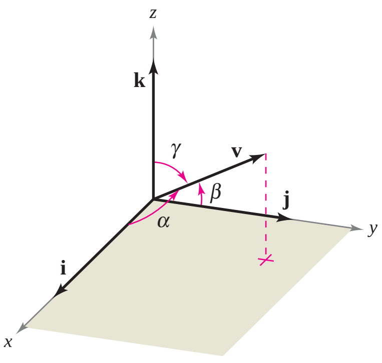
Direction Cosines

For a vector in the plane, you have seen that it is convenient to measure direction in terms of the angle, measured counterclockwise, *from* the positive x-axis *to* the vector. In space, it is more convenient to measure direction in terms of the angles *between* the nonzero vector $\mathbf{v}$ and the three unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$, as shown in Figure 11.26. The angles α, β, and γ are the **direction angles of v,** and $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the **direction cosines of v.** Because



Direction angles
Figure 11.26

REMARK Recall that α, β, and γ are the Greek letters alpha, beta, and gamma, respectively.

$$\mathbf{v}\cdot\mathbf{i} = \|\mathbf{v}\|\,\|\mathbf{i}\|\cos\alpha = \|\mathbf{v}\|\cos\alpha$$

and

$$\mathbf{v}\cdot\mathbf{i} = \langle v_1, v_2, v_3\rangle \cdot \langle 1, 0, 0\rangle = v_1$$

it follows that $\cos\alpha = v_1/\|\mathbf{v}\|$. By similar reasoning with the unit vectors $\mathbf{j}$ and $\mathbf{k}$, you have

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|} \qquad \alpha \text{ is the angle between } \mathbf{v} \text{ and } \mathbf{i}.$$

$$\cos\beta = \frac{v_2}{\|\mathbf{v}\|} \qquad \beta \text{ is the angle between } \mathbf{v} \text{ and } \mathbf{j}.$$

$$\cos\gamma = \frac{v_3}{\|\mathbf{v}\|}. \qquad \gamma \text{ is the angle between } \mathbf{v} \text{ and } \mathbf{k}.$$

Consequently, any nonzero vector $\mathbf{v}$ in space has the normalized form

$$\frac{\mathbf{v}}{\|\mathbf{v}\|} = \frac{v_1}{\|\mathbf{v}\|}\mathbf{i} + \frac{v_2}{\|\mathbf{v}\|}\mathbf{j} + \frac{v_3}{\|\mathbf{v}\|}\mathbf{k} = \cos\alpha\,\mathbf{i} + \cos\beta\,\mathbf{j} + \cos\gamma\,\mathbf{k}$$

and because $\mathbf{v}/\|\mathbf{v}\|$ is a unit vector, it follows that

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1.$$

EXAMPLE 4 Finding Direction Angles

Find the direction cosines and angles for the vector $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$, and show that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$.

Solution Because $\|\mathbf{v}\| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$, you can write the following.

$$\cos\alpha = \frac{v_1}{\|\mathbf{v}\|} = \frac{2}{\sqrt{29}} \Rightarrow \alpha \approx 68.2° \qquad \text{Angle between } \mathbf{v} \text{ and } \mathbf{i}$$

$$\cos\beta = \frac{v_2}{\|\mathbf{v}\|} = \frac{3}{\sqrt{29}} \Rightarrow \beta \approx 56.1° \qquad \text{Angle between } \mathbf{v} \text{ and } \mathbf{j}$$

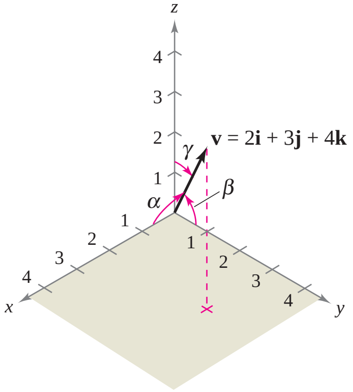
$$\cos\gamma = \frac{v_3}{\|\mathbf{v}\|} = \frac{4}{\sqrt{29}} \Rightarrow \gamma \approx 42.0° \qquad \text{Angle between } \mathbf{v} \text{ and } \mathbf{k}$$

Furthermore, the sum of the squares of the direction cosines is

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = \frac{4}{29} + \frac{9}{29} + \frac{16}{29} = \frac{29}{29} = 1.$$

See Figure 11.27.

α = angle between $\mathbf{v}$ and $\mathbf{i}$
β = angle between $\mathbf{v}$ and $\mathbf{j}$
γ = angle between $\mathbf{v}$ and $\mathbf{k}$



The direction angles of $\mathbf{v}$
Figure 11.27

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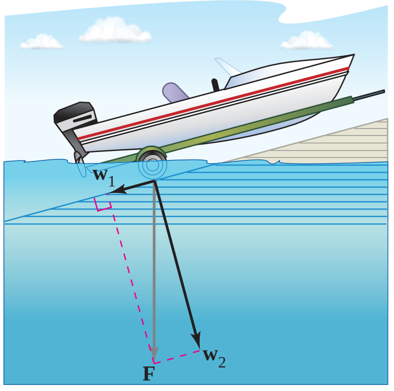
Projections and Vector Components

You have already seen applications in which two vectors are added to produce a resultant vector. Many applications in physics and engineering pose the reverse problem—decomposing a vector into the sum of two **vector components.** The following physical example enables you to see the usefulness of this procedure.

Consider a boat on an inclined ramp, as shown in Figure 11.28. The force $\mathbf{F}$ due to gravity pulls the boat *down* the ramp and *against* the ramp. These two forces, $\mathbf{w}_1$ and $\mathbf{w}_2$, are orthogonal—they are called the vector components of $\mathbf{F}$.

$$\mathbf{F} = \mathbf{w}_1 + \mathbf{w}_2 \qquad \text{Vector components of } \mathbf{F}$$

The forces $\mathbf{w}_1$ and $\mathbf{w}_2$ help you analyze the effect of gravity on the boat. For example, $\mathbf{w}_1$ indicates the force necessary to keep the boat from rolling down the ramp, whereas $\mathbf{w}_2$ indicates the force that the tires must withstand.



The force due to gravity pulls the boat against the ramp and down the ramp.
Figure 11.28

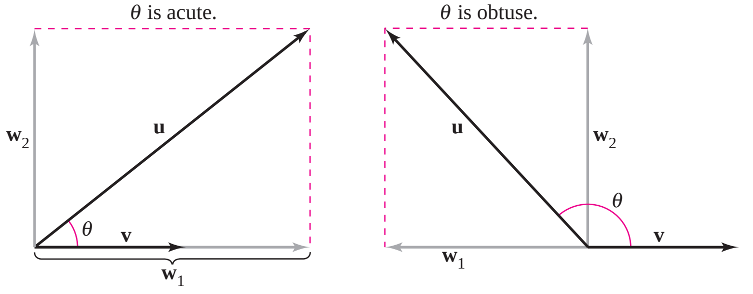
Definitions of Projection and Vector Components

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors. Moreover, let

$$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$$

where $\mathbf{w}_1$ is parallel to $\mathbf{v}$ and $\mathbf{w}_2$ is orthogonal to $\mathbf{v}$, as shown in Figure 11.29.

1. $\mathbf{w}_1$ is called the **projection of u onto v** or the **vector component of u along v,** and is denoted by $\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$.
2. $\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1$ is called the **vector component of u orthogonal to v.**



$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u}$ = projection of $\mathbf{u}$ onto $\mathbf{v}$ = vector component of $\mathbf{u}$ along $\mathbf{v}$
$\mathbf{w}_2$ = vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$
Figure 11.29

EXAMPLE 5 Finding a Vector Component of u Orthogonal to v

Find the vector component of $\mathbf{u} = \langle 5, 10\rangle$ that is orthogonal to $\mathbf{v} = \langle 4, 3\rangle$, given that

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \langle 8, 6\rangle$$

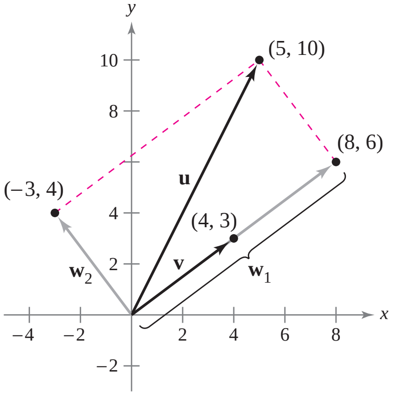
and

$$\mathbf{u} = \langle 5, 10\rangle = \mathbf{w}_1 + \mathbf{w}_2.$$

Solution Because $\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$, where $\mathbf{w}_1$ is parallel to $\mathbf{v}$, it follows that $\mathbf{w}_2$ is the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$. So, you have

$$\begin{aligned}\mathbf{w}_2 &= \mathbf{u} - \mathbf{w}_1\\ &= \langle 5, 10\rangle - \langle 8, 6\rangle\\ &= \langle -3, 4\rangle.\end{aligned}$$

Check to see that $\mathbf{w}_2$ is orthogonal to $\mathbf{v}$, as shown in Figure 11.30.



$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$
Figure 11.30

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From Example 5, you can see that it is easy to find the vector component $\mathbf{w}_2$ once you have found the projection, $\mathbf{w}_1$, of $\mathbf{u}$ onto $\mathbf{v}$. To find this projection, use the dot product in the next theorem, which you will prove in Exercise 78.

THEOREM 11.6 Projection Using the Dot Product

If $\mathbf{u}$ and $\mathbf{v}$ are nonzero vectors, then the projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$\text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v}.$$

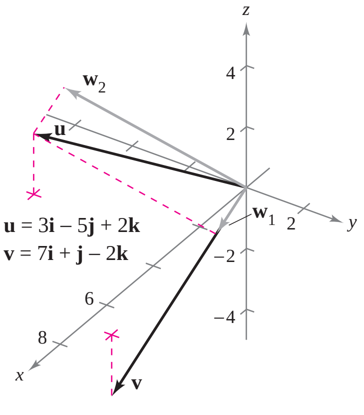
REMARK Note the distinction between the terms "component" and "vector component." For example, using the standard unit vectors with $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$, u_1 is the *component* of $\mathbf{u}$ in the direction of $\mathbf{i}$ and $u_1\mathbf{i}$ is the *vector component* in the direction of $\mathbf{i}$.

The projection of $\mathbf{u}$ onto $\mathbf{v}$ can be written as a scalar multiple of a unit vector in the direction of $\mathbf{v}$. That is,

$$\left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|}\right)\frac{\mathbf{v}}{\|\mathbf{v}\|} = (k)\frac{\mathbf{v}}{\|\mathbf{v}\|}.$$

The scalar k is called the **component of u in the direction of v.** So,

$$\mathbf{k} = \frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|} = \|\mathbf{u}\|\cos\theta.$$



$\mathbf{u} = \mathbf{w}_1 + \mathbf{w}_2$

Figure 11.31

EXAMPLE 6 Decomposing a Vector into Vector Components

Find the projection of $\mathbf{u}$ onto $\mathbf{v}$ and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$ for

$$\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k} \quad \text{and} \quad \mathbf{v} = 7\mathbf{i} + \mathbf{j} - 2\mathbf{k}.$$

Solution The projection of $\mathbf{u}$ onto $\mathbf{v}$ is

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{u} = \left(\frac{\mathbf{u}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = \left(\frac{12}{54}\right)(7\mathbf{i} + \mathbf{j} - 2\mathbf{k}) = \frac{14}{9}\mathbf{i} + \frac{2}{9}\mathbf{j} - \frac{4}{9}\mathbf{k}.$$

The vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$ is the vector

$$\mathbf{w}_2 = \mathbf{u} - \mathbf{w}_1 = (3\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}) - \left(\frac{14}{9}\mathbf{i} + \frac{2}{9}\mathbf{j} - \frac{4}{9}\mathbf{k}\right) = \frac{13}{9}\mathbf{i} - \frac{47}{9}\mathbf{j} + \frac{22}{9}\mathbf{k}.$$

See Figure 11.31.

EXAMPLE 7 Finding a Force

A 600-pound boat sits on a ramp inclined at 30°, as shown in Figure 11.32. What force is required to keep the boat from rolling down the ramp?

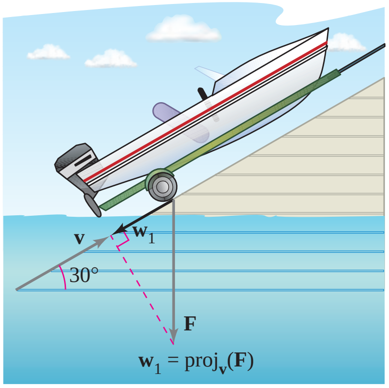


Figure 11.32

Solution Because the force due to gravity is vertical and downward, you can represent the gravitational force by the vector $\mathbf{F} = -600\mathbf{j}$. To find the force required to keep the boat from rolling down the ramp, project $\mathbf{F}$ onto a unit vector $\mathbf{v}$ in the direction of the ramp, as follows.

$$\mathbf{v} = \cos 30°\mathbf{i} + \sin 30°\mathbf{j} = \frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j} \qquad \text{Unit vector along ramp}$$

Therefore, the projection of $\mathbf{F}$ onto $\mathbf{v}$ is

$$\mathbf{w}_1 = \text{proj}_{\mathbf{v}}\mathbf{F} = \left(\frac{\mathbf{F}\cdot\mathbf{v}}{\|\mathbf{v}\|^2}\right)\mathbf{v} = (\mathbf{F}\cdot\mathbf{v})\mathbf{v} = (-600)\left(\frac{1}{2}\right)\mathbf{v} = -300\left(\frac{\sqrt{3}}{2}\mathbf{i} + \frac{1}{2}\mathbf{j}\right).$$

The magnitude of this force is 300, and therefore a force of 300 pounds is required to keep the boat from rolling down the ramp.

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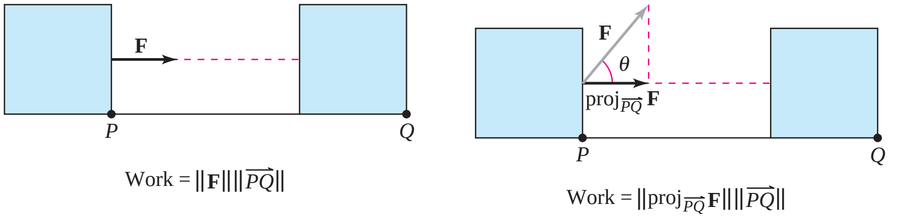
Work

The work W done by the constant force $\mathbf{F}$ acting along the line of motion of an object is given by

$$W = (\text{magnitude of force})(\text{distance}) = \|\mathbf{F}\|\,\|\overrightarrow{PQ}\|$$

as shown in Figure 11.33(a). When the constant force $\mathbf{F}$ is not directed along the line of motion, you can see from Figure 11.33(b) that the work W done by the force is

$$W = \|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|\,\|\overrightarrow{PQ}\| = (\cos\theta)\|\mathbf{F}\|\,\|\overrightarrow{PQ}\| = \mathbf{F}\cdot\overrightarrow{PQ}.$$



(a) Force acts along the line of motion. **(b)** Force acts at angle θ with the line of motion.

Figure 11.33

This notion of work is summarized in the next definition.

> ### Definition of Work
> The work W done by a constant force $\mathbf{F}$ as its point of application moves along the vector $\overrightarrow{PQ}$ is one of the following.
>
> **1.** $W = \|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|\,\|\overrightarrow{PQ}\|$ Projection form
>
> **2.** $W = \mathbf{F}\cdot\overrightarrow{PQ}$ Dot product form

EXAMPLE 8 Finding Work

To close a sliding door, a person pulls on a rope with a constant force of 50 pounds at a constant angle of 60°, as shown in Figure 11.34. Find the work done in moving the door 12 feet to its closed position.

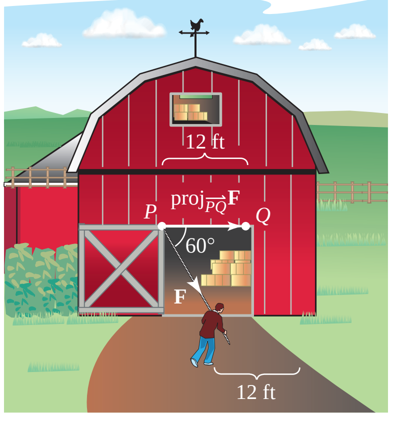


Figure 11.34

Solution Using a projection, you can calculate the work as follows.

$$W = \|\text{proj}_{\overrightarrow{PQ}}\mathbf{F}\|\,\|\overrightarrow{PQ}\| = \cos(60^\circ)\,\|\mathbf{F}\|\,\|\overrightarrow{PQ}\| = \frac{1}{2}(50)(12) = 300 \text{ foot-pounds}$$

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11.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding Dot Products In Exercises 1–8, find (a) $\mathbf{u}\cdot\mathbf{v}$, (b) $\mathbf{u}\cdot\mathbf{u}$, (c) $\|\mathbf{u}\|^2$, (d) $(\mathbf{u}\cdot\mathbf{v})\mathbf{v}$, and (e) $\mathbf{u}\cdot(2\mathbf{v})$.

1. $\mathbf{u} = \langle 3, 4\rangle$, $\mathbf{v} = \langle -1, 5\rangle$
2. $\mathbf{u} = \langle 4, 10\rangle$, $\mathbf{v} = \langle -2, 3\rangle$
3. $\mathbf{u} = \langle 6, -4\rangle$, $\mathbf{v} = \langle -3, 2\rangle$
4. $\mathbf{u} = \langle -4, 8\rangle$, $\mathbf{v} = \langle 7, 5\rangle$
5. $\mathbf{u} = \langle 2, -3, 4\rangle$, $\mathbf{v} = \langle 0, 6, 5\rangle$
6. $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{i}$
7. $\mathbf{u} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{k}$
8. $\mathbf{u} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, $\mathbf{v} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

Finding the Angle Between Two Vectors In Exercises 9–16, find the angle θ between the vectors (a) in radians and (b) in degrees.

9. $\mathbf{u} = \langle 1, 1\rangle$, $\mathbf{v} = \langle 2, -2\rangle$
10. $\mathbf{u} = \langle 3, 1\rangle$, $\mathbf{v} = \langle 2, -1\rangle$
11. $\mathbf{u} = 3\mathbf{i} + \mathbf{j}$, $\mathbf{v} = -2\mathbf{i} + 4\mathbf{j}$
12. $\mathbf{u} = \cos\left(\frac{\pi}{6}\right)\mathbf{i} + \sin\left(\frac{\pi}{6}\right)\mathbf{j}$, $\mathbf{v} = \cos\left(\frac{3\pi}{4}\right)\mathbf{i} + \sin\left(\frac{3\pi}{4}\right)\mathbf{j}$
13. $\mathbf{u} = \langle 1, 1, 1\rangle$, $\mathbf{v} = \langle 2, 1, -1\rangle$
14. $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} - 3\mathbf{j}$
15. $\mathbf{u} = 3\mathbf{i} + 4\mathbf{j}$, $\mathbf{v} = -2\mathbf{j} + 3\mathbf{k}$
16. $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$

Alternative Form of Dot Product In Exercises 17 and 18, use the alternative form of the dot product to find $\mathbf{u}\cdot\mathbf{v}$.

17. $\|\mathbf{u}\| = 8$, $\|\mathbf{v}\| = 5$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\pi/3$.
18. $\|\mathbf{u}\| = 40$, $\|\mathbf{v}\| = 25$, and the angle between $\mathbf{u}$ and $\mathbf{v}$ is $5\pi/6$.

Comparing Vectors In Exercises 19–24, determine whether $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, parallel, or neither.

19. $\mathbf{u} = \langle 4, 3\rangle$, $\mathbf{v} = \left\langle \frac{1}{2}, -\frac{2}{3}\right\rangle$
20. $\mathbf{u} = -\frac{1}{3}(\mathbf{i} - 2\mathbf{j})$, $\mathbf{v} = 2\mathbf{i} - 4\mathbf{j}$
21. $\mathbf{u} = \mathbf{j} + 6\mathbf{k}$, $\mathbf{v} = \mathbf{i} - 2\mathbf{j} - \mathbf{k}$
22. $\mathbf{u} = -2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$
23. $\mathbf{u} = \langle 2, -3, 1\rangle$, $\mathbf{v} = \langle -1, -1, -1\rangle$
24. $\mathbf{u} = \langle \cos\theta, \sin\theta, -1\rangle$, $\mathbf{v} = \langle \sin\theta, -\cos\theta, 0\rangle$

Classifying a Triangle In Exercises 25–28, the vertices of a triangle are given. Determine whether the triangle is an acute triangle, an obtuse triangle, or a right triangle. Explain your reasoning.

25. $(1, 2, 0), (0, 0, 0), (-2, 1, 0)$
26. $(-3, 0, 0), (0, 0, 0), (1, 2, 3)$
27. $(2, 0, 1), (0, 1, 2), (-0.5, 1.5, 0)$
28. $(2, -7, 3), (-1, 5, 8), (4, 6, -1)$

Finding Direction Angles In Exercises 29–34, find the direction cosines and angles of $\mathbf{u}$, and demonstrate that the sum of the squares of the direction cosines is 1.

29. $\mathbf{u} = \mathbf{i} + 2\mathbf{j} + 2\mathbf{k}$
30. $\mathbf{u} = 5\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
31. $\mathbf{u} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$
32. $\mathbf{u} = -4\mathbf{i} + 3\mathbf{j} + 5\mathbf{k}$
33. $\mathbf{u} = \langle 0, 6, -4\rangle$
34. $\mathbf{u} = \langle -1, 5, 2\rangle$

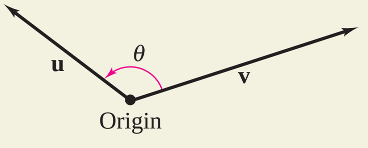
Finding the Projection of u onto v In Exercises 35–42, (a) find the projection of $\mathbf{u}$ onto $\mathbf{v}$, and (b) find the vector component of $\mathbf{u}$ orthogonal to $\mathbf{v}$.

35. $\mathbf{u} = \langle 6, 7\rangle$, $\mathbf{v} = \langle 1, 4\rangle$
36. $\mathbf{u} = \langle 9, 7\rangle$, $\mathbf{v} = \langle 1, 3\rangle$
37. $\mathbf{u} = 2\mathbf{i} + 3\mathbf{j}$, $\mathbf{v} = 5\mathbf{i} + \mathbf{j}$
38. $\mathbf{u} = 2\mathbf{i} - 3\mathbf{j}$, $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j}$
39. $\mathbf{u} = \langle 0, 3, 3\rangle$, $\mathbf{v} = \langle -1, 1, 1\rangle$
40. $\mathbf{u} = \langle 8, 2, 0\rangle$, $\mathbf{v} = \langle 2, 1, -1\rangle$
41. $\mathbf{u} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$, $\mathbf{v} = 3\mathbf{j} + 4\mathbf{k}$
42. $\mathbf{u} = \mathbf{i} + 4\mathbf{k}$, $\mathbf{v} = 3\mathbf{i} + 2\mathbf{k}$

WRITING ABOUT CONCEPTS

43. **Dot Product** Define the dot product of vectors $\mathbf{u}$ and $\mathbf{v}$.
44. **Orthogonal Vectors** State the definition of orthogonal vectors. When vectors are neither parallel nor orthogonal, how do you find the angle between them? Explain.
45. **Using Vectors** Determine which of the following are defined for nonzero vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Explain your reasoning.
    (a) $\mathbf{u}\cdot(\mathbf{v} + \mathbf{w})$ (b) $(\mathbf{u}\cdot\mathbf{v})\mathbf{w}$
    (c) $\mathbf{u}\cdot\mathbf{v} + \mathbf{w}$ (d) $\|\mathbf{u}\|\cdot(\mathbf{v} + \mathbf{w})$
46. **Direction Cosines** Describe direction cosines and direction angles of a vector $\mathbf{v}$.
47. **Projection** Give a geometric description of the projection of $\mathbf{u}$ onto $\mathbf{v}$.
48. **Projection** What can be said about the vectors $\mathbf{u}$ and $\mathbf{v}$ when (a) the projection of $\mathbf{u}$ onto $\mathbf{v}$ equals $\mathbf{u}$ and (b) the projection of $\mathbf{u}$ onto $\mathbf{v}$ equals $\mathbf{0}$?
49. **Projection** When the projection of $\mathbf{u}$ onto $\mathbf{v}$ has the same magnitude as the projection of $\mathbf{v}$ onto $\mathbf{u}$, can you conclude that $\|\mathbf{u}\| = \|\mathbf{v}\|$? Explain.

50. **HOW DO YOU SEE IT?** What is known about θ, the angle between two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$, when
    (a) $\mathbf{u}\cdot\mathbf{v} = 0$? (b) $\mathbf{u}\cdot\mathbf{v} > 0$? (c) $\mathbf{u}\cdot\mathbf{v} < 0$?



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51. Revenue The vector $\mathbf{u} = \langle 3240, 1450, 2235 \rangle$ gives the numbers of hamburgers, chicken sandwiches, and cheeseburgers, respectively, sold at a fast-food restaurant in one week. The vector $\mathbf{v} = \langle 2.25, 2.95, 2.65 \rangle$ gives the prices (in dollars) per unit for the three food items. Find the dot product $\mathbf{u} \cdot \mathbf{v}$, and explain what information it gives.

52. Revenue Repeat Exercise 51 after increasing prices by 4%. Identify the vector operation used to increase prices by 4%.

Orthogonal Vectors **In Exercises 53–56, find two vectors in opposite directions that are orthogonal to the vector u. (The answers are not unique.)**

53. $\mathbf{u} = -\frac{1}{4}\mathbf{i} + \frac{3}{2}\mathbf{j}$

54. $\mathbf{u} = 9\mathbf{i} - 4\mathbf{j}$

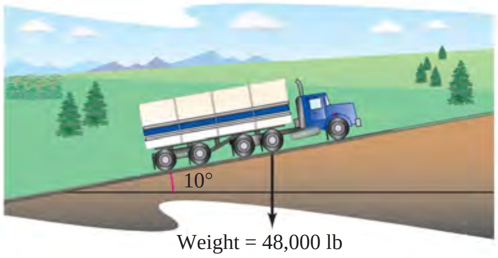
55. $\mathbf{u} = \langle 3, 1, -2 \rangle$

56. $\mathbf{u} = \langle 4, -3, 6 \rangle$

57. Finding an Angle Find the angle between a cube's diagonal and one of its edges.

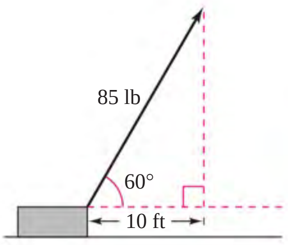
58. Finding an Angle Find the angle between the diagonal of a cube and the diagonal of one of its sides.

59. Braking Load A 48,000-pound truck is parked on a 10° slope (see figure). Assume the only force to overcome is that due to gravity. Find (a) the force required to keep the truck from rolling down the hill and (b) the force perpendicular to the hill.



60. Braking Load A 5400-pound sport utility vehicle is parked on an 18° slope. Assume the only force to overcome is that due to gravity. Find (a) the force required to keep the vehicle from rolling down the hill and (b) the force perpendicular to the hill.

61. Work An object is pulled 10 feet across a floor, using a force of 85 pounds. The direction of the force is 60° above the horizontal (see figure). Find the work done.



Not drawn to scale

Figure for 61

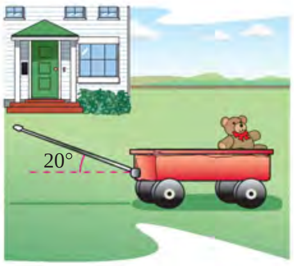


Figure for 62

62. Work A toy wagon is pulled by exerting a force of 25 pounds on a handle that makes a 20° angle with the horizontal (see figure). Find the work done in pulling the wagon 50 feet.

Ziva_K/iStockphoto.com

63. Work A car is towed using a force of 1600 newtons. The chain used to pull the car makes a 25° angle with the horizontal. Find the work done in towing the car 2 kilometers.

64. Work

A sled is pulled by exerting a force of 100 newtons on a rope that makes a 25° angle with the horizontal. Find the work done in pulling the sled 40 meters.

True or False? **In Exercises 65 and 66, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

65. If $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$ and $\mathbf{u} \neq \mathbf{0}$, then $\mathbf{v} = \mathbf{w}$.

66. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to $\mathbf{w}$, then $\mathbf{u} + \mathbf{v}$ is orthogonal to $\mathbf{w}$.

Using Points of Intersection **In Exercises 67–70, (a) find all points of intersection of the graphs of the two equations, (b) find the unit tangent vectors to each curve at their points of intersection, and (c) find the angles ($0° \le \theta \le 90°$) between the curves at their points of intersection.**

67. $y = x^2, \quad y = x^{1/3}$

68. $y = x^3, \quad y = x^{1/3}$

69. $y = 1 - x^2, \quad y = x^2 - 1$

70. $(y + 1)^2 = x, \quad y = x^3 - 1$

71. Proof Use vectors to prove that the diagonals of a rhombus are perpendicular.

72. Proof Use vectors to prove that a parallelogram is a rectangle if and only if its diagonals are equal in length.

73. Bond Angle Consider a regular tetrahedron with vertices $(0, 0, 0)$, $(k, k, 0)$, $(k, 0, k)$, and $(0, k, k)$, where k is a positive real number.

(a) Sketch the graph of the tetrahedron.

(b) Find the length of each edge.

(c) Find the angle between any two edges.

(d) Find the angle between the line segments from the centroid $(k/2, k/2, k/2)$ to two vertices. This is the bond angle for a molecule such as CH_4 or $PbCl_4$, where the structure of the molecule is a tetrahedron.

74. Proof Consider the vectors $\mathbf{u} = \langle \cos\alpha, \sin\alpha, 0 \rangle$ and $\mathbf{v} = \langle \cos\beta, \sin\beta, 0 \rangle$, where $\alpha > \beta$. Find the dot product of the vectors and use the result to prove the identity

$$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

75. Proof Prove that $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\mathbf{u} \cdot \mathbf{v}$.

76. Proof Prove the **Cauchy-Schwarz Inequality,**

$$|\mathbf{u} \cdot \mathbf{v}| \le \|\mathbf{u}\|\,\|\mathbf{v}\|.$$

77. Proof Prove the triangle inequality $\|\mathbf{u} + \mathbf{v}\| \le \|\mathbf{u}\| + \|\mathbf{v}\|$.

78. Proof Prove Theorem 11.6.

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11.4 The Cross Product of Two Vectors in Space

- **Find the cross product of two vectors in space.**
- **Use the triple scalar product of three vectors in space.**

The Cross Product

Many applications in physics, engineering, and geometry involve finding a vector in space that is orthogonal to two given vectors. In this section, you will study a product that will yield such a vector. It is called the **cross product,** and it is most conveniently defined and calculated using the standard unit vector form. Because the cross product yields a vector, it is also called the **vector product.**

Definition of Cross Product of Two Vectors in Space

Let

$$\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \quad \text{and} \quad \mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$$

be vectors in space. The **cross product** of **u** and **v** is the vector

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}.$$

It is important to note that this definition applies only to three-dimensional vectors. The cross product is not defined for two-dimensional vectors.

A convenient way to calculate $\mathbf{u} \times \mathbf{v}$ is to use the *determinant form* with cofactor expansion shown below. (This 3×3 determinant form is used simply to help remember the formula for the cross product—it is technically not a determinant because not all the entries of the corresponding matrix are real numbers.)

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$

Put "**u**" in Row 2.
Put "**v**" in Row 3.

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}\mathbf{k}$$

$$= \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\mathbf{k}$$

$$= (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

Note the minus sign in front of the **j**-component. Each of the three 2×2 determinants can be evaluated by using the diagonal pattern

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$$

Here are a couple of examples.

$$\begin{vmatrix} 2 & 4 \\ 3 & -1 \end{vmatrix} = (2)(-1) - (4)(3) = -2 - 12 = -14$$

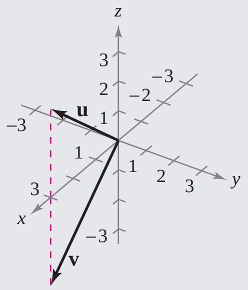
and

$$\begin{vmatrix} 4 & 0 \\ -6 & 3 \end{vmatrix} = (4)(3) - (0)(-6) = 12$$

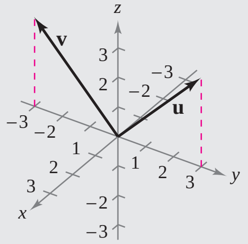
Exploration

Geometric Property of the Cross Product Three pairs of vectors are shown below. Use the definition to find the cross product of each pair. Sketch all three vectors in a three-dimensional system. Describe any relationships among the three vectors. Use your description to write a conjecture about **u**, **v**, and $\mathbf{u} \times \mathbf{v}$.

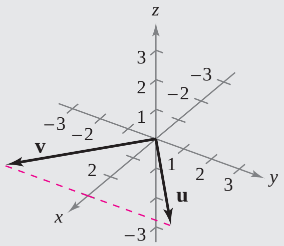
a. $\mathbf{u} = \langle 3, 0, 3\rangle$, $\mathbf{v} = \langle 3, 0, -3\rangle$



b. $\mathbf{u} = \langle 0, 3, 3\rangle$, $\mathbf{v} = \langle 0, -3, 3\rangle$



c. $\mathbf{u} = \langle 3, 3, 0\rangle$, $\mathbf{v} = \langle 3, -3, 0\rangle$



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NOTATION FOR DOT AND CROSS PRODUCTS

The notation for the dot product and cross product of vectors was first introduced by the American physicist Josiah Willard Gibbs (1839–1903). In the early 1880s, Gibbs built a system to represent physical quantities called "vector analysis." The system was a departure from Hamilton's theory of quaternions.

EXAMPLE 1 Finding the Cross Product

For $\mathbf{u} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and $\mathbf{v} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$, find each of the following.

a. $\mathbf{u} \times \mathbf{v}$ **b.** $\mathbf{v} \times \mathbf{u}$ **c.** $\mathbf{v} \times \mathbf{v}$

Solution

a.
$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 3 & 1 & -2 \end{vmatrix}$$
$$= \begin{vmatrix} -2 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 3 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -2 \\ 3 & 1 \end{vmatrix}\mathbf{k}$$
$$= (4 - 1)\mathbf{i} - (-2 - 3)\mathbf{j} + (1 + 6)\mathbf{k}$$
$$= 3\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

b.
$$\mathbf{v} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 1 & -2 & 1 \end{vmatrix}$$
$$= \begin{vmatrix} 1 & -2 \\ -2 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -2 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 1 \\ 1 & -2 \end{vmatrix}\mathbf{k}$$
$$= (1 - 4)\mathbf{i} - (3 + 2)\mathbf{j} + (-6 - 1)\mathbf{k}$$
$$= -3\mathbf{i} - 5\mathbf{j} - 7\mathbf{k}$$

REMARK Note that this result is the negative of that in part (a).

c.
$$\mathbf{v} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 3 & 1 & -2 \end{vmatrix} = \mathbf{0}$$

The results obtained in Example 1 suggest some interesting *algebraic* properties of the cross product. For instance, $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$, and $\mathbf{v} \times \mathbf{v} = \mathbf{0}$. These properties, and several others, are summarized in the next theorem.

THEOREM 11.7 Algebraic Properties of the Cross Product

Let $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ be vectors in space, and let c be a scalar.

1. $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$
2. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$
3. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$
4. $\mathbf{u} \times \mathbf{0} = \mathbf{0} \times \mathbf{u} = \mathbf{0}$
5. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$
6. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

Proof To prove Property 1, let $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$. Then,

$$\mathbf{u} \times \mathbf{v} = (u_2v_3 - u_3v_2)\mathbf{i} - (u_1v_3 - u_3v_1)\mathbf{j} + (u_1v_2 - u_2v_1)\mathbf{k}$$

and

$$\mathbf{v} \times \mathbf{u} = (v_2u_3 - v_3u_2)\mathbf{i} - (v_1u_3 - v_3u_1)\mathbf{j} + (v_1u_2 - v_2u_1)\mathbf{k}$$

which implies that $\mathbf{u} \times \mathbf{v} = -(\mathbf{v} \times \mathbf{u})$. Proofs of Properties 2, 3, 5, and 6 are left as exercises (see Exercises 51–54).

See LarsonCalculus.com for Bruce Edwards's video of this proof.

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Note that Property 1 of Theorem 11.7 indicates that the cross product is *not commutative*. In particular, this property indicates that the vectors $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ have equal lengths but opposite directions. The next theorem lists some other *geometric* properties of the cross product of two vectors.

THEOREM 11.8 Geometric Properties of the Cross Product

Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in space, and let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$.

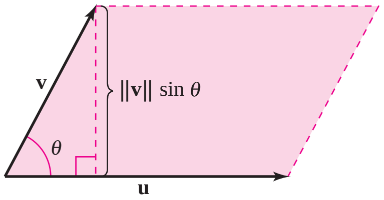
1. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
2. $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\,\|\mathbf{v}\| \sin \theta$
3. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if $\mathbf{u}$ and $\mathbf{v}$ are scalar multiples of each other.
4. $\|\mathbf{u} \times \mathbf{v}\| =$ area of parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides.

REMARK It follows from Properties 1 and 2 in Theorem 11.8 that if $\mathbf{n}$ is a unit vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$, then

$$\mathbf{u} \times \mathbf{v} = \pm(\|\mathbf{u}\|\,\|\mathbf{v}\| \sin \theta)\mathbf{n}.$$

Proof To prove Property 2, note because $\cos \theta = (\mathbf{u} \cdot \mathbf{v})/(\|\mathbf{u}\|\,\|\mathbf{v}\|)$, it follows that

$$\begin{aligned}
\|\mathbf{u}\|\,\|\mathbf{v}\| \sin \theta &= \|\mathbf{u}\|\,\|\mathbf{v}\|\sqrt{1 - \cos^2 \theta} \\
&= \|\mathbf{u}\|\,\|\mathbf{v}\|\sqrt{1 - \frac{(\mathbf{u} \cdot \mathbf{v})^2}{\|\mathbf{u}\|^2\|\mathbf{v}\|^2}} \\
&= \sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2} \\
&= \sqrt{(u_1^2 + u_2^2 + u_3^2)(v_1^2 + v_2^2 + v_3^2) - (u_1v_1 + u_2v_2 + u_3v_3)^2} \\
&= \sqrt{(u_2v_3 - u_3v_2)^2 + (u_1v_3 - u_3v_1)^2 + (u_1v_2 - u_2v_1)^2} \\
&= \|\mathbf{u} \times \mathbf{v}\|.
\end{aligned}$$



The vectors $\mathbf{u}$ and $\mathbf{v}$ form adjacent sides of a parallelogram.
Figure 11.35

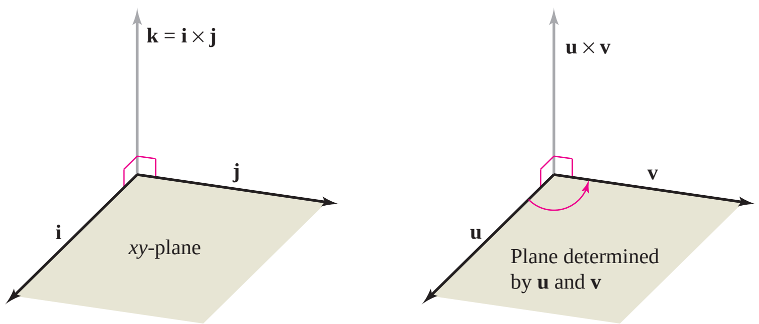
To prove Property 4, refer to Figure 11.35, which is a parallelogram having $\mathbf{v}$ and $\mathbf{u}$ as adjacent sides. Because the height of the parallelogram is $\|\mathbf{v}\| \sin \theta$, the area is

$$\begin{aligned}
\text{Area} &= (\text{base})(\text{height}) \\
&= \|\mathbf{u}\|\,\|\mathbf{v}\| \sin \theta \\
&= \|\mathbf{u} \times \mathbf{v}\|.
\end{aligned}$$

Proofs of Properties 1 and 3 are left as exercises (see Exercises 55 and 56).

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Both $\mathbf{u} \times \mathbf{v}$ and $\mathbf{v} \times \mathbf{u}$ are perpendicular to the plane determined by $\mathbf{u}$ and $\mathbf{v}$. One way to remember the orientations of the vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ is to compare them with the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k} = \mathbf{i} \times \mathbf{j}$, as shown in Figure 11.36. The three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u} \times \mathbf{v}$ form a *right-handed system*, whereas the three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{v} \times \mathbf{u}$ form a *left-handed system*.



Right-handed systems
Figure 11.36

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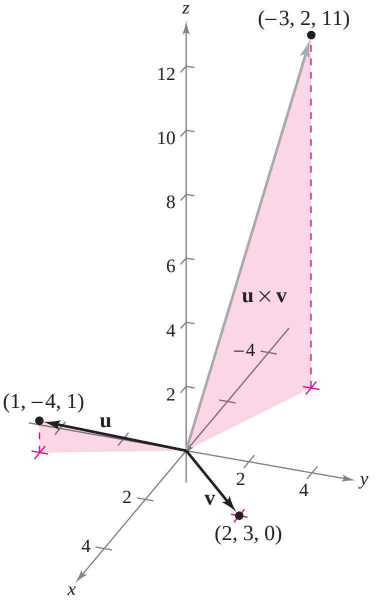
EXAMPLE 2 Using the Cross Product

See LarsonCalculus.com for an interactive version of this type of example.



The vector $\mathbf{u} \times \mathbf{v}$ is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.
Figure 11.37

Find a unit vector that is orthogonal to both

$$\mathbf{u} = \mathbf{i} - 4\mathbf{j} + \mathbf{k}$$

and

$$\mathbf{v} = 2\mathbf{i} + 3\mathbf{j}.$$

Solution The cross product $\mathbf{u} \times \mathbf{v}$, as shown in Figure 11.37, is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -4 & 1 \\ 2 & 3 & 0 \end{vmatrix} \qquad \text{Cross product}$$

$$= -3\mathbf{i} + 2\mathbf{j} + 11\mathbf{k}$$

Because

$$\|\mathbf{u} \times \mathbf{v}\| = \sqrt{(-3)^2 + 2^2 + 11^2} = \sqrt{134}$$

a unit vector orthogonal to both $\mathbf{u}$ and $\mathbf{v}$ is

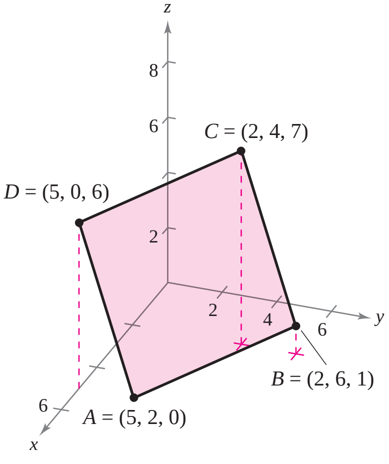
$$\frac{\mathbf{u} \times \mathbf{v}}{\|\mathbf{u} \times \mathbf{v}\|} = -\frac{3}{\sqrt{134}}\mathbf{i} + \frac{2}{\sqrt{134}}\mathbf{j} + \frac{11}{\sqrt{134}}\mathbf{k}.$$

In Example 2, note that you could have used the cross product $\mathbf{v} \times \mathbf{u}$ to form a unit vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$. With that choice, you would have obtained the negative of the unit vector found in the example.

EXAMPLE 3 Geometric Application of the Cross Product

The vertices of a quadrilateral are listed below. Show that the quadrilateral is a parallelogram, and find its area.

$$A = (5, 2, 0) \qquad B = (2, 6, 1)$$
$$C = (2, 4, 7) \qquad D = (5, 0, 6)$$



The area of the parallelogram is approximately 32.19.
Figure 11.38

Solution From Figure 11.38, you can see that the sides of the quadrilateral correspond to the following four vectors.

$$\overrightarrow{AB} = -3\mathbf{i} + 4\mathbf{j} + \mathbf{k} \qquad \overrightarrow{CD} = 3\mathbf{i} - 4\mathbf{j} - \mathbf{k} = -\overrightarrow{AB}$$
$$\overrightarrow{AD} = 0\mathbf{i} - 2\mathbf{j} + 6\mathbf{k} \qquad \overrightarrow{CB} = 0\mathbf{i} + 2\mathbf{j} - 6\mathbf{k} = -\overrightarrow{AD}$$

So, $\overrightarrow{AB}$ is parallel to $\overrightarrow{CD}$ and $\overrightarrow{AD}$ is parallel to $\overrightarrow{CB}$, and you can conclude that the quadrilateral is a parallelogram with $\overrightarrow{AB}$ and $\overrightarrow{AD}$ as adjacent sides. Moreover, because

$$\overrightarrow{AB} \times \overrightarrow{AD} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -3 & 4 & 1 \\ 0 & -2 & 6 \end{vmatrix} \qquad \text{Cross product}$$

$$= 26\mathbf{i} + 18\mathbf{j} + 6\mathbf{k}$$

the area of the parallelogram is

$$\|\overrightarrow{AB} \times \overrightarrow{AD}\| = \sqrt{1036} \approx 32.19.$$

Is the parallelogram a rectangle? You can determine whether it is by finding the angle between the vectors $\overrightarrow{AB}$ and $\overrightarrow{AD}$.

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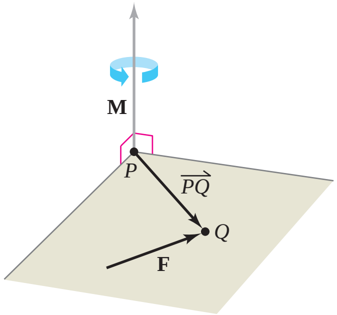
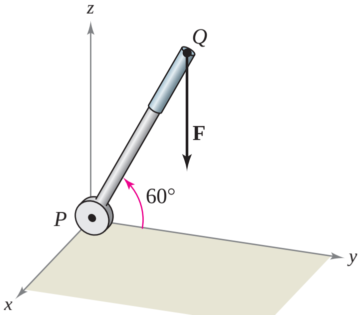


The moment of **F** about *P*

Figure 11.39



A vertical force of 50 pounds is applied at point *Q*.

Figure 11.40

In physics, the cross product can be used to measure **torque**—the **moment M of a force F about a point *P*,** as shown in Figure 11.39. If the point of application of the force is *Q*, then the moment of **F** about *P* is

$$\mathbf{M} = \overrightarrow{PQ} \times \mathbf{F}.$$ Moment of **F** about *P*

The magnitude of the moment **M** measures the tendency of the vector $\overrightarrow{PQ}$ to rotate counterclockwise (using the right-hand rule) about an axis directed along the vector **M**.

EXAMPLE 4 An Application of the Cross Product

A vertical force of 50 pounds is applied to the end of a one-foot lever that is attached to an axle at point *P*, as shown in Figure 11.40. Find the moment of this force about the point *P* when $\theta = 60°$.

Solution Represent the 50-pound force as

$$\mathbf{F} = -50\mathbf{k}$$

and the lever as

$$\overrightarrow{PQ} = \cos(60°)\mathbf{j} + \sin(60°)\mathbf{k} = \frac{1}{2}\mathbf{j} + \frac{\sqrt{3}}{2}\mathbf{k}.$$

The moment of **F** about *P* is

$$\mathbf{M} = \overrightarrow{PQ} \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -50 \end{vmatrix} = -25\mathbf{i}.$$ Moment of **F** about *P*

The magnitude of this moment is 25 foot-pounds. ■

In Example 4, note that the moment (the tendency of the lever to rotate about its axle) is dependent on the angle θ. When $\theta = \pi/2$, the moment is 0. The moment is greatest when $\theta = 0$.

The Triple Scalar Product

For vectors **u**, **v**, and **w** in space, the dot product of **u** and $\mathbf{v} \times \mathbf{w}$

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$$

is called the **triple scalar product,** as defined in Theorem 11.9. The proof of this theorem is left as an exercise (see Exercise 59).

THEOREM 11.9 The Triple Scalar Product

For $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$, $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$, and $\mathbf{w} = w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}$, the triple scalar product is

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

Note that the value of a determinant is multiplied by -1 when two rows are interchanged. After two such interchanges, the value of the determinant will be unchanged. So, the following triple scalar products are equivalent.

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \cdot (\mathbf{w} \times \mathbf{u}) = \mathbf{w} \cdot (\mathbf{u} \times \mathbf{v})$$

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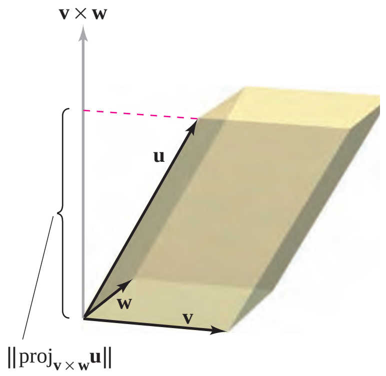


Area of base $= \|\mathbf{v} \times \mathbf{w}\|$
Volume of parallelepiped $= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|$
Figure 11.41

If the vectors **u**, **v**, and **w** do not lie in the same plane, then the triple scalar product $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ can be used to determine the volume of the parallelepiped (a polyhedron, all of whose faces are parallelograms) with **u**, **v**, and **w** as adjacent edges, as shown in Figure 11.41. This is established in the next theorem.

THEOREM 11.10 Geometric Property of the Triple Scalar Product

The volume V of a parallelepiped with vectors **u**, **v**, and **w** as adjacent edges is

$$V = |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|.$$

Proof In Figure 11.41, note that the area of the base is $\|\mathbf{v} \times \mathbf{w}\|$ and the height of the parallelpiped is $\|\text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\|$. Therefore, the volume is

$$\begin{aligned} V &= (\text{height})(\text{area of base}) \\ &= \|\text{proj}_{\mathbf{v}\times\mathbf{w}}\mathbf{u}\| \|\mathbf{v} \times \mathbf{w}\| \\ &= \left|\frac{\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})}{\|\mathbf{v} \times \mathbf{w}\|}\right| \|\mathbf{v} \times \mathbf{w}\| \\ &= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|. \end{aligned}$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 5 Volume by the Triple Scalar Product

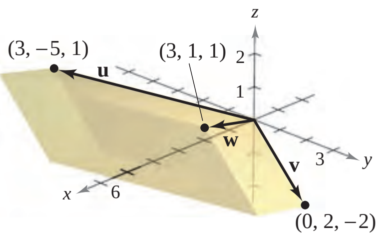
Find the volume of the parallelepiped shown in Figure 11.42 having

$$\mathbf{u} = 3\mathbf{i} - 5\mathbf{j} + \mathbf{k}$$
$$\mathbf{v} = 2\mathbf{j} - 2\mathbf{k}$$

and

$$\mathbf{w} = 3\mathbf{i} + \mathbf{j} + \mathbf{k}$$

as adjacent edges.



The parallelepiped has a volume of 36.
Figure 11.42

Solution By Theorem 11.10, you have

$$\begin{aligned} V &= |\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})| \qquad \text{Triple scalar product} \\ &= \begin{vmatrix} 3 & -5 & 1 \\ 0 & 2 & -2 \\ 3 & 1 & 1 \end{vmatrix} \\ &= 3\begin{vmatrix} 2 & -2 \\ 1 & 1 \end{vmatrix} - (-5)\begin{vmatrix} 0 & -2 \\ 3 & 1 \end{vmatrix} + (1)\begin{vmatrix} 0 & 2 \\ 3 & 1 \end{vmatrix} \\ &= 3(4) + 5(6) + 1(-6) \\ &= 36. \end{aligned}$$

A natural consequence of Theorem 11.10 is that the volume of the parallelepiped is 0 if and only if the three vectors are coplanar. That is, when the vectors $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$, $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$, and $\mathbf{w} = \langle w_1, w_2, w_3 \rangle$ have the same initial point, they lie in the same plane if and only if

$$\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0.$$

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11.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Cross Product of Unit Vectors In Exercises 1–6, find the cross product of the unit vectors and sketch your result.

1. $\mathbf{j} \times \mathbf{i}$
2. $\mathbf{i} \times \mathbf{j}$
3. $\mathbf{j} \times \mathbf{k}$
4. $\mathbf{k} \times \mathbf{j}$
5. $\mathbf{i} \times \mathbf{k}$
6. $\mathbf{k} \times \mathbf{i}$

Finding Cross Products In Exercises 7–10, find (a) $\mathbf{u} \times \mathbf{v}$, (b) $\mathbf{v} \times \mathbf{u}$, and (c) $\mathbf{v} \times \mathbf{v}$.

7. $\mathbf{u} = -2\mathbf{i} + 4\mathbf{j}$, $\mathbf{v} = 3\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$
8. $\mathbf{u} = 3\mathbf{i} + 5\mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$
9. $\mathbf{u} = \langle 7, 3, 2 \rangle$, $\mathbf{v} = \langle 1, -1, 5 \rangle$
10. $\mathbf{u} = \langle 3, -2, -2 \rangle$, $\mathbf{v} = \langle 1, 5, 1 \rangle$

Finding a Cross Product In Exercises 11–16, find $\mathbf{u} \times \mathbf{v}$ and show that it is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

11. $\mathbf{u} = \langle 12, -3, 0 \rangle$, $\mathbf{v} = \langle -2, 5, 0 \rangle$
12. $\mathbf{u} = \langle -1, 1, 2 \rangle$, $\mathbf{v} = \langle 0, 1, 0 \rangle$
13. $\mathbf{u} = \langle 2, -3, 1 \rangle$, $\mathbf{v} = \langle 1, -2, 1 \rangle$
14. $\mathbf{u} = \langle -10, 0, 6 \rangle$, $\mathbf{v} = \langle 5, -3, 0 \rangle$
15. $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = 2\mathbf{i} + \mathbf{j} - \mathbf{k}$
16. $\mathbf{u} = \mathbf{i} + 6\mathbf{j}$, $\mathbf{v} = -2\mathbf{i} + \mathbf{j} + \mathbf{k}$

Finding a Unit Vector In Exercises 17–20, find a unit vector that is orthogonal to both $\mathbf{u}$ and $\mathbf{v}$.

17. $\mathbf{u} = \langle 4, -3, 1 \rangle$, $\mathbf{v} = \langle 2, 5, 3 \rangle$
18. $\mathbf{u} = \langle -8, -6, 4 \rangle$, $\mathbf{v} = \langle 10, -12, -2 \rangle$
19. $\mathbf{u} = -3\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$, $\mathbf{v} = \mathbf{i} - \mathbf{j} + 4\mathbf{k}$
20. $\mathbf{u} = 2\mathbf{k}$, $\mathbf{v} = 4\mathbf{i} + 6\mathbf{k}$

Area In Exercises 21–24, find the area of the parallelogram that has the given vectors as adjacent sides. Use a computer algebra system or a graphing utility to verify your result.

21. $\mathbf{u} = \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$
22. $\mathbf{u} = \mathbf{i} + \mathbf{j} + \mathbf{k}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$
23. $\mathbf{u} = \langle 3, 2, -1 \rangle$, $\mathbf{v} = \langle 1, 2, 3 \rangle$
24. $\mathbf{u} = \langle 2, -1, 0 \rangle$, $\mathbf{v} = \langle -1, 2, 0 \rangle$

Area In Exercises 25 and 26, verify that the points are the vertices of a parallelogram, and find its area.

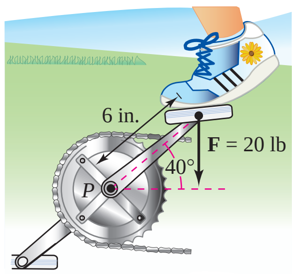
25. $A(0, 3, 2)$, $B(1, 5, 5)$, $C(6, 9, 5)$, $D(5, 7, 2)$
26. $A(2, -3, 1)$, $B(6, 5, -1)$, $C(7, 2, 2)$, $D(3, -6, 4)$

Area In Exercises 27 and 28, find the area of the triangle with the given vertices. $\left(\textit{Hint: } \frac{1}{2}\|\mathbf{u} \times \mathbf{v}\| \text{ is the area of the triangle having } \mathbf{u} \text{ and } \mathbf{v} \text{ as adjacent sides.}\right)$

27. $A(0, 0, 0)$, $B(1, 0, 3)$, $C(-3, 2, 0)$
28. $A(2, -3, 4)$, $B(0, 1, 2)$, $C(-1, 2, 0)$

Elena Elisseeva/Shutterstock.com

29. Torque A child applies the brakes on a bicycle by applying a downward force of 20 pounds on the pedal when the crank makes a 40° angle with the horizontal (see figure). The crank is 6 inches in length. Find the torque at P.



30. Torque Both the magnitude and the direction of the force on a crankshaft change as the crankshaft rotates. Find the torque on the crankshaft using the position and data shown in the figure.

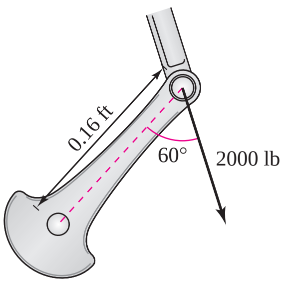


Figure for 30

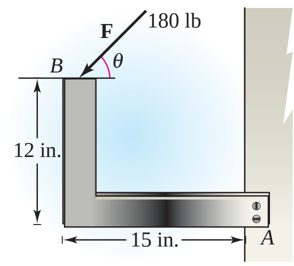


Figure for 31

31. Optimization A force of 180 pounds acts on the bracket shown in the figure.

(a) Determine the vector $\overrightarrow{AB}$ and the vector $\mathbf{F}$ representing the force. ($\mathbf{F}$ will be in terms of θ.)

(b) Find the magnitude of the moment about A by evaluating $\|\overrightarrow{AB} \times \mathbf{F}\|$.

(c) Use the result of part (b) to determine the magnitude of the moment when $\theta = 30°$.

(d) Use the result of part (b) to determine the angle θ when the magnitude of the moment is maximum. At that angle, what is the relationship between the vectors $\mathbf{F}$ and $\overrightarrow{AB}$? Is it what you expected? Why or why not?

(e) Use a graphing utility to graph the function for the magnitude of the moment about A for $0° \le \theta \le 180°$. Find the zero of the function in the given domain. Interpret the meaning of the zero in the context of the problem.

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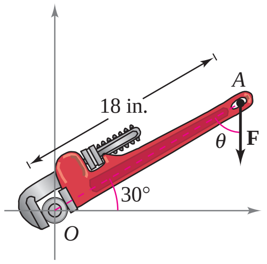
32. Optimization A force of 56 pounds acts on the pipe wrench shown in the figure.



(a) Find the magnitude of the moment about O by evaluating $\|\overrightarrow{OA} \times \mathbf{F}\|$. Use a graphing utility to graph the resulting function of θ.

(b) Use the result of part (a) to determine the magnitude of the moment when $\theta = 45°$.

(c) Use the result of part (a) to determine the angle θ when the magnitude of the moment is maximum. Is the answer what you expected? Why or why not?

Finding a Triple Scalar Product **In Exercises 33–36, find $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$.**

33. $\mathbf{u} = \mathbf{i}$, $\mathbf{v} = \mathbf{j}$, $\mathbf{w} = \mathbf{k}$

34. $\mathbf{u} = \langle 1, 1, 1 \rangle$, $\mathbf{v} = \langle 2, 1, 0 \rangle$, $\mathbf{w} = \langle 0, 0, 1 \rangle$

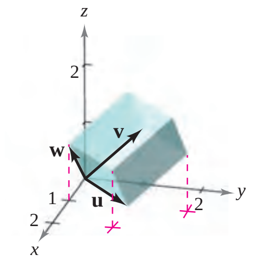
35. $\mathbf{u} = \langle 2, 0, 1 \rangle$, $\mathbf{v} = \langle 0, 3, 0 \rangle$, $\mathbf{w} = \langle 0, 0, 1 \rangle$

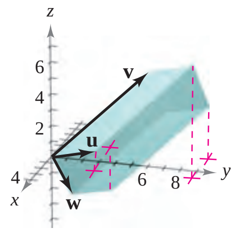
36. $\mathbf{u} = \langle 2, 0, 0 \rangle$, $\mathbf{v} = \langle 1, 1, 1 \rangle$, $\mathbf{w} = \langle 0, 2, 2 \rangle$

Volume **In Exercises 37 and 38, use the triple scalar product to find the volume of the parallelepiped having adjacent edges u, v, and w.**

37. $\mathbf{u} = \mathbf{i} + \mathbf{j}$, $\mathbf{v} = \mathbf{j} + \mathbf{k}$, $\mathbf{w} = \mathbf{i} + \mathbf{k}$

38. $\mathbf{u} = \langle 1, 3, 1 \rangle$, $\mathbf{v} = \langle 0, 6, 6 \rangle$, $\mathbf{w} = \langle -4, 0, -4 \rangle$





Volume **In Exercises 39 and 40, find the volume of the parallelepiped with the given vertices.**

39. (0, 0, 0), (3, 0, 0), (0, 5, 1), (2, 0, 5)
(3, 5, 1), (5, 0, 5), (2, 5, 6), (5, 5, 6)

40. (0, 0, 0), (0, 4, 0), (−3, 0, 0), (−1, 1, 5)
(−3, 4, 0), (−1, 5, 5), (−4, 1, 5), (−4, 5, 5)

41. Comparing Dot Products Identify the dot products that are equal. Explain your reasoning. (Assume **u**, **v**, and **w** are nonzero vectors.)

(a) $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$ (b) $(\mathbf{v} \times \mathbf{w}) \cdot \mathbf{u}$

(c) $(\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ (d) $(\mathbf{u} \times -\mathbf{w}) \cdot \mathbf{v}$

(e) $\mathbf{u} \cdot (\mathbf{w} \times \mathbf{v})$ (f) $\mathbf{w} \cdot (\mathbf{v} \times \mathbf{u})$

(g) $(-\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$ (h) $(\mathbf{w} \times \mathbf{u}) \cdot \mathbf{v}$

42. Using Dot and Cross Products When $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ and $\mathbf{u} \cdot \mathbf{v} = 0$, what can you conclude about **u** and **v**?

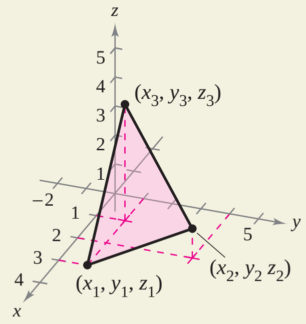
WRITING ABOUT CONCEPTS

43. Cross Product Define the cross product of vectors **u** and **v**.

44. Cross Product State the geometric properties of the cross product.

45. Magnitude When the magnitudes of two vectors are doubled, how will the magnitude of the cross product of the vectors change? Explain.

46. HOW DO YOU SEE IT? The vertices of a triangle in space are (x_1, y_1, z_1), (x_2, y_2, z_2), and (x_3, y_3, z_3). Explain how to find a vector perpendicular to the triangle.



True or False? **In Exercises 47–50, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

47. It is possible to find the cross product of two vectors in a two-dimensional coordinate system.

48. If **u** and **v** are vectors in space that are nonzero and nonparallel, then $\mathbf{u} \times \mathbf{v} = \mathbf{v} \times \mathbf{u}$.

49. If $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

50. If $\mathbf{u} \neq \mathbf{0}$, $\mathbf{u} \cdot \mathbf{v} = \mathbf{u} \cdot \mathbf{w}$, and $\mathbf{u} \times \mathbf{v} = \mathbf{u} \times \mathbf{w}$, then $\mathbf{v} = \mathbf{w}$.

Proof **In Exercises 51–56, prove the property of the cross product.**

51. $\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$

52. $c(\mathbf{u} \times \mathbf{v}) = (c\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (c\mathbf{v})$

53. $\mathbf{u} \times \mathbf{u} = \mathbf{0}$

54. $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \cdot \mathbf{w}$

55. $\mathbf{u} \times \mathbf{v}$ is orthogonal to both **u** and **v**.

56. $\mathbf{u} \times \mathbf{v} = \mathbf{0}$ if and only if **u** and **v** are scalar multiples of each other.

57. Proof Prove that $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\ \|\mathbf{v}\|$ if **u** and **v** are orthogonal.

58. Proof Prove that $\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{u} \cdot \mathbf{v})\mathbf{w}$.

59. Proof Prove Theorem 11.9.

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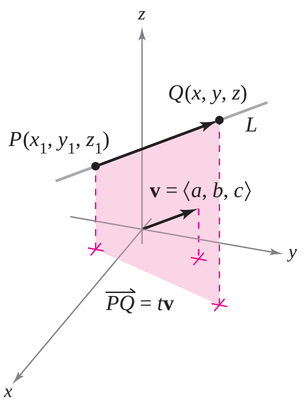
11.5 Lines and Planes in Space

- Write a set of parametric equations for a line in space.
- Write a linear equation to represent a plane in space.
- Sketch the plane given by a linear equation.
- Find the distances between points, planes, and lines in space.

Lines in Space



Line L and its direction vector $\mathbf{v}$

Figure 11.43

In the plane, *slope* is used to determine the equation of a line. In space, it is more convenient to use *vectors* to determine the equation of a line.

In Figure 11.43, consider the line L through the point $P(x_1, y_1, z_1)$ and parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$. The vector $\mathbf{v}$ is a **direction vector** for the line L, and a, b, and c are **direction numbers.** One way of describing the line L is to say that it consists of all points $Q(x, y, z)$ for which the vector $\overrightarrow{PQ}$ is parallel to $\mathbf{v}$. This means that $\overrightarrow{PQ}$ is a scalar multiple of $\mathbf{v}$ and you can write $\overrightarrow{PQ} = t\mathbf{v}$, where t is a scalar (a real number).

$$\overrightarrow{PQ} = \langle x - x_1, y - y_1, z - z_1\rangle = \langle at, bt, ct\rangle = t\mathbf{v}$$

By equating corresponding components, you can obtain **parametric equations** of a line in space.

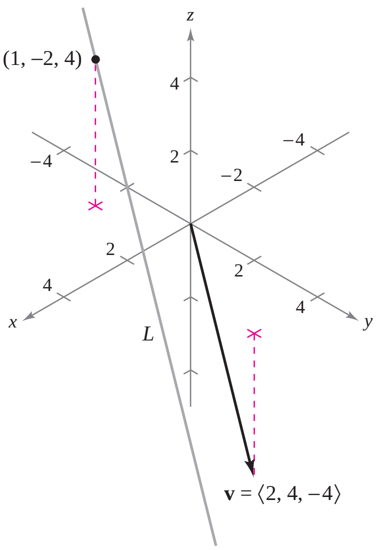
> **THEOREM 11.11 Parametric Equations of a Line in Space**
>
> A line L parallel to the vector $\mathbf{v} = \langle a, b, c\rangle$ and passing through the point $P(x_1, y_1, z_1)$ is represented by the **parametric equations**
>
> $$x = x_1 + at, \quad y = y_1 + bt, \quad \text{and} \quad z = z_1 + ct.$$

If the direction numbers a, b, and c are all nonzero, then you can eliminate the parameter t to obtain **symmetric equations** of the line.

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}$$

Symmetric equations

EXAMPLE 1 Finding Parametric and Symmetric Equations



The vector $\mathbf{v}$ is parallel to the line L.

Figure 11.44

Find parametric and symmetric equations of the line L that passes through the point $(1, -2, 4)$ and is parallel to $\mathbf{v} = \langle 2, 4, -4\rangle$, as shown in Figure 11.44.

Solution To find a set of parametric equations of the line, use the coordinates $x_1 = 1$, $y_1 = -2$, and $z_1 = 4$ and direction numbers $a = 2$, $b = 4$, and $c = -4$.

$$x = 1 + 2t, \quad y = -2 + 4t, \quad z = 4 - 4t$$

Parametric equations

Because a, b, and c are all nonzero, a set of symmetric equations is

$$\frac{x - 1}{2} = \frac{y + 2}{4} = \frac{z - 4}{-4}.$$

Symmetric equations

Neither parametric equations nor symmetric equations of a given line are unique. For instance, in Example 1, by letting $t = 1$ in the parametric equations, you would obtain the point $(3, 2, 0)$. Using this point with the direction numbers $a = 2$, $b = 4$, and $c = -4$ would produce a different set of parametric equations

$$x = 3 + 2t, \quad y = 2 + 4t, \quad \text{and} \quad z = -4t.$$

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EXAMPLE 2 **Parametric Equations of a Line Through Two Points**

See LarsonCalculus.com for an interactive version of this type of example.

Find a set of parametric equations of the line that passes through the points

$$(-2, 1, 0) \quad \text{and} \quad (1, 3, 5).$$

Solution Begin by using the points $P(-2, 1, 0)$ and $Q(1, 3, 5)$ to find a direction vector for the line passing through P and Q.

$$\mathbf{v} = \overrightarrow{PQ} = \langle 1 - (-2), 3 - 1, 5 - 0 \rangle = \langle 3, 2, 5 \rangle = \langle a, b, c \rangle$$

Using the direction numbers $a = 3$, $b = 2$, and $c = 5$ with the point $P(-2, 1, 0)$, you can obtain the parametric equations

$$x = -2 + 3t, \quad y = 1 + 2t, \quad \text{and} \quad z = 5t.$$

REMARK As t varies over all real numbers, the parametric equations in Example 2 determine the points (x, y, z) on the line. In particular, note that $t = 0$ and $t = 1$ give the original points $(-2, 1, 0)$ and $(1, 3, 5)$.

Planes in Space

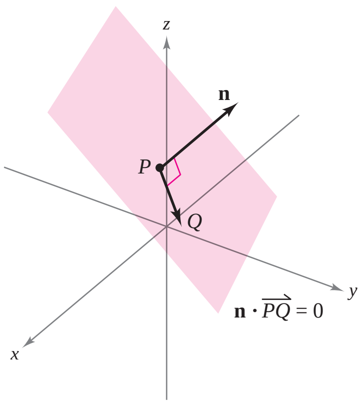
You have seen how an equation of a line in space can be obtained from a point on the line and a vector *parallel* to it. You will now see that an equation of a plane in space can be obtained from a point in the plane and a vector *normal* (perpendicular) to the plane.

Consider the plane containing the point $P(x_1, y_1, z_1)$ having a nonzero normal vector

$$\mathbf{n} = \langle a, b, c \rangle$$

as shown in Figure 11.45. This plane consists of all points $Q(x, y, z)$ for which vector $\overrightarrow{PQ}$ is orthogonal to $\mathbf{n}$. Using the dot product, you can write the following.

$$\mathbf{n} \cdot \overrightarrow{PQ} = 0$$
$$\langle a, b, c \rangle \cdot \langle x - x_1, y - y_1, z - z_1 \rangle = 0$$
$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$



The normal vector $\mathbf{n}$ is orthogonal to each vector $\overrightarrow{PQ}$ in the plane.
Figure 11.45

The third equation of the plane is said to be in **standard form.**

THEOREM 11.12 Standard Equation of a Plane in Space

The plane containing the point (x_1, y_1, z_1) and having normal vector

$$\mathbf{n} = \langle a, b, c \rangle$$

can be represented by the **standard form** of the equation of a plane

$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0.$$

By regrouping terms, you obtain the **general form** of the equation of a plane in space.

$$ax + by + cz + d = 0$$

General form of equation of plane

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Given the general form of the equation of a plane, it is easy to find a normal vector to the plane. Simply use the coefficients of x, y, and z and write

$$\mathbf{n} = \langle a, b, c \rangle.$$

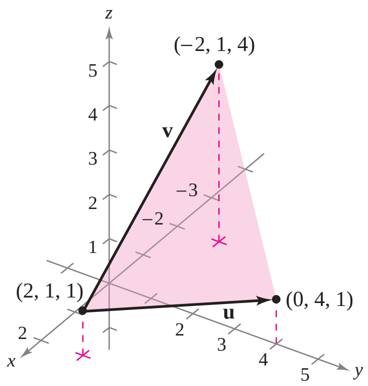
EXAMPLE 3 Finding an Equation of a Plane in Three-Space

Find the general equation of the plane containing the points

$(2, 1, 1)$, $(0, 4, 1)$, and $(-2, 1, 4)$.

Solution To apply Theorem 11.12, you need a point in the plane and a vector that is normal to the plane. There are three choices for the point, but no normal vector is given. To obtain a normal vector, use the cross product of vectors **u** and **v** extending from the point $(2, 1, 1)$ to the points $(0, 4, 1)$ and $(-2, 1, 4)$, as shown in Figure 11.46. The component forms of **u** and **v** are

$$\mathbf{u} = \langle 0 - 2, 4 - 1, 1 - 1 \rangle = \langle -2, 3, 0 \rangle$$
$$\mathbf{v} = \langle -2 - 2, 1 - 1, 4 - 1 \rangle = \langle -4, 0, 3 \rangle$$



A plane determined by **u** and **v**
Figure 11.46

and it follows that

$$\begin{aligned}\mathbf{n} &= \mathbf{u} \times \mathbf{v}\\ &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -2 & 3 & 0 \\ -4 & 0 & 3 \end{vmatrix}\\ &= 9\mathbf{i} + 6\mathbf{j} + 12\mathbf{k}\\ &= \langle a, b, c \rangle\end{aligned}$$

is normal to the given plane. Using the direction numbers for **n** and the point $(x_1, y_1, z_1) = (2, 1, 1)$, you can determine an equation of the plane to be

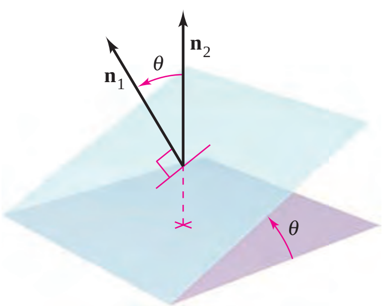
$$a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$$
$$9(x - 2) + 6(y - 1) + 12(z - 1) = 0 \qquad \text{Standard form}$$
$$9x + 6y + 12z - 36 = 0 \qquad \text{General form}$$
$$3x + 2y + 4z - 12 = 0. \qquad \text{Simplified general form}$$

REMARK In Example 3, check to see that each of the three original points satisfies the equation

$$3x + 2y + 4z - 12 = 0.$$

Two distinct planes in three-space either are parallel or intersect in a line. For two planes that intersect, you can determine the angle $(0 \le \theta \le \pi/2)$ between them from the angle between their normal vectors, as shown in Figure 11.47. Specifically, if vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are normal to two intersecting planes, then the angle θ between the normal vectors is equal to the angle between the two planes and is

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|}. \qquad \text{Angle between two planes}$$



The angle θ between two planes
Figure 11.47

Consequently, two planes with normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ are

1. *perpendicular* when $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$.
2. *parallel* when $\mathbf{n}_1$ is a scalar multiple of $\mathbf{n}_2$.

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EXAMPLE 4 Finding the Line of Intersection of Two Planes

Find the angle between the two planes

$$x - 2y + z = 0 \quad \text{and} \quad 2x + 3y - 2z = 0.$$

Then find parametric equations of their line of intersection (see Figure 11.48).

REMARK The three-dimensional rotatable graphs that are available at *LarsonCalculus.com* can help you visualize surfaces such as those shown in Figure 11.48. If you have access to these graphs, you should use them to help your spatial intuition when studying this section and other sections in the text that deal with vectors, curves, or surfaces in space.

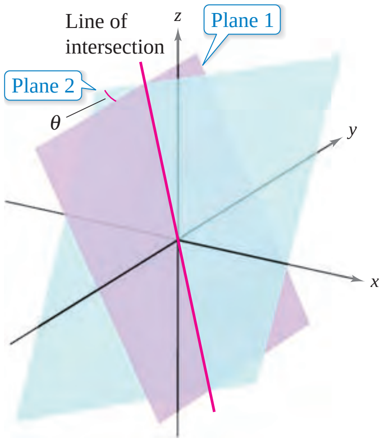

Figure 11.48

Solution Normal vectors for the planes are $\mathbf{n}_1 = \langle 1, -2, 1\rangle$ and $\mathbf{n}_2 = \langle 2, 3, -2\rangle$. Consequently, the angle between the two planes is determined as follows.

$$\cos\theta = \frac{|\mathbf{n}_1 \cdot \mathbf{n}_2|}{\|\mathbf{n}_1\|\,\|\mathbf{n}_2\|} = \frac{|-6|}{\sqrt{6}\,\sqrt{17}} = \frac{6}{\sqrt{102}} \approx 0.59409$$

This implies that the angle between the two planes is $\theta \approx 53.55°$. You can find the line of intersection of the two planes by simultaneously solving the two linear equations representing the planes. One way to do this is to multiply the first equation by -2 and add the result to the second equation.

$$\begin{array}{ll} x - 2y + z = 0 & \Rightarrow \quad -2x + 4y - 2z = 0 \\ 2x + 3y - 2z = 0 & \qquad \underline{2x + 3y - 2z = 0} \\ & \qquad 7y - 4z = 0 \quad \Rightarrow \quad y = \dfrac{4z}{7} \end{array}$$

Substituting $y = 4z/7$ back into one of the original equations, you can determine that $x = z/7$. Finally, by letting $t = z/7$, you obtain the parametric equations

$$x = t, \quad y = 4t, \quad \text{and} \quad z = 7t \qquad \text{Line of intersection}$$

which indicate that 1, 4, and 7 are direction numbers for the line of intersection.

Note that the direction numbers in Example 4 can be obtained from the cross product of the two normal vectors as follows.

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 1 \\ 2 & 3 & -2 \end{vmatrix}$$

$$= \begin{vmatrix} -2 & 1 \\ 3 & -2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 2 & -2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -2 \\ 2 & 3 \end{vmatrix}\mathbf{k}$$

$$= \mathbf{i} + 4\mathbf{j} + 7\mathbf{k}$$

This means that the line of intersection of the two planes is parallel to the cross product of their normal vectors.

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Sketching Planes in Space

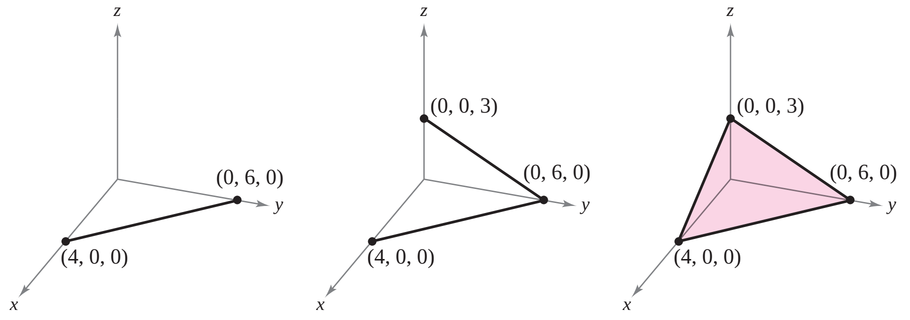
If a plane in space intersects one of the coordinate planes, then the line of intersection is called the **trace** of the given plane in the coordinate plane. To sketch a plane in space, it is helpful to find its points of intersection with the coordinate axes and its traces in the coordinate planes. For example, consider the plane

$3x + 2y + 4z = 12.$ Equation of plane

You can find the *xy*-trace by letting $z = 0$ and sketching the line

$3x + 2y = 12$ *xy*-trace

in the *xy*-plane. This line intersects the *x*-axis at $(4, 0, 0)$ and the *y*-axis at $(0, 6, 0)$. In Figure 11.49, this process is continued by finding the *yz*-trace and the *xz*-trace, and then shading the triangular region lying in the first octant.



xy-trace ($z = 0$): $3x + 2y = 12$

yz-trace ($x = 0$): $2y + 4z = 12$

xz-trace ($y = 0$): $3x + 4z = 12$

Traces of the plane $3x + 2y + 4z = 12$

Figure 11.49

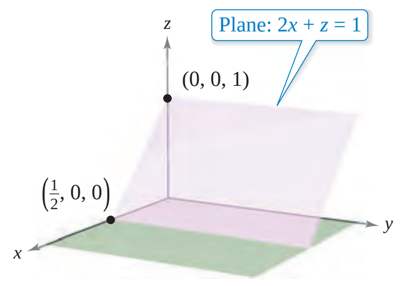
If an equation of a plane has a missing variable, such as

$2x + z = 1$

then the plane must be *parallel to the axis* represented by the missing variable, as shown in Figure 11.50. If two variables are missing from an equation of a plane, such as

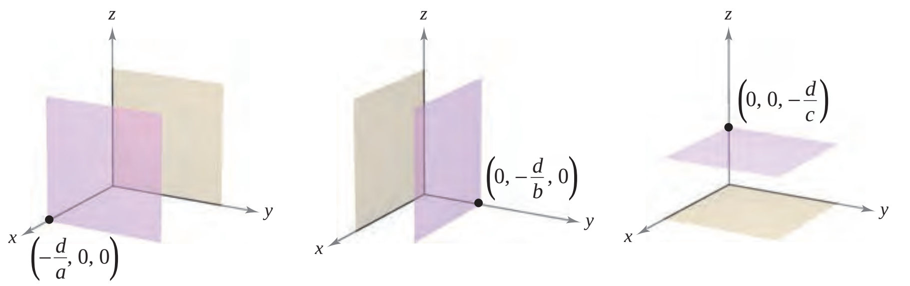
$ax + d = 0$

then it is *parallel to the coordinate plane* represented by the missing variables, as shown in Figure 11.51.



Plane $2x + z = 1$ is parallel to the *y*-axis.

Figure 11.50



Plane $ax + d = 0$ is parallel to the *yz*-plane.

Plane $by + d = 0$ is parallel to the *xz*-plane.

Plane $cz + d = 0$ is parallel to the *xy*-plane.

Figure 11.51

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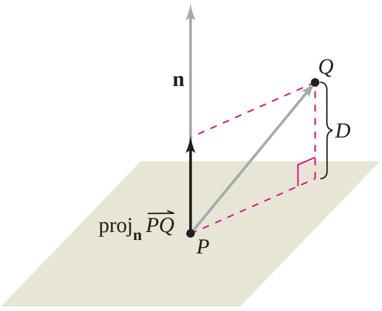

$D = \|\text{proj}_{\mathbf{n}} \overrightarrow{PQ}\|$

The distance between a point and a plane

Figure 11.52

Distances Between Points, Planes, and Lines

Consider two types of problems involving distance in space: (1) finding the distance between a point and a plane, and (2) finding the distance between a point and a line. The solutions of these problems illustrate the versatility and usefulness of vectors in coordinate geometry: the first problem uses the *dot product* of two vectors, and the second problem uses the *cross product.*

The distance D between a point Q and a plane is the length of the shortest line segment connecting Q to the plane, as shown in Figure 11.52. For *any* point P in the plane, you can find this distance by projecting the vector $\overrightarrow{PQ}$ onto the normal vector $\mathbf{n}$. The length of this projection is the desired distance.

THEOREM 11.13 Distance Between a Point and a Plane

The distance between a plane and a point Q (not in the plane) is

$$D = \|\text{proj}_{\mathbf{n}} \overrightarrow{PQ}\| = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|}$$

where P is a point in the plane and $\mathbf{n}$ is normal to the plane.

To find a point in the plane $ax + by + cz + d = 0$, where $a \neq 0$, let $y = 0$ and $z = 0$. Then, from the equation $ax + d = 0$, you can conclude that the point

$$\left(-\frac{d}{a}, 0, 0\right)$$

lies in the plane.

EXAMPLE 5 Finding the Distance Between a Point and a Plane

Find the distance between the point $Q(1, 5, -4)$ and the plane $3x - y + 2z = 6$.

Solution You know that $\mathbf{n} = \langle 3, -1, 2\rangle$ is normal to the plane. To find a point in the plane, let $y = 0$ and $z = 0$, and obtain the point $P(2, 0, 0)$. The vector from P to Q is

$$\overrightarrow{PQ} = \langle 1 - 2, 5 - 0, -4 - 0\rangle = \langle -1, 5, -4\rangle.$$

Using the Distance Formula given in Theorem 11.13 produces

$$D = \frac{|\overrightarrow{PQ} \cdot \mathbf{n}|}{\|\mathbf{n}\|} = \frac{|\langle -1, 5, -4\rangle \cdot \langle 3, -1, 2\rangle|}{\sqrt{9 + 1 + 4}} = \frac{|-3 - 5 - 8|}{\sqrt{14}} = \frac{16}{\sqrt{14}} \approx 4.28.$$

REMARK In the solution to Example 5, note that the choice of the point P is arbitrary. Try choosing a different point in the plane to verify that you obtain the same distance.

From Theorem 11.13, you can determine that the distance between the point $Q(x_0, y_0, z_0)$ and the plane $ax + by + cz + d = 0$ is

$$D = \frac{|a(x_0 - x_1) + b(y_0 - y_1) + c(z_0 - z_1)|}{\sqrt{a^2 + b^2 + c^2}}$$

or

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Distance between a point and a plane

where $P(x_1, y_1, z_1)$ is a point in the plane and $d = -(ax_1 + by_1 + cz_1)$.

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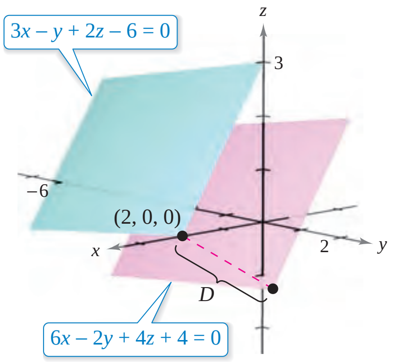


The distance between the parallel planes is approximately 2.14.
Figure 11.53

EXAMPLE 6 Finding the Distance Between Two Parallel Planes

Two parallel planes, $3x - y + 2z - 6 = 0$ and $6x - 2y + 4z + 4 = 0$, are shown in Figure 11.53. To find the distance between the planes, choose a point in the first plane, such as $(x_0, y_0, z_0) = (2, 0, 0)$. Then, from the second plane, you can determine that $a = 6$, $b = -2$, $c = 4$, and $d = 4$, and conclude that the distance is

$$D = \frac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

$$= \frac{|6(2) + (-2)(0) + (4)(0) + 4|}{\sqrt{6^2 + (-2)^2 + 4^2}}$$

$$= \frac{16}{\sqrt{56}} = \frac{8}{\sqrt{14}} \approx 2.14.$$

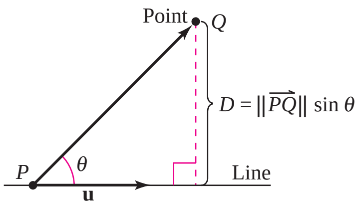
The formula for the distance between a point and a line in space resembles that for the distance between a point and a plane—except that you replace the dot product with the length of the cross product and the normal vector **n** with a direction vector for the line.

THEOREM 11.14 Distance Between a Point and a Line in Space

The distance between a point Q and a line in space is

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|}$$

where **u** is a direction vector for the line and P is a point on the line.



The distance between a point and a line
Figure 11.54

Proof In Figure 11.54, let D be the distance between the point Q and the line. Then $D = \|\overrightarrow{PQ}\| \sin \theta$, where θ is the angle between $\mathbf{u}$ and $\overrightarrow{PQ}$. By Property 2 of Theorem 11.8, you have $\|\mathbf{u}\| \|\overrightarrow{PQ}\| \sin \theta = \|\mathbf{u} \times \overrightarrow{PQ}\| = \|\overrightarrow{PQ} \times \mathbf{u}\|$. Consequently,

$$D = \|\overrightarrow{PQ}\| \sin \theta = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|}.$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 7 Finding the Distance Between a Point and a Line

Find the distance between the point $Q(3, -1, 4)$ and the line

$$x = -2 + 3t, \quad y = -2t, \quad \text{and} \quad z = 1 + 4t.$$

Solution Using the direction numbers 3, -2, and 4, a direction vector for the line is $\mathbf{u} = \langle 3, -2, 4 \rangle$. To find a point on the line, let $t = 0$ and obtain $P = (-2, 0, 1)$. So,

$$\overrightarrow{PQ} = \langle 3 - (-2), -1 - 0, 4 - 1 \rangle = \langle 5, -1, 3 \rangle$$

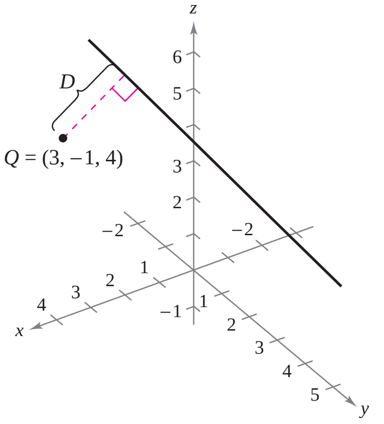
and you can form the cross product

$$\overrightarrow{PQ} \times \mathbf{u} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & -1 & 3 \\ 3 & -2 & 4 \end{vmatrix} = 2\mathbf{i} - 11\mathbf{j} - 7\mathbf{k} = \langle 2, -11, -7 \rangle.$$

Finally, using Theorem 11.14, you can find the distance to be

$$D = \frac{\|\overrightarrow{PQ} \times \mathbf{u}\|}{\|\mathbf{u}\|} = \frac{\sqrt{174}}{\sqrt{29}} = \sqrt{6} \approx 2.45.$$

See Figure 11.55.



The distance between the point Q and the line is $\sqrt{6} \approx 2.45$.
Figure 11.55

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11.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Checking Points on a Line **In Exercises 1 and 2, determine whether each point lies on the line.**

1. $x = -2 + t, y = 3t, z = 4 + t$

(a) $(0, 6, 6)$ (b) $(2, 3, 5)$

2. $\dfrac{x-3}{2} = \dfrac{y-7}{8} = z + 2$

(a) $(7, 23, 0)$ (b) $(1, -1, -3)$

Finding Parametric and Symmetric Equations **In Exercises 3–8, find sets of (a) parametric equations and (b) symmetric equations of the line through the point parallel to the given vector or line (if possible). (For each line, write the direction numbers as integers.)**

	Point	Parallel to
3.	$(0, 0, 0)$	$\mathbf{v} = \langle 3, 1, 5\rangle$
4.	$(0, 0, 0)$	$\mathbf{v} = \left\langle -2, \frac{5}{2}, 1\right\rangle$
5.	$(-2, 0, 3)$	$\mathbf{v} = 2\mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$
6.	$(-3, 0, 2)$	$\mathbf{v} = 6\mathbf{j} + 3\mathbf{k}$
7.	$(1, 0, 1)$	$x = 3 + 3t, y = 5 - 2t, z = -7 + t$
8.	$(-3, 5, 4)$	$\dfrac{x-1}{3} = \dfrac{y+1}{-2} = z - 3$

Finding Parametric and Symmetric Equations **In Exercises 9–12, find sets of (a) parametric equations and (b) symmetric equations of the line through the two points (if possible). (For each line, write the direction numbers as integers.)**

9. $(5, -3, -2), \left(-\frac{2}{3}, \frac{2}{3}, 1\right)$ **10.** $(0, 4, 3), (-1, 2, 5)$

11. $(7, -2, 6), (-3, 0, 6)$ **12.** $(0, 0, 25), (10, 10, 0)$

Finding Parametric Equations **In Exercises 13–20, find a set of parametric equations of the line.**

13. The line passes through the point $(2, 3, 4)$ and is parallel to the xz-plane and the yz-plane.

14. The line passes through the point $(-4, 5, 2)$ and is parallel to the xy-plane and the yz-plane.

15. The line passes through the point $(2, 3, 4)$ and is perpendicular to the plane given by $3x + 2y - z = 6$.

16. The line passes through the point $(-4, 5, 2)$ and is perpendicular to the plane given by $-x + 2y + z = 5$.

17. The line passes through the point $(5, -3, -4)$ and is parallel to $\mathbf{v} = \langle 2, -1, 3\rangle$.

18. The line passes through the point $(-1, 4, -3)$ and is parallel to $\mathbf{v} = 5\mathbf{i} - \mathbf{j}$.

19. The line passes through the point $(2, 1, 2)$ and is parallel to the line $x = -t, y = 1 + t, z = -2 + t$.

20. The line passes through the point $(-6, 0, 8)$ and is parallel to the line $x = 5 - 2t, y = -4 + 2t, z = 0$.

Using Parametric and Symmetric Equations **In Exercises 21–24, find the coordinates of a point *P* on the line and a vector v parallel to the line.**

21. $x = 3 - t, \quad y = -1 + 2t, \quad z = -2$

22. $x = 4t, \quad y = 5 - t, \quad z = 4 + 3t$

23. $\dfrac{x-7}{4} = \dfrac{y+6}{2} = z + 2$ **24.** $\dfrac{x+3}{5} = \dfrac{y}{8} = \dfrac{z-3}{6}$

Determining Parallel Lines **In Exercises 25–28, determine whether any of the lines are parallel or identical.**

25. L_1: $x = 6 - 3t, \quad y = -2 + 2t, \quad z = 5 + 4t$

L_2: $x = 6t, \quad y = 2 - 4t, \quad z = 13 - 8t$

L_3: $x = 10 - 6t, \quad y = 3 + 4t, \quad z = 7 + 8t$

L_4: $x = -4 + 6t, \quad y = 3 + 4t, \quad z = 5 - 6t$

26. L_1: $x = 3 + 2t, \quad y = -6t, \quad z = 1 - 2t$

L_2: $x = 1 + 2t, \quad y = -1 - t, \quad z = 3t$

L_3: $x = -1 + 2t, \quad y = 3 - 10t, \quad z = 1 - 4t$

L_4: $x = 5 + 2t, \quad y = 1 - t, \quad z = 8 + 3t$

27. L_1: $\dfrac{x-8}{4} = \dfrac{y+5}{-2} = \dfrac{z+9}{3}$

L_2: $\dfrac{x+7}{2} = \dfrac{y-4}{1} = \dfrac{z+6}{5}$

L_3: $\dfrac{x+4}{-8} = \dfrac{y-1}{4} = \dfrac{z+18}{-6}$

L_4: $\dfrac{x-2}{-2} = \dfrac{y+3}{1} = \dfrac{z-4}{1.5}$

28. L_1: $\dfrac{x-3}{2} = \dfrac{y-2}{1} = \dfrac{z+2}{2}$

L_2: $\dfrac{x-1}{4} = \dfrac{y-1}{2} = \dfrac{z+3}{4}$

L_3: $\dfrac{x+2}{1} = \dfrac{y-1}{0.5} = \dfrac{z-3}{1}$

L_4: $\dfrac{x-3}{2} = \dfrac{y+1}{4} = \dfrac{z-2}{-1}$

Finding a Point of Intersection **In Exercises 29–32, determine whether the lines intersect, and if so, find the point of intersection and the cosine of the angle of intersection.**

29. $x = 4t + 2, \quad y = 3, \quad z = -t + 1$

$x = 2s + 2, \quad y = 2s + 3, \quad z = s + 1$

30. $x = -3t + 1, \quad y = 4t + 1, \quad z = 2t + 4$

$x = 3s + 1, \quad y = 2s + 4, \quad z = -s + 1$

31. $\dfrac{x}{3} = \dfrac{y-2}{-1} = z + 1, \quad \dfrac{x-1}{4} = y + 2 = \dfrac{z+3}{-3}$

32. $\dfrac{x-2}{-3} = \dfrac{y-2}{6} = z - 3, \quad \dfrac{x-3}{2} = y + 5 = \dfrac{z+2}{4}$

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Checking Points on a Plane **In Exercises 33 and 34, determine whether the plane passes through each point.**

33. $x + 2y - 4z - 1 = 0$

(a) $(-7, 2, -1)$ (b) $(5, 2, 2)$

34. $2x + y + 3z - 6 = 0$

(a) $(3, 6, -2)$ (b) $(-1, 5, -1)$

Finding an Equation of a Plane **In Exercises 35–40, find an equation of the plane passing through the point perpendicular to the given vector or line.**

	Point	Perpendicular to
35.	$(1, 3, -7)$	$\mathbf{n} = \mathbf{j}$
36.	$(0, -1, 4)$	$\mathbf{n} = \mathbf{k}$
37.	$(3, 2, 2)$	$\mathbf{n} = 2\mathbf{i} + 3\mathbf{j} - \mathbf{k}$
38.	$(0, 0, 0)$	$\mathbf{n} = -3\mathbf{i} + 2\mathbf{k}$
39.	$(-1, 4, 0)$	$x = -1 + 2t, y = 5 - t, z = 3 - 2t$
40.	$(3, 2, 2)$	$\frac{x-1}{4} = y + 2 = \frac{z+3}{-3}$

Finding an Equation of a Plane **In Exercises 41–52, find an equation of the plane.**

41. The plane passes through $(0, 0, 0)$, $(2, 0, 3)$, and $(-3, -1, 5)$.

42. The plane passes through $(3, -1, 2)$, $(2, 1, 5)$, and $(1, -2, -2)$.

43. The plane passes through $(1, 2, 3)$, $(3, 2, 1)$, and $(-1, -2, 2)$.

44. The plane passes through the point $(1, 2, 3)$ and is parallel to the yz-plane.

45. The plane passes through the point $(1, 2, 3)$ and is parallel to the xy-plane.

46. The plane contains the y-axis and makes an angle of $\pi/6$ with the positive x-axis.

47. The plane contains the lines given by

$$\frac{x-1}{-2} = y - 4 = z$$

and

$$\frac{x-2}{-3} = \frac{y-1}{4} = \frac{z-2}{-1}.$$

48. The plane passes through the point $(2, 2, 1)$ and contains the line given by

$$\frac{x}{2} = \frac{y-4}{-1} = z.$$

49. The plane passes through the points $(2, 2, 1)$ and $(-1, 1, -1)$ and is perpendicular to the plane $2x - 3y + z = 3$.

50. The plane passes through the points $(3, 2, 1)$ and $(3, 1, -5)$ and is perpendicular to the plane $6x + 7y + 2z = 10$.

51. The plane passes through the points $(1, -2, -1)$ and $(2, 5, 6)$ and is parallel to the x-axis.

52. The plane passes through the points $(4, 2, 1)$ and $(-3, 5, 7)$ and is parallel to the z-axis.

Finding an Equation of a Plane **In Exercises 53–56, find an equation of the plane that contains all the points that are equidistant from the given points.**

53. $(2, 2, 0)$, $(0, 2, 2)$

54. $(1, 0, 2)$, $(2, 0, 1)$

55. $(-3, 1, 2)$, $(6, -2, 4)$

56. $(-5, 1, -3)$, $(2, -1, 6)$

Comparing Planes **In Exercises 57–62, determine whether the planes are parallel, orthogonal, or neither. If they are neither parallel nor orthogonal, find the angle of intersection.**

57. $5x - 3y + z = 4$
$x + 4y + 7z = 1$

58. $3x + y - 4z = 3$
$-9x - 3y + 12z = 4$

59. $x - 3y + 6z = 4$
$5x + y - z = 4$

60. $3x + 2y - z = 7$
$x - 4y + 2z = 0$

61. $x - 5y - z = 1$
$5x - 25y - 5z = -3$

62. $2x - z = 1$
$4x + y + 8z = 10$

Sketching a Graph of a Plane **In Exercises 63–70, sketch a graph of the plane and label any intercepts.**

63. $4x + 2y + 6z = 12$

64. $3x + 6y + 2z = 6$

65. $2x - y + 3z = 4$

66. $2x - y + z = 4$

67. $x + z = 6$

68. $2x + y = 8$

69. $x = 5$

70. $z = 8$

Parallel Planes **In Exercises 71–74, determine whether any of the planes are parallel or identical.**

71. P_1: $-5x + 2y - 8z = 6$
P_2: $15x - 6y + 24z = 17$
P_3: $6x - 4y + 4z = 9$
P_4: $3x - 2y - 2z = 4$

72. P_1: $2x - y + 3z = 8$
P_2: $3x - 5y - 2z = 6$
P_3: $8x - 4y + 12z = 5$
P_4: $-4x - 2y + 6z = 11$

73. P_1: $3x - 2y + 5z = 10$
P_2: $-6x + 4y - 10z = 5$
P_3: $-3x + 2y + 5z = 8$
P_4: $75x - 50y + 125z = 250$

74. P_1: $-60x + 90y + 30z = 27$
P_2: $6x - 9y - 3z = 2$
P_3: $-20x + 30y + 10z = 9$
P_4: $12x - 18y + 6z = 5$

Intersection of Planes **In Exercises 75 and 76, (a) find the angle between the two planes, and (b) find a set of parametric equations for the line of intersection of the planes.**

75. $3x + 2y - z = 7$
$x - 4y + 2z = 0$

76. $6x - 3y + z = 5$
$-x + y + 5z = 5$

Intersection of a Plane and a Line **In Exercises 77–80, find the point(s) of intersection (if any) of the plane and the line. Also, determine whether the line lies in the plane.**

77. $2x - 2y + z = 12, \quad x - \frac{1}{2} = \frac{y + (3/2)}{-1} = \frac{z+1}{2}$

78. $2x + 3y = -5, \quad \frac{x-1}{4} = \frac{y}{2} = \frac{z-3}{6}$

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79. $2x + 3y = 10, \quad \dfrac{x-1}{3} = \dfrac{y+1}{-2} = z - 3$

80. $5x + 3y = 17, \quad \dfrac{x-4}{2} = \dfrac{y+1}{-3} = \dfrac{z+2}{5}$

Finding the Distance Between a Point and a Plane **In Exercises 81–84, find the distance between the point and the plane.**

81. $(0, 0, 0)$
$2x + 3y + z = 12$

82. $(0, 0, 0)$
$5x + y - z = 9$

83. $(2, 8, 4)$
$2x + y + z = 5$

84. $(1, 3, -1)$
$3x - 4y + 5z = 6$

Finding the Distance Between Two Parallel Planes **In Exercises 85–88, verify that the two planes are parallel, and find the distance between the planes.**

85. $x - 3y + 4z = 10$
$x - 3y + 4z = 6$

86. $4x - 4y + 9z = 7$
$4x - 4y + 9z = 18$

87. $-3x + 6y + 7z = 1$
$6x - 12y - 14z = 25$

88. $2x - 4z = 4$
$2x - 4z = 10$

Finding the Distance Between a Point and a Line **In Exercises 89–92, find the distance between the point and the line given by the set of parametric equations.**

89. $(1, 5, -2); \quad x = 4t - 2, \quad y = 3, \quad z = -t + 1$

90. $(1, -2, 4); \quad x = 2t, \quad y = t - 3, \quad z = 2t + 2$

91. $(-2, 1, 3); \quad x = 1 - t, \quad y = 2 + t, \quad z = -2t$

92. $(4, -1, 5); \quad x = 3, \quad y = 1 + 3t, \quad z = 1 + t$

Finding the Distance Between Two Parallel Lines **In Exercises 93 and 94, verify that the lines are parallel, and find the distance between them.**

93. L_1: $x = 2 - t, \quad y = 3 + 2t, \quad z = 4 + t$
L_2: $x = 3t, \quad y = 1 - 6t, \quad z = 4 - 3t$

94. L_1: $x = 3 + 6t, \quad y = -2 + 9t, \quad z = 1 - 12t$
L_2: $x = -1 + 4t, \quad y = 3 + 6t, \quad z = -8t$

WRITING ABOUT CONCEPTS

95. Parametric and Symmetric Equations Give the parametric equations and the symmetric equations of a line in space. Describe what is required to find these equations.

96. Standard Equation of a Plane in Space Give the standard equation of a plane in space. Describe what is required to find this equation.

97. Intersection of Two Planes Describe a method of finding the line of intersection of two planes.

98. Parallel and Perpendicular Planes Describe a method for determining when two planes, $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$, are (a) parallel and (b) perpendicular. Explain your reasoning.

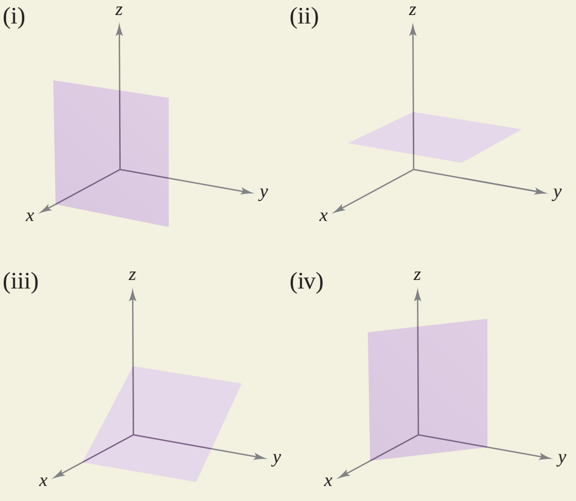
WRITING ABOUT CONCEPTS (continued)

99. Normal Vector Let L_1 and L_2 be nonparallel lines that do not intersect. Is it possible to find a nonzero vector **v** such that **v** is normal to both L_1 and L_2? Explain your reasoning.

100. HOW DO YOU SEE IT? Match the general equation with its graph. Then state what axis or plane the equation is parallel to.

(a) $ax + by + d = 0$ (b) $ax + d = 0$

(c) $cz + d = 0$ (d) $ax + cz + d = 0$



101. Modeling Data Personal consumption expenditures (in billions of dollars) for several types of recreation from 2005 through 2010 are shown in the table, where x is the expenditures on amusement parks and campgrounds, y is the expenditures on live entertainment (excluding sports), and z is the expenditures on spectator sports. *(Source: U.S. Bureau of Economic Analysis)*

Year	2005	2006	2007	2008	2009	2010
x	36.4	39.0	42.4	44.7	43.0	45.2
y	15.3	16.6	17.4	17.5	17.0	17.3
z	16.4	18.1	20.0	20.5	20.1	21.4

A model for the data is given by

$$0.46x + 0.30y - z = 4.94.$$

(a) Complete a fourth row in the table using the model to approximate z for the given values of x and y. Compare the approximations with the actual values of z.

(b) According to this model, increases in expenditures on recreation types x and y would correspond to what kind of change in expenditures on recreation type z?

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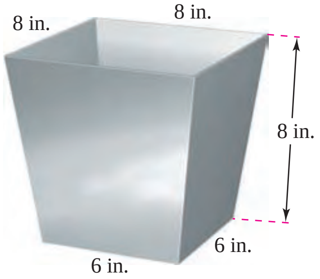
102. Mechanical Design The figure shows a chute at the top of a grain elevator of a combine that funnels the grain into a bin. Find the angle between two adjacent sides.



103. Distance Two insects are crawling along different lines in three-space. At time t (in minutes), the first insect is at the point (x, y, z) on the line $x = 6 + t$, $y = 8 - t$, $z = 3 + t$. Also, at time t, the second insect is at the point (x, y, z) on the line $x = 1 + t$, $y = 2 + t$, $z = 2t$. Assume that distances are given in inches.

(a) Find the distance between the two insects at time $t = 0$.

(b) Use a graphing utility to graph the distance between the insects from $t = 0$ to $t = 10$.

(c) Using the graph from part (b), what can you conclude about the distance between the insects?

(d) How close to each other do the insects get?

104. Finding an Equation of a Sphere Find the standard equation of the sphere with center $(-3, 2, 4)$ that is tangent to the plane given by $2x + 4y - 3z = 8$.

105. Finding a Point of Intersection Find the point of intersection of the plane $3x - y + 4z = 7$ and the line through $(5, 4, -3)$ that is perpendicular to this plane.

106. Finding the Distance Between a Plane and a Line Show that the plane $2x - y - 3z = 4$ is parallel to the line $x = -2 + 2t$, $y = -1 + 4t$, $z = 4$, and find the distance between them.

107. Finding a Point of Intersection Find the point of intersection of the line through $(1, -3, 1)$ and $(3, -4, 2)$ and the plane given by $x - y + z = 2$.

108. Finding Parametric Equations Find a set of parametric equations for the line passing through the point $(1, 0, 2)$ that is parallel to the plane given by $x + y + z = 5$ and perpendicular to the line $x = t$, $y = 1 + t$, $z = 1 + t$.

True or False? **In Exercises 109–114, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

109. If $\mathbf{v} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ is any vector in the plane given by $a_2x + b_2y + c_2z + d_2 = 0$, then $a_1a_2 + b_1b_2 + c_1c_2 = 0$.

110. Every two lines in space are either intersecting or parallel.

111. Two planes in space are either intersecting or parallel.

112. If two lines L_1 and L_2 are parallel to a plane P, then L_1 and L_2 are parallel.

113. Two planes perpendicular to a third plane in space are parallel.

114. A plane and a line in space are either intersecting or parallel.

SECTION PROJECT

Distances in Space

You have learned two distance formulas in this section—one for the distance between a point and a plane, and one for the distance between a point and a line. In this project, you will study a third distance problem—the distance between two skew lines. Two lines in space are *skew* if they are neither parallel nor intersecting (see figure).

(a) Consider the following two lines in space.

L_1: $x = 4 + 5t$, $y = 5 + 5t$, $z = 1 - 4t$

L_2: $x = 4 + s$, $y = -6 + 8s$, $z = 7 - 3s$

(i) Show that these lines are not parallel.

(ii) Show that these lines do not intersect, and therefore are skew lines.

(iii) Show that the two lines lie in parallel planes.

(iv) Find the distance between the parallel planes from part (iii). This is the distance between the original skew lines.

(b) Use the procedure in part (a) to find the distance between the lines.

L_1: $x = 2t$, $y = 4t$, $z = 6t$

L_2: $x = 1 - s$, $y = 4 + s$, $z = -1 + s$

(c) Use the procedure in part (a) to find the distance between the lines.

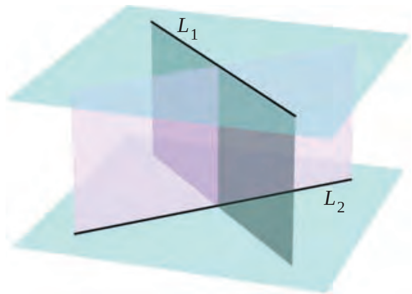
L_1: $x = 3t$, $y = 2 - t$, $z = -1 + t$

L_2: $x = 1 + 4s$, $y = -2 + s$, $z = -3 - 3s$

(d) Develop a formula for finding the distance between the skew lines.

L_1: $x = x_1 + a_1t$, $y = y_1 + b_1t$, $z = z_1 + c_1t$

L_2: $x = x_2 + a_2s$, $y = y_2 + b_2s$, $z = z_2 + c_2s$



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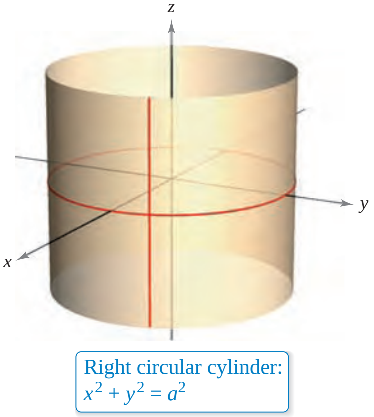
11.6 Surfaces in Space

- Recognize and write equations of cylindrical surfaces.
- Recognize and write equations of quadric surfaces.
- Recognize and write equations of surfaces of revolution.

Cylindrical Surfaces

The first five sections of this chapter contained the vector portion of the preliminary work necessary to study vector calculus and the calculus of space. In this and the next section, you will study surfaces in space and alternative coordinate systems for space. You have already studied two special types of surfaces.

1. Spheres: $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$ Section 11.2
2. Planes: $ax + by + cz + d = 0$ Section 11.5



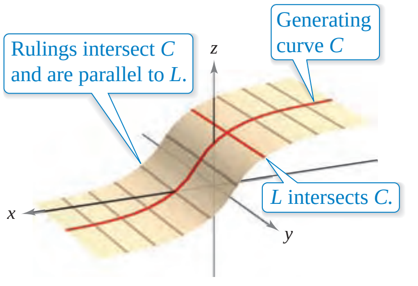
Rulings are parallel to z-axis
Figure 11.56

A third type of surface in space is a **cylindrical surface,** or simply a **cylinder.** To define a cylinder, consider the familiar right circular cylinder shown in Figure 11.56. The cylinder was generated by a vertical line moving around the circle $x^2 + y^2 = a^2$ in the xy-plane. This circle is a **generating curve** for the cylinder, as indicated in the next definition.

Definition of a Cylinder

Let C be a curve in a plane and let L be a line not in a parallel plane. The set of all lines parallel to L and intersecting C is a **cylinder**. The curve C is the **generating curve** (or **directrix**) of the cylinder, and the parallel lines are **rulings.**

Without loss of generality, you can assume that C lies in one of the three coordinate planes. Moreover, this text restricts the discussion to *right* cylinders—cylinders whose rulings are perpendicular to the coordinate plane containing C, as shown in Figure 11.57. Note that the rulings intersect C and are parallel to the line L.



Right cylinder: A cylinder whose rulings are perpendicular to the coordinate plane containing C
Figure 11.57

For the right circular cylinder shown in Figure 11.56, the equation of the generating curve in the xy-plane is

$$x^2 + y^2 = a^2.$$

To find an equation of the cylinder, note that you can generate any one of the rulings by fixing the values of x and y and then allowing z to take on all real values. In this sense, the value of z is arbitrary and is, therefore, not included in the equation. In other words, the equation of this cylinder is simply the equation of its generating curve.

$$x^2 + y^2 = a^2$$ Equation of cylinder in space

Equations of Cylinders

The equation of a cylinder whose rulings are parallel to one of the coordinate axes contains only the variables corresponding to the other two axes.

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EXAMPLE 1 **Sketching a Cylinder**

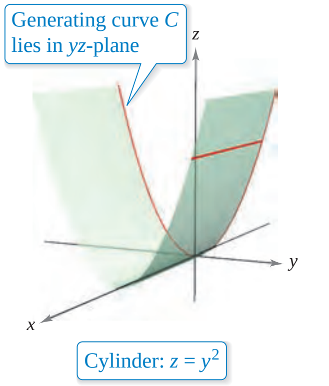
Sketch the surface represented by each equation.

a. $z = y^2$ **b.** $z = \sin x, \quad 0 \le x \le 2\pi$

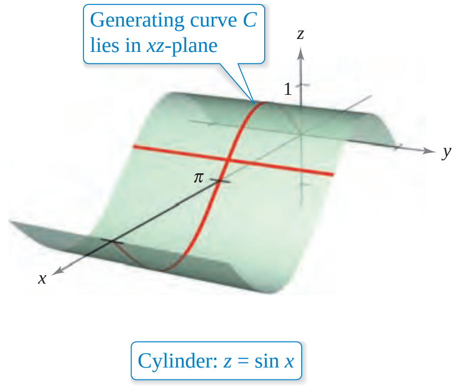
Solution

a. The graph is a cylinder whose generating curve, $z = y^2$, is a parabola in the yz-plane. The rulings of the cylinder are parallel to the x-axis, as shown in Figure 11.58(a).

b. The graph is a cylinder generated by the sine curve in the xz-plane. The rulings are parallel to the y-axis, as shown in Figure 11.58(b).



(a) Rulings are parallel to x-axis.



(b) Rulings are parallel to y-axis.

Figure 11.58

Quadric Surfaces

The fourth basic type of surface in space is a **quadric surface.** Quadric surfaces are the three-dimensional analogs of conic sections.

Quadric Surface

The equation of a **quadric surface** in space is a second-degree equation in three variables. The **general form** of the equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz + Gx + Hy + Iz + J = 0.$$

There are six basic types of quadric surfaces: **ellipsoid, hyperboloid of one sheet, hyperboloid of two sheets, elliptic cone, elliptic paraboloid, and hyperbolic paraboloid.**

The intersection of a surface with a plane is called the **trace of the surface** in the plane. To visualize a surface in space, it is helpful to determine its traces in some well-chosen planes. The traces of quadric surfaces are conics. These traces, together with the **standard form** of the equation of each quadric surface, are shown in the table on the next two pages.

In the table on the next two pages, only one of several orientations of each quadric surface is shown. When the surface is oriented along a different axis, its standard equation will change accordingly, as illustrated in Examples 2 and 3. The fact that the two types of paraboloids have one variable raised to the first power can be helpful in classifying quadric surfaces. The other four types of basic quadric surfaces have equations that are of *second degree* in all three variables.

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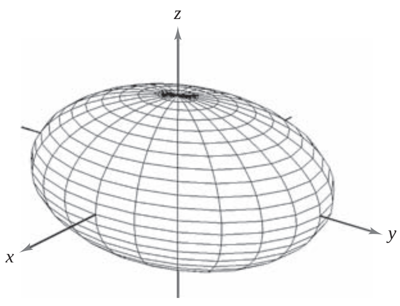
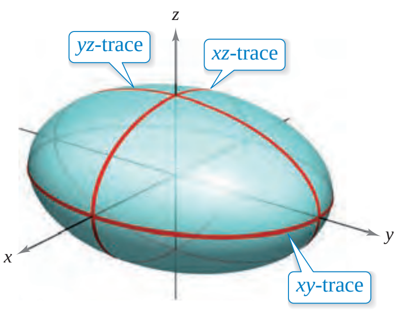


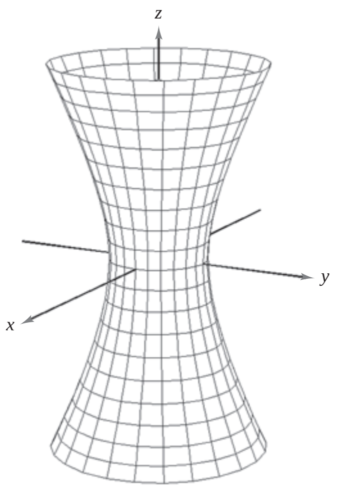
Ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$$

Trace	**Plane**
Ellipse	Parallel to *xy*-plane
Ellipse	Parallel to *xz*-plane
Ellipse	Parallel to *yz*-plane

The surface is a sphere when $a = b = c \neq 0$.



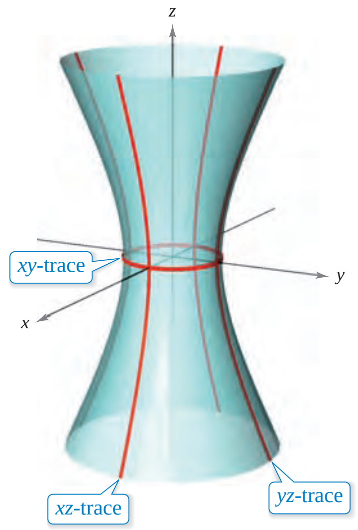


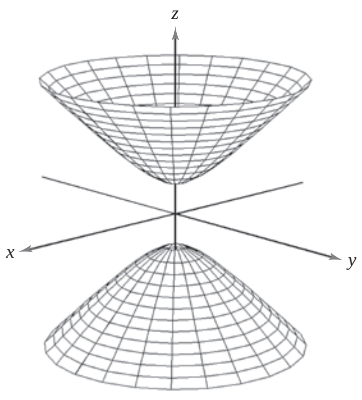
Hyperboloid of One Sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

Trace	**Plane**
Ellipse	Parallel to *xy*-plane
Hyperbola	Parallel to *xz*-plane
Hyperbola	Parallel to *yz*-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is negative.



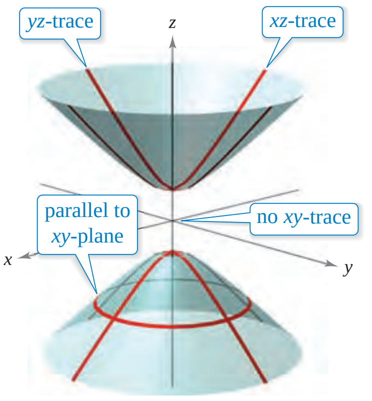


Hyperboloid of Two Sheets

$$\frac{z^2}{c^2} - \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Trace	**Plane**
Ellipse	Parallel to *xy*-plane
Hyperbola	Parallel to *xz*-plane
Hyperbola	Parallel to *yz*-plane

The axis of the hyperboloid corresponds to the variable whose coefficient is positive. There is no trace in the coordinate plane perpendicular to this axis.



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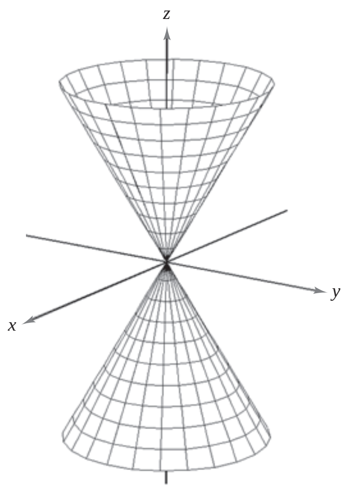
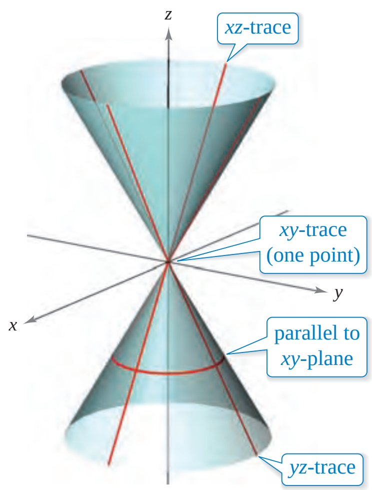


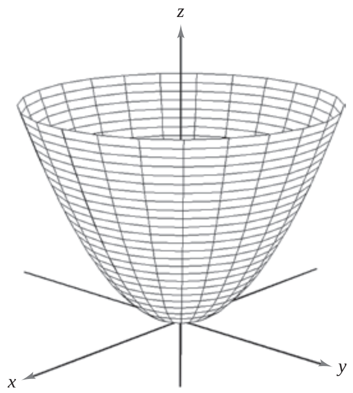
Elliptic Cone

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$$

Trace	**Plane**
Ellipse	Parallel to *xy*-plane
Hyperbola	Parallel to *xz*-plane
Hyperbola	Parallel to *yz*-plane

The axis of the cone corresponds to the variable whose coefficient is negative. The traces in the coordinate planes parallel to this axis are intersecting lines.



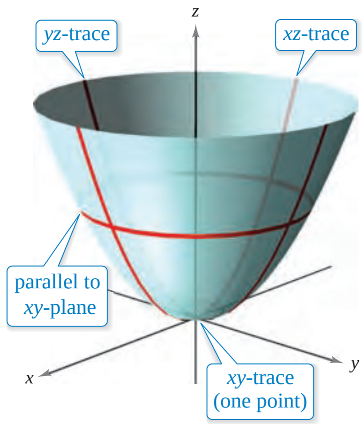


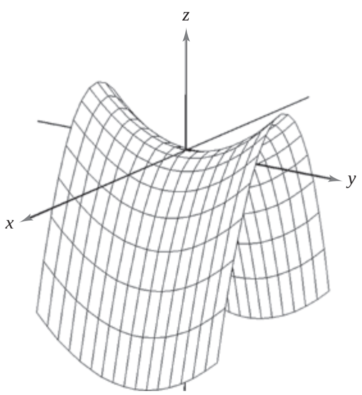
Elliptic Paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

Trace	**Plane**
Ellipse	Parallel to *xy*-plane
Parabola	Parallel to *xz*-plane
Parabola	Parallel to *yz*-plane

The axis of the paraboloid corresponds to the variable raised to the first power.



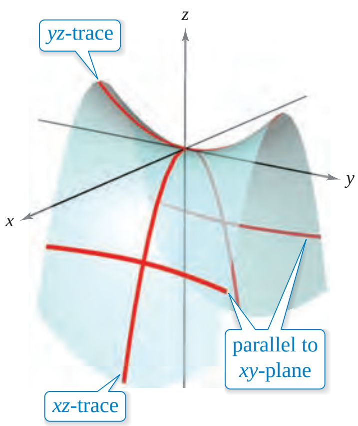


Hyperbolic Paraboloid

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

Trace	**Plane**
Hyperbola	Parallel to *xy*-plane
Parabola	Parallel to *xz*-plane
Parabola	Parallel to *yz*-plane

The axis of the paraboloid corresponds to the variable raised to the first power.



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To classify a quadric surface, begin by writing the equation of the surface in standard form. Then, determine several traces taken in the coordinate planes *or* taken in planes that are parallel to the coordinate planes.

EXAMPLE 2 Sketching a Quadric Surface

Classify and sketch the surface

$$4x^2 - 3y^2 + 12z^2 + 12 = 0.$$

Solution Begin by writing the equation in standard form.

$$4x^2 - 3y^2 + 12z^2 + 12 = 0 \qquad \text{Write original equation.}$$

$$\frac{x^2}{-3} + \frac{y^2}{4} - z^2 - 1 = 0 \qquad \text{Divide by } -12.$$

$$\frac{y^2}{4} - \frac{x^2}{3} - \frac{z^2}{1} = 1 \qquad \text{Standard form}$$

From the table on pages 796 and 797, you can conclude that the surface is a hyperboloid of two sheets with the y-axis as its axis. To sketch the graph of this surface, it helps to find the traces in the coordinate planes.

$$xy\text{-trace } (z = 0): \quad \frac{y^2}{4} - \frac{x^2}{3} = 1 \qquad \text{Hyperbola}$$

$$xz\text{-trace } (y = 0): \quad \frac{x^2}{3} + \frac{z^2}{1} = -1 \qquad \text{No trace}$$

$$yz\text{-trace } (x = 0): \quad \frac{y^2}{4} - \frac{z^2}{1} = 1 \qquad \text{Hyperbola}$$

The graph is shown in Figure 11.59.

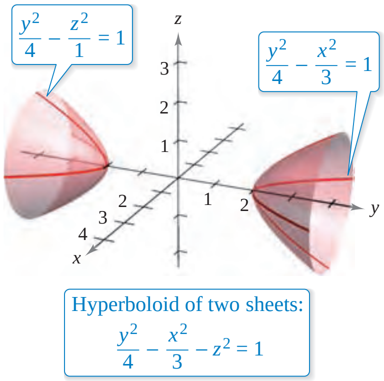

Figure 11.59

EXAMPLE 3 Sketching a Quadric Surface

Classify and sketch the surface

$$x - y^2 - 4z^2 = 0.$$

Solution Because x is raised only to the first power, the surface is a paraboloid. The axis of the paraboloid is the x-axis. In standard form, the equation is

$$x = y^2 + 4z^2. \qquad \text{Standard form}$$

Some convenient traces are listed below.

$$xy\text{-trace } (z = 0): \quad x = y^2 \qquad \text{Parabola}$$

$$xz\text{-trace } (y = 0): \quad x = 4z^2 \qquad \text{Parabola}$$

$$\text{parallel to } yz\text{-plane } (x = 4): \quad \frac{y^2}{4} + \frac{z^2}{1} = 1 \qquad \text{Ellipse}$$

The surface is an *elliptic* paraboloid, as shown in Figure 11.60.

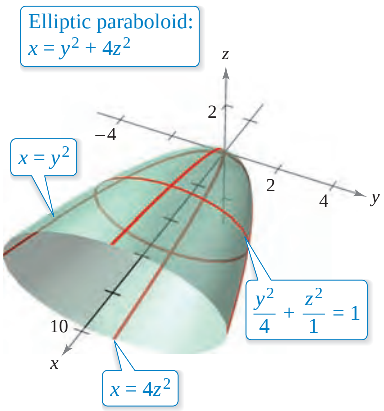

Figure 11.60

Some second-degree equations in x, y, and z do not represent any of the basic types of quadric surfaces. For example, the graph of

$$x^2 + y^2 + z^2 = 0 \qquad \text{Single point}$$

is a single point, and the graph of

$$x^2 + y^2 = 1 \qquad \text{Right circular cylinder}$$

is a right circular cylinder.

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For a quadric surface not centered at the origin, you can form the standard equation by completing the square, as demonstrated in Example 4.

EXAMPLE 4 A Quadric Surface Not Centered at the Origin

See LarsonCalculus.com for an interactive version of this type of example.

Classify and sketch the surface

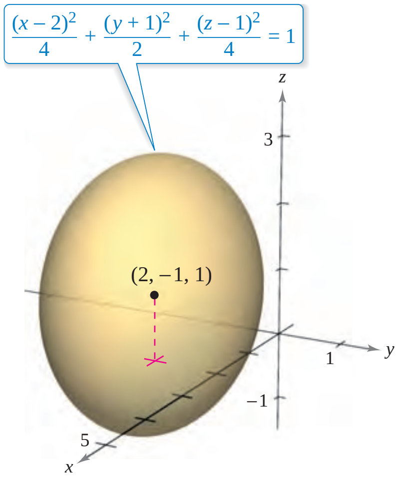
$$x^2 + 2y^2 + z^2 - 4x + 4y - 2z + 3 = 0.$$

Solution Begin by grouping terms and factoring where possible.

$$x^2 - 4x + 2(y^2 + 2y) + z^2 - 2z = -3$$

Next, complete the square for each variable and write the equation in standard form.

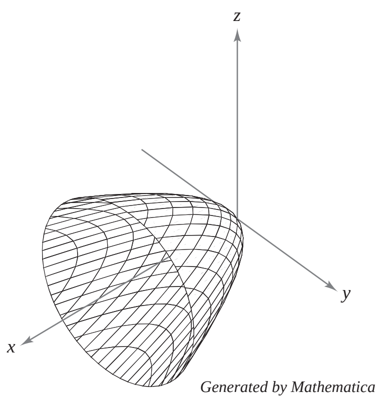
$$(x^2 - 4x + \quad) + 2(y^2 + 2y + \quad) + (z^2 - 2z + \quad) = -3$$
$$(x^2 - 4x + 4) + 2(y^2 + 2y + 1) + (z^2 - 2z + 1) = -3 + 4 + 2 + 1$$
$$(x - 2)^2 + 2(y + 1)^2 + (z - 1)^2 = 4$$
$$\frac{(x - 2)^2}{4} + \frac{(y + 1)^2}{2} + \frac{(z - 1)^2}{4} = 1$$



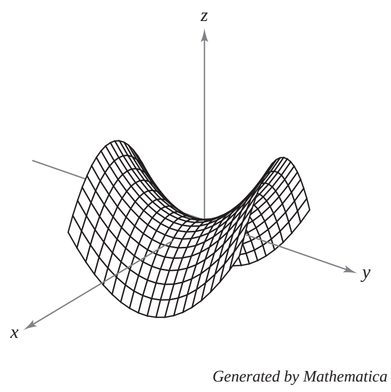
An ellipsoid centered at $(2, -1, 1)$
Figure 11.61

From this equation, you can see that the quadric surface is an ellipsoid that is centered at $(2, -1, 1)$. Its graph is shown in Figure 11.61.

TECHNOLOGY A 3-D graphing utility can help you visualize a surface in space.* Such a graphing utility may create a three-dimensional graph by sketching several traces of the surface and then applying a "hidden-line" routine that blocks out portions of the surface that lie behind other portions of the surface. Two examples of figures that were generated by *Mathematica* are shown below.



Elliptic paraboloid
$x = \frac{y^2}{2} + \frac{z^2}{2}$



Hyperbolic paraboloid
$z = \frac{y^2}{16} - \frac{x^2}{16}$

Using a graphing utility to graph a surface in space requires practice. For one thing, you must know enough about the surface to be able to specify a *viewing window* that gives a representative view of the surface. Also, you can often improve the view of a surface by rotating the axes. For instance, note that the elliptic paraboloid in the figure is seen from a line of sight that is "higher" than the line of sight used to view the hyperbolic paraboloid.

* Some 3-D graphing utilities require surfaces to be entered with parametric equations. For a discussion of this technique, see Section 15.5.

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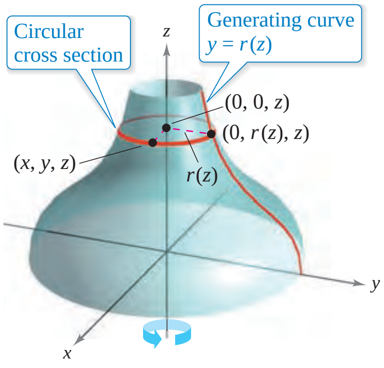

Figure 11.62

Surfaces of Revolution

The fifth special type of surface you will study is a **surface of revolution.** In Section 7.4, you studied a method for finding the *area* of such a surface. You will now look at a procedure for finding its *equation.* Consider the graph of the **radius function**

$$y = r(z) \qquad \text{Generating curve}$$

in the yz-plane. When this graph is revolved about the z-axis, it forms a surface of revolution, as shown in Figure 11.62. The trace of the surface in the plane $z = z_0$ is a circle whose radius is $r(z_0)$ and whose equation is

$$x^2 + y^2 = [r(z_0)]^2. \qquad \text{Circular trace in plane: } z = z_0$$

Replacing z_0 with z produces an equation that is valid for all values of z. In a similar manner, you can obtain equations for surfaces of revolution for the other two axes, and the results are summarized as follows.

Surface of Revolution

If the graph of a radius function r is revolved about one of the coordinate axes, then the equation of the resulting surface of revolution has one of the forms listed below.

1. Revolved about the x-axis: $y^2 + z^2 = [r(x)]^2$
2. Revolved about the y-axis: $x^2 + z^2 = [r(y)]^2$
3. Revolved about the z-axis: $x^2 + y^2 = [r(z)]^2$

EXAMPLE 5 Finding an Equation for a Surface of Revolution

Find an equation for the surface of revolution formed by revolving (a) the graph of $y = 1/z$ about the z-axis and (b) the graph of $9x^2 = y^3$ about the y-axis.

Solution

a. An equation for the surface of revolution formed by revolving the graph of

$$y = \frac{1}{z} \qquad \text{Radius function}$$

about the z-axis is

$$x^2 + y^2 = [r(z)]^2 \qquad \text{Revolved about the } z\text{-axis}$$

$$x^2 + y^2 = \left(\frac{1}{z}\right)^2. \qquad \text{Substitute } 1/z \text{ for } r(z).$$

b. To find an equation for the surface formed by revolving the graph of $9x^2 = y^3$ about the y-axis, solve for x in terms of y to obtain

$$x = \frac{1}{3}y^{3/2} = r(y). \qquad \text{Radius function}$$

So, the equation for this surface is

$$x^2 + z^2 = [r(y)]^2 \qquad \text{Revolved about the } y\text{-axis}$$

$$x^2 + z^2 = \left(\frac{1}{3}y^{3/2}\right)^2 \qquad \text{Substitute } \tfrac{1}{3}y^{3/2} \text{ for } r(y).$$

$$x^2 + z^2 = \frac{1}{9}y^3. \qquad \text{Equation of surface}$$

The graph is shown in Figure 11.63.

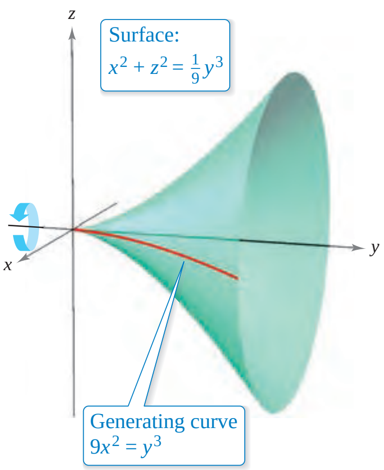

Figure 11.63

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The generating curve for a surface of revolution is not unique. For instance, the surface

$$x^2 + z^2 = e^{-2y}$$

can be formed by revolving either the graph of

$$x = e^{-y}$$

about the y-axis or the graph of

$$z = e^{-y}$$

about the y-axis, as shown in Figure 11.64.

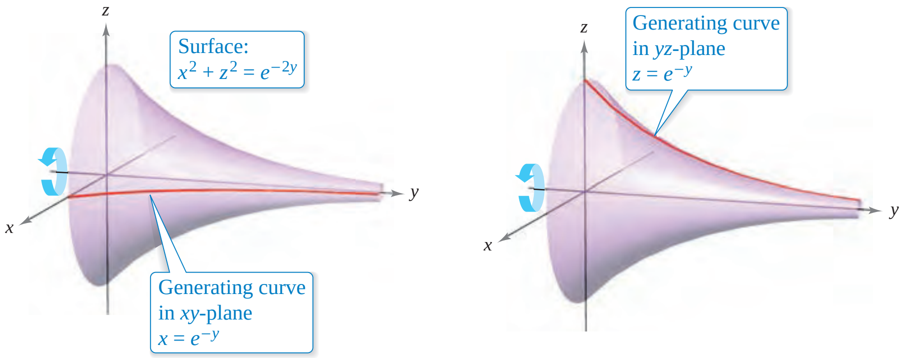

Figure 11.64

EXAMPLE 6 Finding a Generating Curve

Find a generating curve and the axis of revolution for the surface

$$x^2 + 3y^2 + z^2 = 9.$$

Solution The equation has one of the forms listed below.

$$x^2 + y^2 = [r(z)]^2 \qquad \text{Revolved about } z\text{-axis}$$
$$y^2 + z^2 = [r(x)]^2 \qquad \text{Revolved about } x\text{-axis}$$
$$x^2 + z^2 = [r(y)]^2 \qquad \text{Revolved about } y\text{-axis}$$

Because the coefficients of x^2 and z^2 are equal, you should choose the third form and write

$$x^2 + z^2 = 9 - 3y^2.$$

The y-axis is the axis of revolution. You can choose a generating curve from either of the traces

$$x^2 = 9 - 3y^2 \qquad \text{Trace in } xy\text{-plane}$$

or

$$z^2 = 9 - 3y^2. \qquad \text{Trace in } yz\text{-plane}$$

For instance, using the first trace, the generating curve is the semiellipse

$$x = \sqrt{9 - 3y^2}. \qquad \text{Generating curve}$$

The graph of this surface is shown in Figure 11.65.

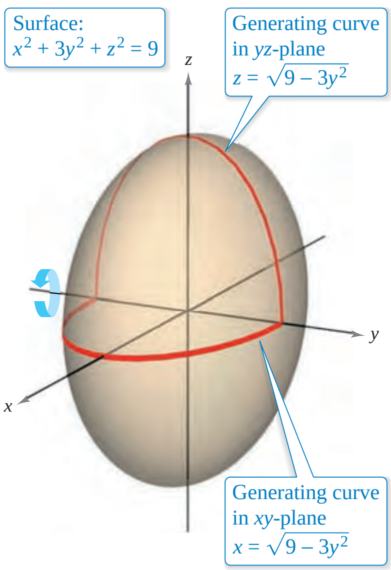

Figure 11.65

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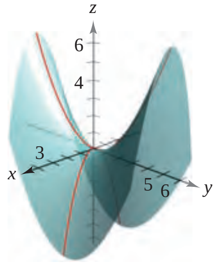
11.6 Exercises

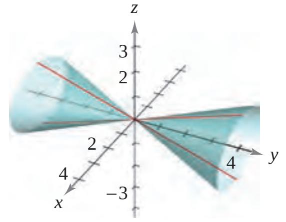
See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Matching **In Exercises 1–6, match the equation with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]**

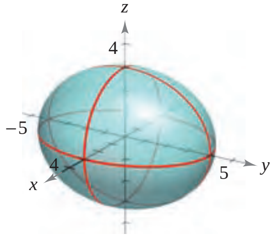
(a)



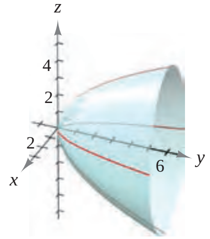
(b)



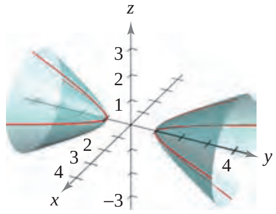
(c)



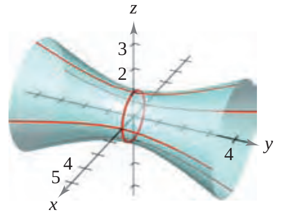
(d)



(e)



(f)



1. $\frac{x^2}{9} + \frac{y^2}{16} + \frac{z^2}{9} = 1$
2. $15x^2 - 4y^2 + 15z^2 = -4$
3. $4x^2 - y^2 + 4z^2 = 4$
4. $y^2 = 4x^2 + 9z^2$
5. $4x^2 - 4y + z^2 = 0$
6. $4x^2 - y^2 + 4z = 0$

Sketching a Surface in Space **In Exercises 7–12, describe and sketch the surface.**

7. $y = 5$
8. $z = 2$
9. $y^2 + z^2 = 9$
10. $y^2 + z = 6$
11. $4x^2 + y^2 = 4$
12. $y^2 - z^2 = 16$

Sketching a Quadric Surface **In Exercises 13–24, classify and sketch the quadric surface. Use a computer algebra system or a graphing utility to confirm your sketch.**

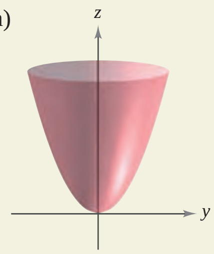
13. $x^2 + \frac{y^2}{4} + z^2 = 1$
14. $\frac{x^2}{16} + \frac{y^2}{25} + \frac{z^2}{25} = 1$
15. $16x^2 - y^2 + 16z^2 = 4$
16. $-8x^2 + 18y^2 + 18z^2 = 2$
17. $4x^2 - y^2 - z^2 = 1$
18. $z^2 - x^2 - \frac{y^2}{4} = 1$
19. $x^2 - y + z^2 = 0$
20. $z = x^2 + 4y^2$
21. $x^2 - y^2 + z = 0$
22. $3z = -y^2 + x^2$
23. $z^2 = x^2 + \frac{y^2}{9}$
24. $x^2 = 2y^2 + 2z^2$

WRITING ABOUT CONCEPTS

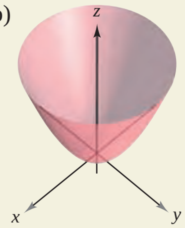
25. **Cylinder** State the definition of a cylinder.
26. **Trace of a Surface** What is meant by the trace of a surface? How do you find a trace?
27. **Quadric Surfaces** Identify the six quadric surfaces and give the standard form of each.
28. **Classifying an Equation** What does the equation $z = x^2$ represent in the xz-plane? What does it represent in three-space?
29. **Classifying an Equation** What does the equation $4x^2 + 6y^2 - 3z^2 = 12$ represent in the xy-plane? What does it represent in three-space?

30. **HOW DO YOU SEE IT?** The four figures are graphs of the quadric surface $z = x^2 + y^2$. Match each of the four graphs with the point in space from which the paraboloid is viewed. The four points are (0, 0, 20), (0, 20, 0), (20, 0, 0), and (10, 10, 20).

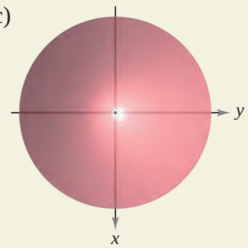
(a)



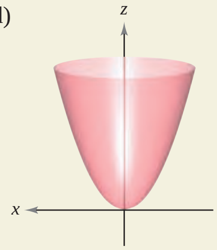
(b)



(c)



(d)



Finding an Equation of a Surface of Revolution **In Exercises 31–36, find an equation for the surface of revolution formed by revolving the curve in the indicated coordinate plane about the given axis.**

	Equation of Curve	Coordinate Plane	Axis of Revolution
31.	$z^2 = 4y$	yz-plane	y-axis
32.	$z = 3y$	yz-plane	y-axis
33.	$z = 2y$	yz-plane	z-axis

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	Equation of Curve	Coordinate Plane	Axis of Revolution
34.	$2z = \sqrt{4 - x^2}$	xz-plane	x-axis
35.	$xy = 2$	xy-plane	x-axis
36.	$z = \ln y$	yz-plane	z-axis

Finding a Generating Curve **In Exercises 37 and 38, find an equation of a generating curve given the equation of its surface of revolution.**

37. $x^2 + y^2 - 2z = 0$ **38.** $x^2 + z^2 = \cos^2 y$

Finding the Volume of a Solid **In Exercises 39 and 40, use the shell method to find the volume of the solid below the surface of revolution and above the *xy*-plane.**

39. The curve $z = 4x - x^2$ in the xz-plane is revolved about the z-axis.

40. The curve $z = \sin y\ (0 \le y \le \pi)$ in the yz-plane is revolved about the z-axis.

Analyzing a Trace **In Exercises 41 and 42, analyze the trace when the surface**

$$z = \tfrac{1}{2}x^2 + \tfrac{1}{4}y^2$$

is intersected by the indicated planes.

41. Find the lengths of the major and minor axes and the coordinates of the foci of the ellipse generated when the surface is intersected by the planes given by

(a) $z = 2$ and (b) $z = 8$.

42. Find the coordinates of the focus of the parabola formed when the surface is intersected by the planes given by

(a) $y = 4$ and (b) $x = 2$.

Finding an Equation of a Surface **In Exercises 43 and 44, find an equation of the surface satisfying the conditions, and identify the surface.**

43. The set of all points equidistant from the point $(0, 2, 0)$ and the plane $y = -2$

44. The set of all points equidistant from the point $(0, 0, 4)$ and the xy-plane

45. Geography

Because of the forces caused by its rotation, Earth is an oblate ellipsoid rather than a sphere. The equatorial radius is 3963 miles and the polar radius is 3950 miles. Find an equation of the ellipsoid. (Assume that the center of Earth is at the origin and that the trace formed by the plane $z = 0$ corresponds to the equator.)

46. Machine Design The top of a rubber bushing designed to absorb vibrations in an automobile is the surface of revolution generated by revolving the curve

$$z = \tfrac{1}{2}y^2 + 1$$

for $0 \le y \le 2$ in the yz-plane about the z-axis.

(a) Find an equation for the surface of revolution.

(b) All measurements are in centimeters and the bushing is set on the xy-plane. Use the shell method to find its volume.

(c) The bushing has a hole of diameter 1 centimeter through its center and parallel to the axis of revolution. Find the volume of the rubber bushing.

47. Using a Hyperbolic Paraboloid Determine the intersection of the hyperbolic paraboloid

$$z = \frac{y^2}{b^2} - \frac{x^2}{a^2}$$

with the plane $bx + ay - z = 0$. (Assume $a, b > 0$.)

48. Intersection of Surfaces Explain why the curve of intersection of the surfaces

$$x^2 + 3y^2 - 2z^2 + 2y = 4$$

and

$$2x^2 + 6y^2 - 4z^2 - 3x = 2$$

lies in a plane.

True or False? **In Exercises 49–52, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

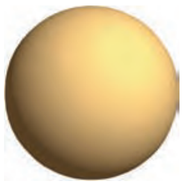
49. A sphere is an ellipsoid.

50. The generating curve for a surface of revolution is unique.

51. All traces of an ellipsoid are ellipses.

52. All traces of a hyperboloid of one sheet are hyperboloids.

53. Think About It Three types of classic "topological" surfaces are shown below. The sphere and torus have both an "inside" and an "outside." Does the Klein bottle have both an inside and an outside? Explain.



Sphere

Torus

Klein bottle

Klein bottle

Denis Tabler/Shutterstock.com

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11.7 Cylindrical and Spherical Coordinates

- **Use cylindrical coordinates to represent surfaces in space.**
- **Use spherical coordinates to represent surfaces in space.**

Cylindrical Coordinates

You have already seen that some two-dimensional graphs are easier to represent in polar coordinates than in rectangular coordinates. A similar situation exists for surfaces in space. In this section, you will study two alternative space-coordinate systems. The first, the **cylindrical coordinate system,** is an extension of polar coordinates in the plane to three-dimensional space.

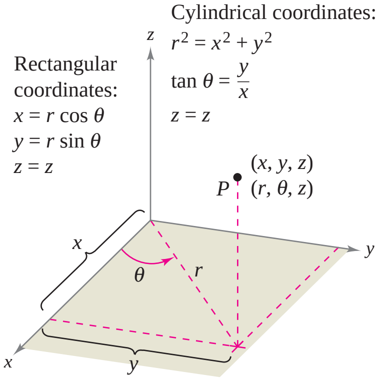

Figure 11.66

The Cylindrical Coordinate System

In a **cylindrical coordinate system,** a point P in space is represented by an ordered triple (r, θ, z).

1. (r, θ) is a polar representation of the projection of P in the xy-plane.
2. z is the directed distance from (r, θ) to P.

To convert from rectangular to cylindrical coordinates (or vice versa), use the conversion guidelines for polar coordinates listed below and illustrated in Figure 11.66.

Cylindrical to rectangular:

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z$$

Rectangular to cylindrical:

$$r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}, \qquad z = z$$

The point $(0, 0, 0)$ is called the **pole.** Moreover, because the representation of a point in the polar coordinate system is not unique, it follows that the representation in the cylindrical coordinate system is also not unique.

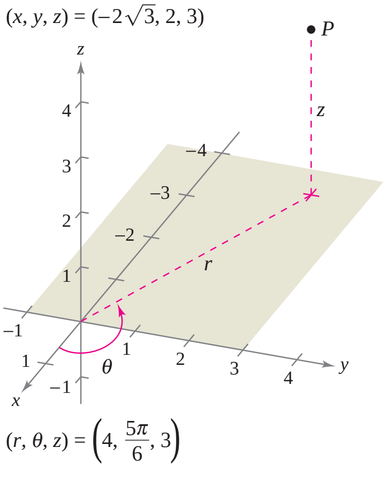

Figure 11.67

EXAMPLE 1 Cylindrical-to-Rectangular Conversion

Convert the point $(r, \theta, z) = (4, 5\pi/6, 3)$ to rectangular coordinates.

Solution Using the cylindrical-to-rectangular conversion equations produces

$$x = 4\cos\frac{5\pi}{6} = 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$$

$$y = 4\sin\frac{5\pi}{6} = 4\left(\frac{1}{2}\right) = 2$$

$$z = 3.$$

So, in rectangular coordinates, the point is $(x, y, z) = \left(-2\sqrt{3}, 2, 3\right)$, as shown in Figure 11.67.

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EXAMPLE 2 Rectangular-to-Cylindrical Conversion

Convert the point

$$(x, y, z) = (1, \sqrt{3}, 2)$$

to cylindrical coordinates.

Solution Use the rectangular-to-cylindrical conversion equations.

$$r = \pm\sqrt{1 + 3} = \pm 2$$

$$\tan\theta = \sqrt{3} \quad \Rightarrow \quad \theta = \arctan\left(\sqrt{3}\right) + n\pi = \frac{\pi}{3} + n\pi$$

$$z = 2$$

You have two choices for r and infinitely many choices for θ. As shown in Figure 11.68, two convenient representations of the point are

$$\left(2, \frac{\pi}{3}, 2\right) \qquad r > 0 \text{ and } \theta \text{ in Quadrant I}$$

and

$$\left(-2, \frac{4\pi}{3}, 2\right). \qquad r < 0 \text{ and } \theta \text{ in Quadrant III}$$

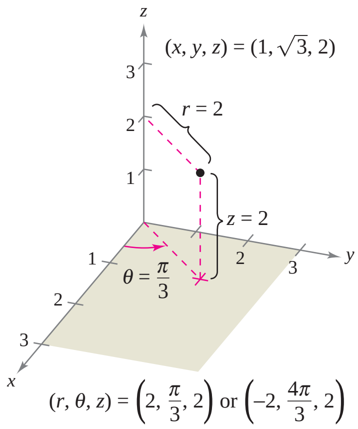
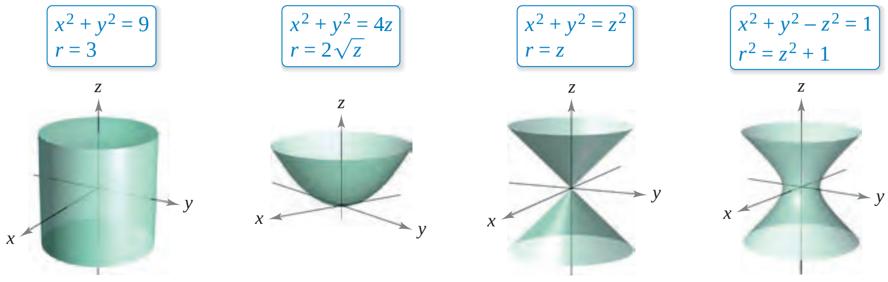


Figure 11.68

Cylindrical coordinates are especially convenient for representing cylindrical surfaces and surfaces of revolution with the z-axis as the axis of symmetry, as shown in Figure 11.69.



Cylinder Paraboloid Cone Hyperboloid

Figure 11.69

Vertical planes containing the z-axis and horizontal planes also have simple cylindrical coordinate equations, as shown in Figure 11.70.

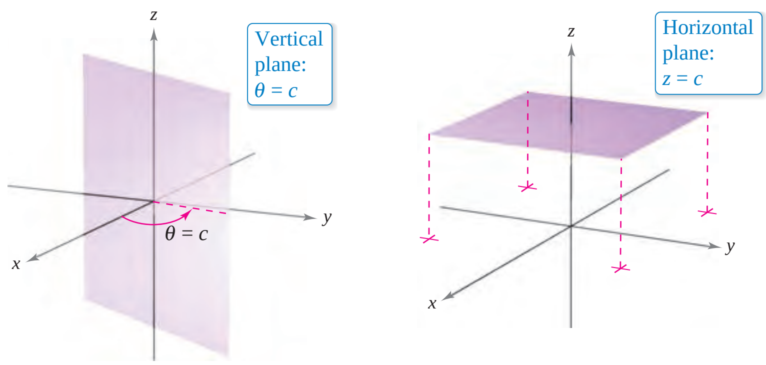


Figure 11.70

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EXAMPLE 3 Rectangular-to-Cylindrical Conversion

Find an equation in cylindrical coordinates for the surface represented by each rectangular equation.

a. $x^2 + y^2 = 4z^2$

b. $y^2 = x$

Solution

a. From Section 11.6, you know that the graph of

$$x^2 + y^2 = 4z^2$$

is an elliptic cone with its axis along the z-axis, as shown in Figure 11.71. When you replace $x^2 + y^2$ with r^2, the equation in cylindrical coordinates is

$$x^2 + y^2 = 4z^2 \quad \text{Rectangular equation}$$
$$r^2 = 4z^2. \quad \text{Cylindrical equation}$$

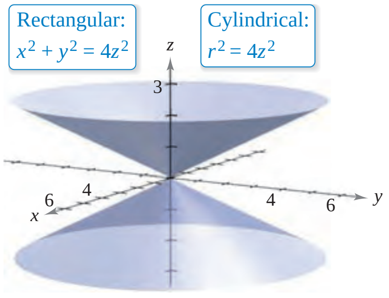

Figure 11.71

b. The graph of the surface

$$y^2 = x$$

is a parabolic cylinder with rulings parallel to the z-axis, as shown in Figure 11.72. To obtain the equation in cylindrical coordinates, replace y^2 with $r^2 \sin^2 \theta$ and x with $r \cos \theta$, as shown.

$$y^2 = x \quad \text{Rectangular equation}$$
$$r^2 \sin^2 \theta = r \cos \theta \quad \text{Substitute } r \sin \theta \text{ for } y \text{ and } r \cos \theta \text{ for } x.$$
$$r(r \sin^2 \theta - \cos \theta) = 0 \quad \text{Collect terms and factor.}$$
$$r \sin^2 \theta - \cos \theta = 0 \quad \text{Divide each side by } r.$$
$$r = \frac{\cos \theta}{\sin^2 \theta} \quad \text{Solve for } r.$$
$$r = \csc \theta \cot \theta \quad \text{Cylindrical equation}$$

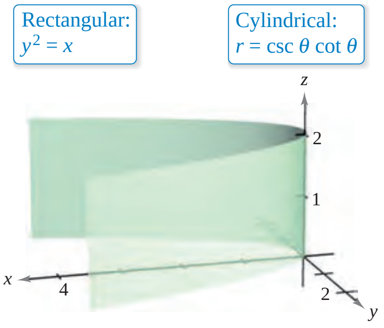

Figure 11.72

Note that this equation includes a point for which $r = 0$, so nothing was lost by dividing each side by the factor r.

Converting from cylindrical coordinates to rectangular coordinates is less straightforward than converting from rectangular coordinates to cylindrical coordinates, as demonstrated in Example 4.

EXAMPLE 4 Cylindrical-to-Rectangular Conversion

Find an equation in rectangular coordinates for the surface represented by the cylindrical equation

$$r^2 \cos 2\theta + z^2 + 1 = 0.$$

Solution

$$r^2 \cos 2\theta + z^2 + 1 = 0 \quad \text{Cylindrical equation}$$
$$r^2(\cos^2 \theta - \sin^2 \theta) + z^2 + 1 = 0 \quad \text{Trigonometric identity}$$
$$r^2 \cos^2 \theta - r^2 \sin^2 \theta + z^2 = -1$$
$$x^2 - y^2 + z^2 = -1 \quad \text{Replace } r \cos \theta \text{ with } x \text{ and } r \sin \theta \text{ with } y.$$
$$y^2 - x^2 - z^2 = 1 \quad \text{Rectangular equation}$$

This is a hyperboloid of two sheets whose axis lies along the y-axis, as shown in Figure 11.73.

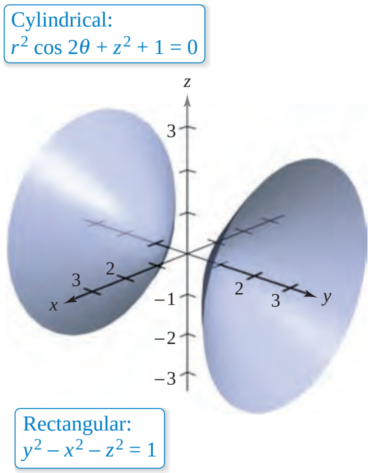

Figure 11.73

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Spherical Coordinates

In the **spherical coordinate system,** each point is represented by an ordered triple: the first coordinate is a distance, and the second and third coordinates are angles. This system is similar to the latitude-longitude system used to identify points on the surface of Earth. For example, the point on the surface of Earth whose latitude is 40° North (of the equator) and whose longitude is 80° West (of the prime meridian) is shown in Figure 11.74. Assuming that Earth is spherical and has a radius of 4000 miles, you would label this point as

$$(4000, -80°, 50°).$$

Radius — 80° clockwise from prime meridian — 50° down from North Pole

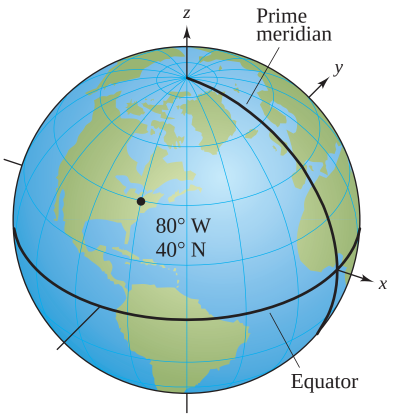


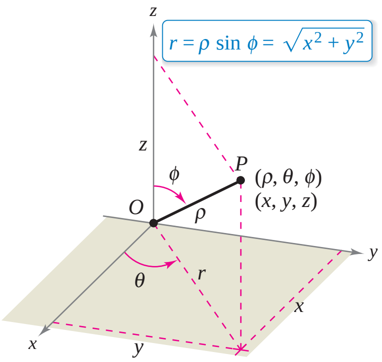
Figure 11.74

The Spherical Coordinate System

In a **spherical coordinate system,** a point P in space is represented by an ordered triple (ρ, θ, ϕ), where ρ is the lowercase Greek letter *rho* and ϕ is the lowercase Greek letter *phi.*

1. ρ is the distance between P and the origin, $\rho \geq 0$.
2. θ is the same angle used in cylindrical coordinates for $r \geq 0$.
3. ϕ is the angle *between* the positive z-axis and the line segment $\overrightarrow{OP}$, $0 \leq \phi \leq \pi$.

Note that the first and third coordinates, ρ and ϕ, are nonnegative.



Spherical coordinates
Figure 11.75

The relationship between rectangular and spherical coordinates is illustrated in Figure 11.75. To convert from one system to the other, use the conversion guidelines listed below.

Spherical to rectangular:

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad z = \rho \cos \phi$$

Rectangular to spherical:

$$\rho^2 = x^2 + y^2 + z^2, \quad \tan \theta = \frac{y}{x}, \quad \phi = \arccos\left(\frac{z}{\sqrt{x^2 + y^2 + z^2}}\right)$$

To change coordinates between the cylindrical and spherical systems, use the conversion guidelines listed below.

Spherical to cylindrical $(r \geq 0)$***:***

$$r^2 = \rho^2 \sin^2 \phi, \quad \theta = \theta, \quad z = \rho \cos \phi$$

Cylindrical to spherical $(r \geq 0)$***:***

$$\rho = \sqrt{r^2 + z^2}, \quad \theta = \theta, \quad \phi = \arccos\left(\frac{z}{\sqrt{r^2 + z^2}}\right)$$

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The spherical coordinate system is useful primarily for surfaces in space that have a *point* or *center* of symmetry. For example, Figure 11.76 shows three surfaces with simple spherical equations.

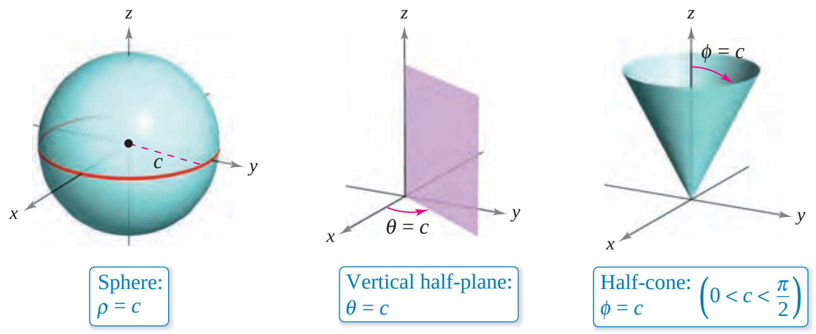

Figure 11.76

EXAMPLE 5 Rectangular-to-Spherical Conversion

See LarsonCalculus.com for an interactive version of this type of example.

Find an equation in spherical coordinates for the surface represented by each rectangular equation.

a. Cone: $x^2 + y^2 = z^2$ **b.** Sphere: $x^2 + y^2 + z^2 - 4z = 0$

Solution

a. Use the spherical-to-rectangular equations

$$x = \rho \sin \phi \cos \theta, \quad y = \rho \sin \phi \sin \theta, \quad \text{and} \quad z = \rho \cos \phi$$

and substitute in the rectangular equation as shown.

$$x^2 + y^2 = z^2$$
$$\rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta = \rho^2 \cos^2 \phi$$
$$\rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) = \rho^2 \cos^2 \phi$$
$$\rho^2 \sin^2 \phi = \rho^2 \cos^2 \phi$$
$$\frac{\sin^2 \phi}{\cos^2 \phi} = 1 \qquad \rho \geq 0$$
$$\tan^2 \phi = 1$$
$$\tan \phi = \pm 1$$

So, you can conclude that

$$\phi = \frac{\pi}{4} \quad \text{or} \quad \phi = \frac{3\pi}{4}.$$

The equation $\phi = \pi/4$ represents the *upper* half-cone, and the equation $\phi = 3\pi/4$ represents the *lower* half-cone.

b. Because $\rho^2 = x^2 + y^2 + z^2$ and $z = \rho \cos \phi$, the rectangular equation has the following spherical form.

$$\rho^2 - 4\rho \cos \phi = 0 \quad \Rightarrow \quad \rho(\rho - 4 \cos \phi) = 0$$

Temporarily discarding the possibility that $\rho = 0$, you have the spherical equation

$$\rho - 4 \cos \phi = 0 \quad \text{or} \quad \rho = 4 \cos \phi.$$

Note that the solution set for this equation includes a point for which $\rho = 0$, so nothing is lost by discarding the factor ρ. The sphere represented by the equation $\rho = 4 \cos \phi$ is shown in Figure 11.77.

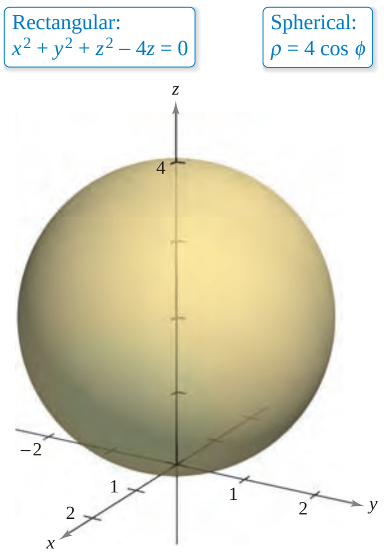

Figure 11.77

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11.7 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Cylindrical-to-Rectangular Conversion In Exercises 1–6, convert the point from cylindrical coordinates to rectangular coordinates.

1. $(-7, 0, 5)$
2. $(2, -\pi, -4)$
3. $\left(3, \frac{\pi}{4}, 1\right)$
4. $\left(6, -\frac{\pi}{4}, 2\right)$
5. $\left(4, \frac{7\pi}{6}, 3\right)$
6. $\left(-0.5, \frac{4\pi}{3}, 8\right)$

Rectangular-to-Cylindrical Conversion In Exercises 7–12, convert the point from rectangular coordinates to cylindrical coordinates.

7. $(0, 5, 1)$
8. $(2\sqrt{2}, -2\sqrt{2}, 4)$
9. $(2, -2, -4)$
10. $(3, -3, 7)$
11. $(1, \sqrt{3}, 4)$
12. $(2\sqrt{3}, -2, 6)$

Rectangular-to-Cylindrical Conversion In Exercises 13–20, find an equation in cylindrical coordinates for the equation given in rectangular coordinates.

13. $z = 4$
14. $x = 9$
15. $x^2 + y^2 + z^2 = 17$
16. $z = x^2 + y^2 - 11$
17. $y = x^2$
18. $x^2 + y^2 = 8x$
19. $y^2 = 10 - z^2$
20. $x^2 + y^2 + z^2 - 3z = 0$

Cylindrical-to-Rectangular Conversion In Exercises 21–28, find an equation in rectangular coordinates for the equation given in cylindrical coordinates, and sketch its graph.

21. $r = 3$
22. $z = 2$
23. $\theta = \frac{\pi}{6}$
24. $r = \frac{1}{2}z$
25. $r^2 + z^2 = 5$
26. $z = r^2 \cos^2 \theta$
27. $r = 2 \sin \theta$
28. $r = 2 \cos \theta$

Rectangular-to-Spherical Conversion In Exercises 29–34, convert the point from rectangular coordinates to spherical coordinates.

29. $(4, 0, 0)$
30. $(-4, 0, 0)$
31. $(-2, 2\sqrt{3}, 4)$
32. $(2, 2, 4\sqrt{2})$
33. $(\sqrt{3}, 1, 2\sqrt{3})$
34. $(-1, 2, 1)$

Spherical-to-Rectangular Conversion In Exercises 35–40, convert the point from spherical coordinates to rectangular coordinates.

35. $\left(4, \frac{\pi}{6}, \frac{\pi}{4}\right)$
36. $\left(12, \frac{3\pi}{4}, \frac{\pi}{9}\right)$
37. $\left(12, -\frac{\pi}{4}, 0\right)$
38. $\left(9, \frac{\pi}{4}, \pi\right)$
39. $\left(5, \frac{\pi}{4}, \frac{3\pi}{4}\right)$
40. $\left(6, \pi, \frac{\pi}{2}\right)$

Rectangular-to-Spherical Conversion In Exercises 41–48, find an equation in spherical coordinates for the equation given in rectangular coordinates.

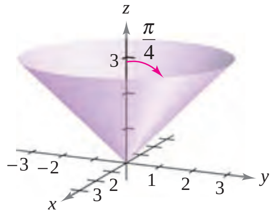
41. $y = 2$
42. $z = 6$
43. $x^2 + y^2 + z^2 = 49$
44. $x^2 + y^2 - 3z^2 = 0$
45. $x^2 + y^2 = 16$
46. $x = 13$
47. $x^2 + y^2 = 2z^2$
48. $x^2 + y^2 + z^2 - 9z = 0$

Spherical-to-Rectangular Conversion In Exercises 49–56, find an equation in rectangular coordinates for the equation given in spherical coordinates, and sketch its graph.

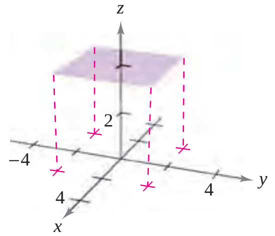
49. $\rho = 5$
50. $\theta = \frac{3\pi}{4}$
51. $\phi = \frac{\pi}{6}$
52. $\phi = \frac{\pi}{2}$
53. $\rho = 4 \cos \phi$
54. $\rho = 2 \sec \phi$
55. $\rho = \csc \phi$
56. $\rho = 4 \csc \phi \sec \theta$

Matching In Exercises 57–62, match the equation (written in terms of cylindrical or spherical coordinates) with its graph. [The graphs are labeled (a), (b), (c), (d), (e), and (f).]

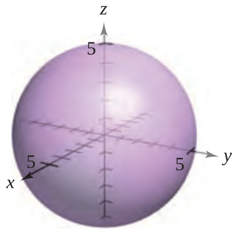
(a)



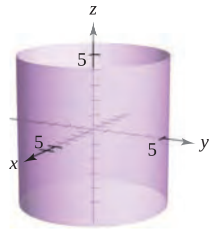
(b)



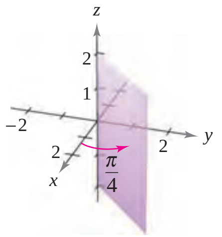
(c)



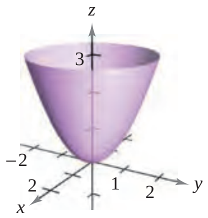
(d)



(e)



(f)



57. $r = 5$
58. $\theta = \frac{\pi}{4}$
59. $\rho = 5$
60. $\phi = \frac{\pi}{4}$
61. $r^2 = z$
62. $\rho = 4 \sec \phi$

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Cylindrical-to-Spherical Conversion **In Exercises 63–70, convert the point from cylindrical coordinates to spherical coordinates.**

63. $\left(4, \frac{\pi}{4}, 0\right)$

64. $\left(3, -\frac{\pi}{4}, 0\right)$

65. $\left(4, \frac{\pi}{2}, 4\right)$

66. $\left(2, \frac{2\pi}{3}, -2\right)$

67. $\left(4, -\frac{\pi}{6}, 6\right)$

68. $\left(-4, \frac{\pi}{3}, 4\right)$

69. $(12, \pi, 5)$

70. $\left(4, \frac{\pi}{2}, 3\right)$

Spherical-to-Cylindrical Conversion **In Exercises 71–78, convert the point from spherical coordinates to cylindrical coordinates.**

71. $\left(10, \frac{\pi}{6}, \frac{\pi}{2}\right)$

72. $\left(4, \frac{\pi}{18}, \frac{\pi}{2}\right)$

73. $\left(36, \pi, \frac{\pi}{2}\right)$

74. $\left(18, \frac{\pi}{3}, \frac{\pi}{3}\right)$

75. $\left(6, -\frac{\pi}{6}, \frac{\pi}{3}\right)$

76. $\left(5, -\frac{5\pi}{6}, \pi\right)$

77. $\left(8, \frac{7\pi}{6}, \frac{\pi}{6}\right)$

78. $\left(7, \frac{\pi}{4}, \frac{3\pi}{4}\right)$

WRITING ABOUT CONCEPTS

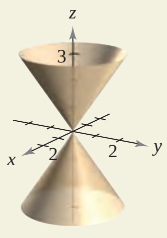
79. **Rectangular and Cylindrical Coordinates** Give the equations for the coordinate conversion from rectangular to cylindrical coordinates and vice versa.

80. **Spherical Coordinates** Explain why in spherical coordinates the graph of $\theta = c$ is a half-plane and not an entire plane.

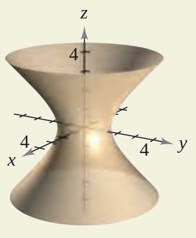
81. **Rectangular and Spherical Coordinates** Give the equations for the coordinate conversion from rectangular to spherical coordinates and vice versa.

82. **HOW DO YOU SEE IT?** Identify the surface graphed and match the graph with its rectangular equation. Then find an equation in cylindrical coordinates for the equation given in rectangular coordinates.

(a)



(b)



(i) $x^2 + y^2 = \frac{4}{9}z^2$

(ii) $x^2 + y^2 - z^2 = 2$

Converting a Rectangular Equation **In Exercises 83–90, convert the rectangular equation to an equation in (a) cylindrical coordinates and (b) spherical coordinates.**

83. $x^2 + y^2 + z^2 = 25$

84. $4(x^2 + y^2) = z^2$

85. $x^2 + y^2 + z^2 - 2z = 0$

86. $x^2 + y^2 = z$

87. $x^2 + y^2 = 4y$

88. $x^2 + y^2 = 36$

89. $x^2 - y^2 = 9$

90. $y = 4$

Sketching a Solid **In Exercises 91–94, sketch the solid that has the given description in cylindrical coordinates.**

91. $0 \le \theta \le \pi/2, 0 \le r \le 2, 0 \le z \le 4$

92. $-\pi/2 \le \theta \le \pi/2, 0 \le r \le 3, 0 \le z \le r\cos\theta$

93. $0 \le \theta \le 2\pi, 0 \le r \le a, r \le z \le a$

94. $0 \le \theta \le 2\pi, 2 \le r \le 4, z^2 \le -r^2 + 6r - 8$

Sketching a Solid **In Exercises 95–98, sketch the solid that has the given description in spherical coordinates.**

95. $0 \le \theta \le 2\pi, 0 \le \phi \le \pi/6, 0 \le \rho \le a\sec\phi$

96. $0 \le \theta \le 2\pi, \pi/4 \le \phi \le \pi/2, 0 \le \rho \le 1$

97. $0 \le \theta \le \pi/2, 0 \le \phi \le \pi/2, 0 \le \rho \le 2$

98. $0 \le \theta \le \pi, 0 \le \phi \le \pi/2, 1 \le \rho \le 3$

Think About It **In Exercises 99–104, find inequalities that describe the solid, and state the coordinate system used. Position the solid on the coordinate system such that the inequalities are as simple as possible.**

99. A cube with each edge 10 centimeters long

100. A cylindrical shell 8 meters long with an inside diameter of 0.75 meter and an outside diameter of 1.25 meters

101. A spherical shell with inside and outside radii of 4 inches and 6 inches, respectively

102. The solid that remains after a hole 1 inch in diameter is drilled through the center of a sphere 6 inches in diameter

103. The solid inside both $x^2 + y^2 + z^2 = 9$ and $\left(x - \frac{3}{2}\right)^2 + y^2 = \frac{9}{4}$

104. The solid between the spheres $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 + z^2 = 9$, and inside the cone $z^2 = x^2 + y^2$

True or False? **In Exercises 105–108, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

105. In cylindrical coordinates, the equation $r = z$ is a cylinder.

106. The equations $\rho = 2$ and $x^2 + y^2 + z^2 = 4$ represent the same surface.

107. The cylindrical coordinates of a point (x, y, z) are unique.

108. The spherical coordinates of a point (x, y, z) are unique.

109. **Intersection of Surfaces** Identify the curve of intersection of the surfaces (in cylindrical coordinates) $z = \sin\theta$ and $r = 1$.

110. **Intersection of Surfaces** Identify the curve of intersection of the surfaces (in spherical coordinates) $\rho = 2\sec\phi$ and $\rho = 4$.

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Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Writing Vectors in Different Forms **In Exercises 1 and 2, let $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{PR}$, and (a) write u and v in component form, (b) write u and v as the linear combination of the standard unit vectors i and j, (c) find the magnitudes of u and v, and (d) find 2u + v.**

1. $P = (1, 2), Q = (4, 1), R = (5, 4)$
2. $P = (-2, -1), Q = (5, -1), R = (2, 4)$

Finding a Vector **In Exercises 3 and 4, find the component form of v given its magnitude and the angle it makes with the positive *x*-axis.**

3. $\|\mathbf{v}\| = 8,\ \theta = 60°$
4. $\|\mathbf{v}\| = \frac{1}{2},\ \theta = 225°$

5. **Finding Coordinates of a Point** Find the coordinates of the point located in the *xy*-plane, four units to the right of the *xz*-plane, and five units behind the *yz*-plane.
6. **Finding Coordinates of a Point** Find the coordinates of the point located on the *y*-axis and seven units to the left of the *xz*-plane.

Finding the Distance Between Two Points in Space **In Exercises 7 and 8, find the distance between the points.**

7. $(1, 6, 3), (-2, 3, 5)$
8. $(-2, 1, -5), (4, -1, -1)$

Finding the Equation of a Sphere **In Exercises 9 and 10, find the standard equation of the sphere.**

9. Center: $(3, -2, 6)$; Diameter: 15
10. Endpoints of a diameter: $(0, 0, 4), (4, 6, 0)$

Finding the Equation of a Sphere **In Exercises 11 and 12, complete the square to write the equation of the sphere in standard form. Find the center and radius.**

11. $x^2 + y^2 + z^2 - 4x - 6y + 4 = 0$
12. $x^2 + y^2 + z^2 - 10x + 6y - 4z + 34 = 0$

Writing a Vector in Different Forms **In Exercises 13 and 14, the initial and terminal points of a vector are given. (a) Sketch the directed line segment, (b) find the component form of the vector, (c) write the vector using standard unit vector notation, and (d) sketch the vector with its initial point at the origin.**

13. Initial point: $(2, -1, 3)$
    Terminal point: $(4, 4, -7)$
14. Initial point: $(6, 2, 0)$
    Terminal point: $(3, -3, 8)$

Using Vectors to Determine Collinear Points **In Exercises 15 and 16, use vectors to determine whether the points are collinear.**

15. $(3, 4, -1),\ (-1, 6, 9),\ (5, 3, -6)$
16. $(5, -4, 7),\ (8, -5, 5),\ (11, 6, 3)$

17. **Finding a Unit Vector** Find a unit vector in the direction of $\mathbf{u} = \langle 2, 3, 5 \rangle$.
18. **Finding a Vector** Find the vector **v** of magnitude 8 in the direction $\langle 6, -3, 2 \rangle$.

Finding Dot Products **In Exercises 19 and 20, let $\mathbf{u} = \overrightarrow{PQ}$ and $\mathbf{v} = \overrightarrow{PR}$, and find (a) the component forms of u and v, (b) u · v, and (c) v · v.**

19. $P = (5, 0, 0),\ Q = (4, 4, 0),\ R = (2, 0, 6)$
20. $P = (2, -1, 3),\ Q = (0, 5, 1),\ R = (5, 5, 0)$

Finding the Angle Between Two Vectors **In Exercises 21–24, find the angle θ between the vectors (a) in radians and (b) in degrees.**

21. $\mathbf{u} = 5[\cos(3\pi/4)\mathbf{i} + \sin(3\pi/4)\mathbf{j}]$
    $\mathbf{v} = 2[\cos(2\pi/3)\mathbf{i} + \sin(2\pi/3)\mathbf{j}]$
22. $\mathbf{u} = 6\mathbf{i} + 2\mathbf{j} - 3\mathbf{k},\quad \mathbf{v} = -\mathbf{i} + 5\mathbf{j}$
23. $\mathbf{u} = \langle 10, -5, 15 \rangle,\quad \mathbf{v} = \langle -2, 1, -3 \rangle$
24. $\mathbf{u} = \langle 1, 0, -3 \rangle,\quad \mathbf{v} = \langle 2, -2, 1 \rangle$

Comparing Vectors **In Exercises 25 and 26, determine whether u and v are orthogonal, parallel, or neither.**

25. $\mathbf{u} = \langle 7, -2, 3 \rangle$
    $\mathbf{v} = \langle -1, 4, 5 \rangle$
26. $\mathbf{u} = \langle -4, 3, -6 \rangle$
    $\mathbf{v} = \langle 16, -12, 24 \rangle$

Finding the Projection of u onto v **In Exercises 27–30, find the projection of u onto v.**

27. $\mathbf{u} = \langle 7, 9 \rangle,\quad \mathbf{v} = \langle 1, 5 \rangle$
28. $\mathbf{u} = 4\mathbf{i} + 2\mathbf{j},\quad \mathbf{v} = 3\mathbf{i} + 4\mathbf{j}$
29. $\mathbf{u} = \langle 1, -1, 1 \rangle,\quad \mathbf{v} = \langle 2, 0, 2 \rangle$
30. $\mathbf{u} = 5\mathbf{i} + \mathbf{j} + 3\mathbf{k},\quad \mathbf{v} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$

31. **Orthogonal Vectors** Find two vectors in opposite directions that are orthogonal to the vector $\mathbf{u} = \langle 5, 6, -3 \rangle$.
32. **Work** An object is pulled 8 feet across a floor using a force of 75 pounds. The direction of the force is 30° above the horizontal. Find the work done.

Finding Cross Products **In Exercises 33–36, find (a) u × v, (b) v × u, and (c) v × v.**

33. $\mathbf{u} = 4\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$
    $\mathbf{v} = 5\mathbf{i} + 2\mathbf{j} + \mathbf{k}$
34. $\mathbf{u} = 6\mathbf{i} - 5\mathbf{j} + 2\mathbf{k}$
    $\mathbf{v} = -4\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$
35. $\mathbf{u} = \langle 2, -4, -4 \rangle$
    $\mathbf{v} = \langle 1, 1, 3 \rangle$
36. $\mathbf{u} = \langle 0, 2, 1 \rangle$
    $\mathbf{v} = \langle 1, -3, 4 \rangle$

37. **Finding a Unit Vector** Find a unit vector that is orthogonal to both $\mathbf{u} = \langle 2, -10, 8 \rangle$ and $\mathbf{v} = \langle 4, 6, -8 \rangle$.
38. **Area** Find the area of the parallelogram that has the vectors $\mathbf{u} = \langle 3, -1, 5 \rangle$ and $\mathbf{v} = \langle 2, -4, 1 \rangle$ as adjacent sides.

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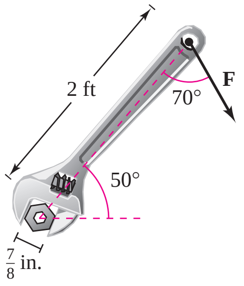
39. **Torque** The specifications for a tractor state that the torque on a bolt with head size $\frac{7}{8}$ inch cannot exceed 200 foot-pounds. Determine the maximum force $\|\mathbf{F}\|$ that can be applied to the wrench in the figure.



40. **Volume** Use the triple scalar product to find the volume of the parallelepiped having adjacent edges $\mathbf{u} = 2\mathbf{i} + \mathbf{j}$, $\mathbf{v} = 2\mathbf{j} + \mathbf{k}$, and $\mathbf{w} = -\mathbf{j} + 2\mathbf{k}$.

Finding Parametric and Symmetric Equations **In Exercises 41 and 42, find sets of (a) parametric equations and (b) symmetric equations of the line through the two points. (For each line, write the direction numbers as integers.)**

41. $(3, 0, 2)$, $(9, 11, 6)$ **42.** $(-1, 4, 3)$, $(8, 10, 5)$

Finding Parametric Equations **In Exercises 43–46, find a set of parametric equations of the line.**

43. The line passes through the point $(1, 2, 3)$ and is perpendicular to the xz-plane.

44. The line passes through the point $(1, 2, 3)$ and is parallel to the line given by $x = y = z$.

45. The line is the intersection of the planes $3x - 3y - 7z = -4$ and $x - y + 2z = 3$.

46. The line passes through the point $(0, 1, 4)$ and is perpendicular to $\mathbf{u} = \langle 2, -5, 1 \rangle$ and $\mathbf{v} = \langle -3, 1, 4 \rangle$.

Finding an Equation of a Plane **In Exercises 47–50, find an equation of the plane.**

47. The plane passes through $(-3, -4, 2)$, $(-3, 4, 1)$, and $(1, 1, -2)$.

48. The plane passes through the point $(-2, 3, 1)$ and is perpendicular to $\mathbf{n} = 3\mathbf{i} - \mathbf{j} + \mathbf{k}$.

49. The plane contains the lines given by

$$\frac{x-1}{-2} = y = z + 1 \quad \text{and} \quad \frac{x+1}{-2} = y - 1 = z - 2.$$

50. The plane passes through the points $(5, 1, 3)$ and $(2, -2, 1)$ and is perpendicular to the plane $2x + y - z = 4$.

51. **Distance** Find the distance between the point $(1, 0, 2)$ and the plane $2x - 3y + 6z = 6$.

52. **Distance** Find the distance between the point $(3, -2, 4)$ and the plane $2x - 5y + z = 10$.

53. **Distance** Find the distance between the planes $5x - 3y + z = 2$ and $5x - 3y + z = -3$.

54. **Distance** Find the distance between the point $(-5, 1, 3)$ and the line given by $x = 1 + t$, $y = 3 - 2t$, and $z = 5 - t$.

Sketching a Surface in Space **In Exercises 55–64, describe and sketch the surface.**

55. $x + 2y + 3z = 6$ **56.** $y = z^2$

57. $y = \frac{1}{2}z$ **58.** $y = \cos z$

59. $\frac{x^2}{16} + \frac{y^2}{9} + z^2 = 1$ **60.** $16x^2 + 16y^2 - 9z^2 = 0$

61. $\frac{x^2}{16} - \frac{y^2}{9} + z^2 = -1$ **62.** $\frac{x^2}{25} + \frac{y^2}{4} - \frac{z^2}{100} = 1$

63. $x^2 + z^2 = 4$ **64.** $y^2 + z^2 = 16$

65. **Surface of Revolution** Find an equation for the surface of revolution formed by revolving the curve $z^2 = 2y$ in the yz-plane about the y-axis.

66. **Surface of Revolution** Find an equation for the surface of revolution formed by revolving the curve $2x + 3z = 1$ in the xz-plane about the x-axis.

Converting Rectangular Coordinates **In Exercises 67 and 68, convert the point from rectangular coordinates to (a) cylindrical coordinates and (b) spherical coordinates.**

67. $\left(-2\sqrt{2}, 2\sqrt{2}, 2\right)$ **68.** $\left(\frac{\sqrt{3}}{4}, \frac{3}{4}, \frac{3\sqrt{3}}{2}\right)$

Cylindrical-to-Spherical Conversion **In Exercises 69 and 70, convert the point from cylindrical coordinates to spherical coordinates.**

69. $\left(100, -\frac{\pi}{6}, 50\right)$ **70.** $\left(81, -\frac{5\pi}{6}, 27\sqrt{3}\right)$

Spherical-to-Cylindrical Conversion **In Exercises 71 and 72, convert the point from spherical coordinates to cylindrical coordinates.**

71. $\left(25, -\frac{\pi}{4}, \frac{3\pi}{4}\right)$ **72.** $\left(12, -\frac{\pi}{2}, \frac{2\pi}{3}\right)$

Converting a Rectangular Equation **In Exercises 73 and 74, convert the rectangular equation to an equation in (a) cylindrical coordinates and (b) spherical coordinates.**

73. $x^2 - y^2 = 2z$ **74.** $x^2 + y^2 + z^2 = 16$

Cylindrical-to-Rectangular Conversion **In Exercises 75 and 76, find an equation in rectangular coordinates for the equation given in cylindrical coordinates, and sketch its graph.**

75. $r = 5\cos\theta$ **76.** $z = 4$

Spherical-to-Rectangular Conversion **In Exercises 77 and 78, find an equation in rectangular coordinates for the equation given in spherical coordinates, and sketch its graph.**

77. $\theta = \frac{\pi}{4}$ **78.** $\rho = 3\cos\phi$

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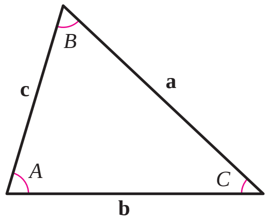
P.S. Problem Solving

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

1. Proof Using vectors, prove the Law of Sines: If **a**, **b**, and **c** are the three sides of the triangle shown in the figure, then

$$\frac{\sin A}{\|\mathbf{a}\|} = \frac{\sin B}{\|\mathbf{b}\|} = \frac{\sin C}{\|\mathbf{c}\|}.$$



2. Using an Equation Consider the function

$$f(x) = \int_0^x \sqrt{t^4 + 1}\, dt.$$

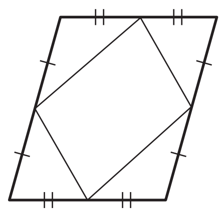
(a) Use a graphing utility to graph the function on the interval $-2 \le x \le 2$.

(b) Find a unit vector parallel to the graph of f at the point $(0, 0)$.

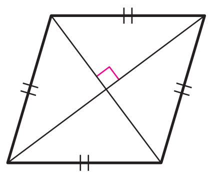
(c) Find a unit vector perpendicular to the graph of f at the point $(0, 0)$.

(d) Find the parametric equations of the tangent line to the graph of f at the point $(0, 0)$.

3. Proof Using vectors, prove that the line segments joining the midpoints of the sides of a parallelogram form a parallelogram (see figure).



4. Proof Using vectors, prove that the diagonals of a rhombus are perpendicular (see figure).



5. Distance

(a) Find the shortest distance between the point $Q(2, 0, 0)$ and the line determined by the points $P_1(0, 0, 1)$ and $P_2(0, 1, 2)$.

(b) Find the shortest distance between the point $Q(2, 0, 0)$ and the line segment joining the points $P_1(0, 0, 1)$ and $P_2(0, 1, 2)$.

6. Orthogonal Vectors Let P_0 be a point in the plane with normal vector **n**. Describe the set of points P in the plane for which $(\mathbf{n} + \overrightarrow{PP_0})$ is orthogonal to $(\mathbf{n} - \overrightarrow{PP_0})$.

7. Volume

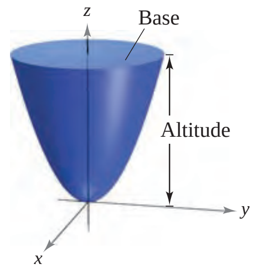
(a) Find the volume of the solid bounded below by the paraboloid $z = x^2 + y^2$ and above by the plane $z = 1$.

(b) Find the volume of the solid bounded below by the elliptic paraboloid

$$z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$$

and above by the plane $z = k$, where $k > 0$.

(c) Show that the volume of the solid in part (b) is equal to one-half the product of the area of the base times the altitude, as shown in the figure.



8. Volume

(a) Use the disk method to find the volume of the sphere $x^2 + y^2 + z^2 = r^2$.

(b) Find the volume of the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.

9. Proof Prove the following property of the cross product.

$$(\mathbf{u} \times \mathbf{v}) \times (\mathbf{w} \times \mathbf{z}) = (\mathbf{u} \times \mathbf{v} \cdot \mathbf{z})\mathbf{w} - (\mathbf{u} \times \mathbf{v} \cdot \mathbf{w})\mathbf{z}$$

10. Using Parametric Equations Consider the line given by the parametric equations

$$x = -t + 3, \quad y = \tfrac{1}{2}t + 1, \quad z = 2t - 1$$

and the point $(4, 3, s)$ for any real number s.

(a) Write the distance between the point and the line as a function of s.

(b) Use a graphing utility to graph the function in part (a). Use the graph to find the value of s such that the distance between the point and the line is minimum.

(c) Use the *zoom* feature of a graphing utility to zoom out several times on the graph in part (b). Does it appear that the graph has slant asymptotes? Explain. If it appears to have slant asymptotes, find them.

11. Sketching Graphs Sketch the graph of each equation given in spherical coordinates.

(a) $\rho = 2 \sin \phi$ (b) $\rho = 2 \cos \phi$

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12. Sketching Graphs Sketch the graph of each equation given in cylindrical coordinates.

(a) $r = 2 \cos \theta$ (b) $z = r^2 \cos 2\theta$

13. Tetherball A tetherball weighing 1 pound is pulled outward from the pole by a horizontal force $\mathbf{u}$ until the rope makes an angle of θ degrees with the pole (see figure).

(a) Determine the resulting tension in the rope and the magnitude of $\mathbf{u}$ when $\theta = 30°$.

(b) Write the tension T in the rope and the magnitude of $\mathbf{u}$ as functions of θ. Determine the domains of the functions.

(c) Use a graphing utility to complete the table.

θ	0°	10°	20°	30°	40°	50°	60°
T							
$\lVert\mathbf{u}\rVert$							

(d) Use a graphing utility to graph the two functions for $0° \le \theta \le 60°$.

(e) Compare T and $\lVert\mathbf{u}\rVert$ as θ increases.

(f) Find (if possible) $\lim_{\theta \to \pi/2^-} T$ and $\lim_{\theta \to \pi/2^-} \lVert\mathbf{u}\rVert$. Are the results what you expected? Explain.

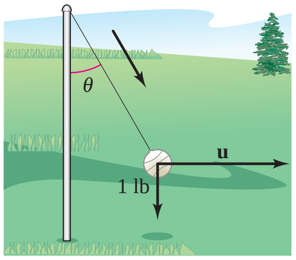


Figure for 13

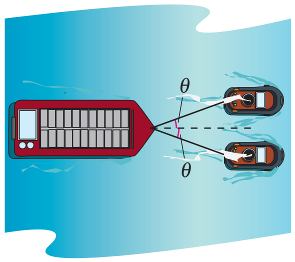


Figure for 14

14. Towing A loaded barge is being towed by two tugboats, and the magnitude of the resultant is 6000 pounds directed along the axis of the barge (see figure). Each towline makes an angle of θ degrees with the axis of the barge.

(a) Find the tension in the towlines when $\theta = 20°$.

(b) Write the tension T of each line as a function of θ. Determine the domain of the function.

(c) Use a graphing utility to complete the table.

θ	10°	20°	30°	40°	50°	60°
T						

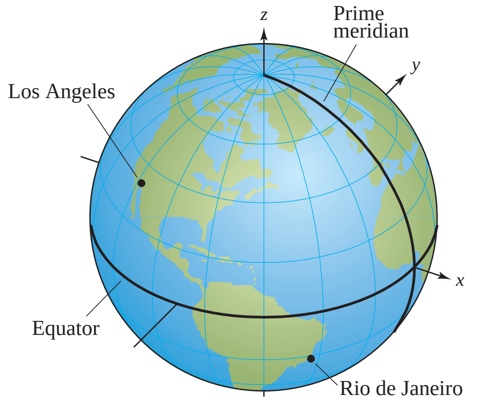
(d) Use a graphing utility to graph the tension function.

(e) Explain why the tension increases as θ increases.

15. Proof Consider the vectors $\mathbf{u} = \langle \cos \alpha, \sin \alpha, 0 \rangle$ and $\mathbf{v} = \langle \cos \beta, \sin \beta, 0 \rangle$, where $\alpha > \beta$. Find the cross product of the vectors and use the result to prove the identity

$$\sin(\alpha - \beta) = \sin \alpha \cos \beta - \cos \alpha \sin \beta.$$

16. Latitude-Longitude System Los Angeles is located at 34.05° North latitude and 118.24° West longitude, and Rio de Janeiro, Brazil, is located at 22.90° South latitude and 43.23° West longitude (see figure). Assume that Earth is spherical and has a radius of 4000 miles.



(a) Find the spherical coordinates for the location of each city.

(b) Find the rectangular coordinates for the location of each city.

(c) Find the angle (in radians) between the vectors from the center of Earth to the two cities.

(d) Find the great-circle distance s between the cities. (*Hint:* $s = r\theta$)

(e) Repeat parts (a)–(d) for the cities of Boston, located at 42.36° North latitude and 71.06° West longitude, and Honolulu, located at 21.31° North latitude and 157.86° West longitude.

17. Distance Between a Point and a Plane Consider the plane that passes through the points P, R, and S. Show that the distance from a point Q to this plane is

$$\text{Distance} = \frac{|\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})|}{\lVert\mathbf{u} \times \mathbf{v}\rVert}$$

where $\mathbf{u} = \overrightarrow{PR}$, $\mathbf{v} = \overrightarrow{PS}$, and $\mathbf{w} = \overrightarrow{PQ}$.

18. Distance Between Parallel Planes Show that the distance between the parallel planes

$$ax + by + cz + d_1 = 0 \quad \text{and} \quad ax + by + cz + d_2 = 0$$

is

$$\text{Distance} = \frac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}.$$

19. Intersection of Planes Show that the curve of intersection of the plane $z = 2y$ and the cylinder $x^2 + y^2 = 1$ is an ellipse.

20. Vector Algebra Read the article "Tooth Tables: Solution of a Dental Problem by Vector Algebra" by Gary Hosler Meisters in *Mathematics Magazine.* (To view this article, go to *MathArticles.com*.) Then write a paragraph explaining how vectors and vector algebra can be used in the construction of dental inlays.

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