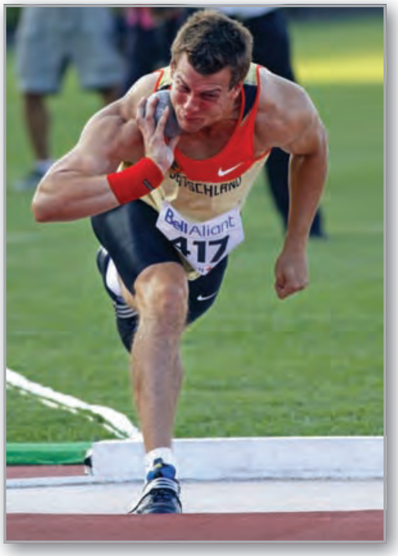
12 Vector-Valued Functions

Speed *(Exercise 68, p. 861)*

Air Traffic Control
(Exercise 65, p. 850)

Football *(Exercise 32, p. 839)*



Shot-Put Throw
(Exercise 42, p. 839)

Playground Slide *(Exercise 81, p. 823)*

Clockwise from top left, 06photo/Shutterstock.com; Elena Aliaga/Shutterstock.com;
Jamie Roach/Shutterstock.com; Jack.Q/Shutterstock.com; Nicholas Moore/Shutterstock.com

12.1 Vector-Valued Functions

- **Analyze and sketch a space curve given by a vector-valued function.**
- **Extend the concepts of limits and continuity to vector-valued functions.**

Space Curves and Vector-Valued Functions

In Section 10.2, a *plane curve* was defined as the set of ordered pairs $(f(t), g(t))$ together with their defining parametric equations

$$x = f(t) \quad \text{and} \quad y = g(t)$$

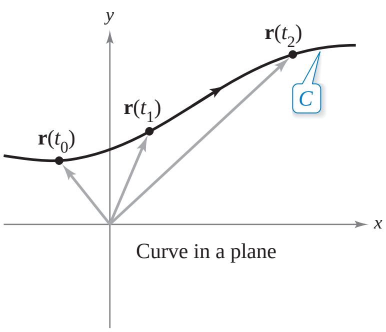
where f and g are continuous functions of t on an interval I. This definition can be extended naturally to three-dimensional space. A **space curve** C is the set of all ordered triples $(f(t), g(t), h(t))$ together with their defining parametric equations

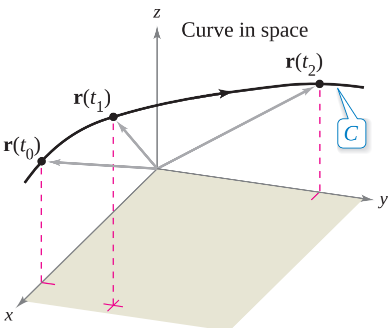
$$x = f(t), \quad y = g(t), \quad \text{and} \quad z = h(t)$$

where f, g, and h are continuous functions of t on an interval I.

Before looking at examples of space curves, a new type of function, called a **vector-valued function,** is introduced. This type of function maps real numbers to vectors.

> **Definition of Vector-Valued Function**
>
> A function of the form
>
> $$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} \qquad \text{Plane}$$
>
> or
>
> $$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k} \qquad \text{Space}$$
>
> is a **vector-valued function,** where the **component functions** f, g, and h are real-valued functions of the parameter t. Vector-valued functions are sometimes denoted as
>
> $$\mathbf{r}(t) = \langle f(t), g(t) \rangle \qquad \text{Plane}$$
>
> or
>
> $$\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle. \qquad \text{Space}$$





Curve C is traced out by the terminal point of position vector $\mathbf{r}(t)$.
Figure 12.1

Technically, a curve in a plane or in space consists of a collection of points and the defining parametric equations. Two different curves can have the same graph. For instance, each of the curves

$$\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} \quad \text{and} \quad \mathbf{r}(t) = \sin t^2\,\mathbf{i} + \cos t^2\,\mathbf{j}$$

has the unit circle as its graph, but these equations do not represent the same curve—because the circle is traced out in different ways on the graphs.

Be sure you see the distinction between the vector-valued function $\mathbf{r}$ and the real-valued functions f, g, and h. All are functions of the real variable t, but $\mathbf{r}(t)$ is a vector, whereas $f(t)$, $g(t)$, and $h(t)$ are real numbers (for each specific value of t).

Vector-valued functions serve dual roles in the representation of curves. By letting the parameter t represent time, you can use a vector-valued function to represent *motion* along a curve. Or, in the more general case, you can use a vector-valued function to *trace the graph* of a curve. In either case, the terminal point of the position vector $\mathbf{r}(t)$ coincides with the point (x, y) or (x, y, z) on the curve given by the parametric equations, as shown in Figure 12.1. The arrowhead on the curve indicates the curve's *orientation* by pointing in the direction of increasing values of t.

Unless stated otherwise, the **domain** of a vector-valued function $\mathbf{r}$ is considered to be the intersection of the domains of the component functions f, g, and h. For instance, the domain of $\mathbf{r}(t) = \ln t\,\mathbf{i} + \sqrt{1-t}\,\mathbf{j} + t\mathbf{k}$ is the interval $(0, 1]$.

EXAMPLE 1 Sketching a Plane Curve

Sketch the plane curve represented by the vector-valued function

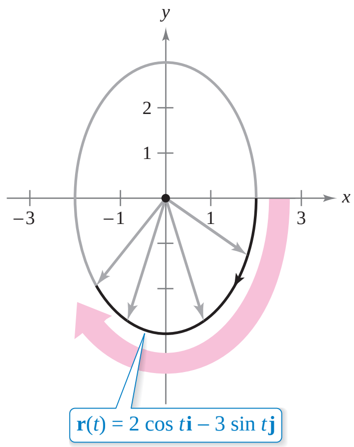
$$\mathbf{r}(t) = 2\cos t\mathbf{i} - 3\sin t\mathbf{j}, \quad 0 \le t \le 2\pi. \qquad \text{Vector-valued function}$$

Solution From the position vector $\mathbf{r}(t)$, you can write the parametric equations

$$x = 2\cos t \quad \text{and} \quad y = -3\sin t.$$

Solving for $\cos t$ and $\sin t$ and using the identity $\cos^2 t + \sin^2 t = 1$ produces the rectangular equation

$$\frac{x^2}{2^2} + \frac{y^2}{3^2} = 1. \qquad \text{Rectangular equation}$$



The ellipse is traced clockwise as t increases from 0 to 2π.
Figure 12.2

The graph of this rectangular equation is the ellipse shown in Figure 12.2. The curve has a *clockwise* orientation. That is, as t increases from 0 to 2π, the position vector $\mathbf{r}(t)$ moves clockwise, and its terminal point traces the ellipse.

EXAMPLE 2 Sketching a Space Curve

See LarsonCalculus.com for an interactive version of this type of example.

Sketch the space curve represented by the vector-valued function

$$\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + t\mathbf{k}, \quad 0 \le t \le 4\pi. \qquad \text{Vector-valued function}$$

Solution From the first two parametric equations

$$x = 4\cos t \quad \text{and} \quad y = 4\sin t$$

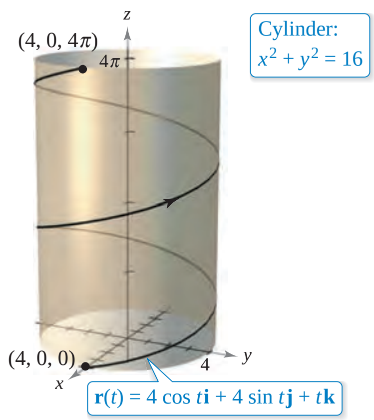
you can obtain

$$x^2 + y^2 = 16. \qquad \text{Rectangular equation}$$

This means that the curve lies on a right circular cylinder of radius 4, centered about the z-axis. To locate the curve on this cylinder, you can use the third parametric equation

$$z = t.$$

In Figure 12.3, note that as t increases from 0 to 4π, the point (x, y, z) spirals up the cylinder to produce a **helix.** A real-life example of a helix is shown in the drawing at the left.



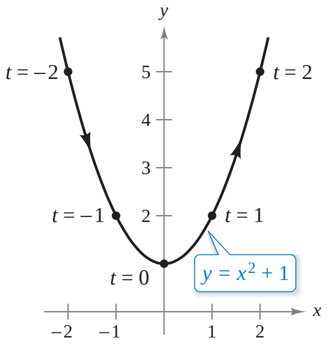
As t increases from 0 to 4π, two spirals on the helix are traced out.
Figure 12.3

In 1953, Francis Crick and James D. Watson discovered the double helix structure of DNA.

In Examples 1 and 2, you were given a vector-valued function and were asked to sketch the corresponding curve. The next two examples address the reverse problem—finding a vector-valued function to represent a given graph. Of course, when the graph is described parametrically, representation by a vector-valued function is straightforward. For instance, to represent the line in space given by $x = 2 + t$, $y = 3t$, and $z = 4 - t$, you can simply use the vector-valued function

$$\mathbf{r}(t) = (2 + t)\mathbf{i} + 3t\mathbf{j} + (4 - t)\mathbf{k}.$$

When a set of parametric equations for the graph is not given, the problem of representing the graph by a vector-valued function boils down to finding a set of parametric equations.



There are many ways to parametrize this graph. One way is to let $x = t$.
Figure 12.4

EXAMPLE 3 Representing a Graph: Vector-Valued Function

Represent the parabola

$$y = x^2 + 1$$

by a vector-valued function.

Solution Although there are many ways to choose the parameter t, a natural choice is to let $x = t$. Then $y = t^2 + 1$ and you have

$$\mathbf{r}(t) = t\mathbf{i} + (t^2 + 1)\mathbf{j}.$$ Vector-valued function

Note in Figure 12.4 the orientation produced by this particular choice of parameter. Had you chosen $x = -t$ as the parameter, the curve would have been oriented in the opposite direction.

EXAMPLE 4 Representing a Graph: Vector-Valued Function

Sketch the space curve C represented by the intersection of the semiellipsoid

$$\frac{x^2}{12} + \frac{y^2}{24} + \frac{z^2}{4} = 1, \quad z \geq 0$$

and the parabolic cylinder $y = x^2$. Then find a vector-valued function to represent the graph.

Solution The intersection of the two surfaces is shown in Figure 12.5. As in Example 3, a natural choice of parameter is $x = t$. For this choice, you can use the given equation $y = x^2$ to obtain $y = t^2$. Then it follows that

$$\frac{z^2}{4} = 1 - \frac{x^2}{12} - \frac{y^2}{24} = 1 - \frac{t^2}{12} - \frac{t^4}{24} = \frac{24 - 2t^2 - t^4}{24} = \frac{(6 + t^2)(4 - t^2)}{24}.$$

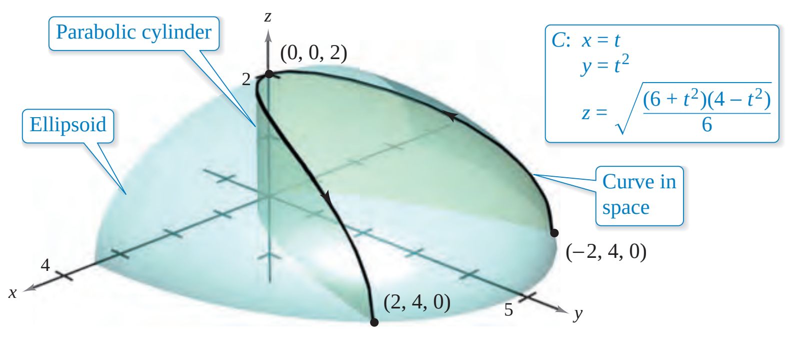
Because the curve lies above the xy-plane, you should choose the positive square root for z and obtain the parametric equations

$$x = t, \quad y = t^2, \quad \text{and} \quad z = \sqrt{\frac{(6 + t^2)(4 - t^2)}{6}}.$$

The resulting vector-valued function is

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \sqrt{\frac{(6 + t^2)(4 - t^2)}{6}}\,\mathbf{k}, \quad -2 \leq t \leq 2.$$ Vector-valued function

(Note that the $\mathbf{k}$-component of $\mathbf{r}(t)$ implies $-2 \leq t \leq 2$.) From the points $(-2, 4, 0)$ and $(2, 4, 0)$ shown in Figure 12.5, you can see that the curve is traced as t increases from -2 to 2.



The curve C is the intersection of the semiellipsoid and the parabolic cylinder.
Figure 12.5

REMARK Curves in space can be specified in various ways. For instance, the curve in Example 4 is described as the intersection of two surfaces in space.

Limits and Continuity

Many techniques and definitions used in the calculus of real-valued functions can be applied to vector-valued functions. For instance, you can add and subtract vector-valued functions, multiply a vector-valued function by a scalar, take the limit of a vector-valued function, differentiate a vector-valued function, and so on. The basic approach is to capitalize on the linearity of vector operations by extending the definitions on a component-by-component basis. For example, to add two vector-valued functions (in the plane), you can write

$$\begin{aligned}\mathbf{r}_1(t) + \mathbf{r}_2(t) &= [f_1(t)\mathbf{i} + g_1(t)\mathbf{j}] + [f_2(t)\mathbf{i} + g_2(t)\mathbf{j}] \\ &= [f_1(t) + f_2(t)]\mathbf{i} + [g_1(t) + g_2(t)]\mathbf{j}.\end{aligned}$$

Sum

To subtract two vector-valued functions, you can write

$$\begin{aligned}\mathbf{r}_1(t) - \mathbf{r}_2(t) &= [f_1(t)\mathbf{i} + g_1(t)\mathbf{j}] - [f_2(t)\mathbf{i} + g_2(t)\mathbf{j}] \\ &= [f_1(t) - f_2(t)]\mathbf{i} + [g_1(t) - g_2(t)]\mathbf{j}.\end{aligned}$$

Difference

Similarly, to multiply a vector-valued function by a scalar, you can write

$$\begin{aligned}c\mathbf{r}(t) &= c[f_1(t)\mathbf{i} + g_1(t)\mathbf{j}] \\ &= cf_1(t)\mathbf{i} + cg_1(t)\mathbf{j}.\end{aligned}$$

Scalar multiplication

To divide a vector-valued function by a scalar, you can write

$$\begin{aligned}\frac{\mathbf{r}(t)}{c} &= \frac{[f_1(t)\mathbf{i} + g_1(t)\mathbf{j}]}{c}, \quad c \neq 0 \\ &= \frac{f_1(t)}{c}\mathbf{i} + \frac{g_1(t)}{c}\mathbf{j}.\end{aligned}$$

Scalar division

This component-by-component extension of operations with real-valued functions to vector-valued functions is further illustrated in the definition of the limit of a vector-valued function.

Definition of the Limit of a Vector-Valued Function

1. If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, then

$$\lim_{t \to a} \mathbf{r}(t) = \left[\lim_{t \to a} f(t)\right]\mathbf{i} + \left[\lim_{t \to a} g(t)\right]\mathbf{j}$$

Plane

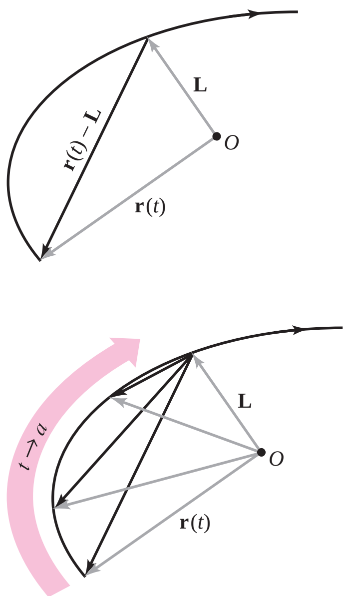
provided f and g have limits as $t \to a$.

2. If $\mathbf{r}$ is a vector-valued function such that $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, then

$$\lim_{t \to a} \mathbf{r}(t) = \left[\lim_{t \to a} f(t)\right]\mathbf{i} + \left[\lim_{t \to a} g(t)\right]\mathbf{j} + \left[\lim_{t \to a} h(t)\right]\mathbf{k}$$

Space

provided f, g, and h have limits as $t \to a$.



As t approaches a, $\mathbf{r}(t)$ approaches the limit $\mathbf{L}$. For the limit $\mathbf{L}$ to exist, it is not necessary that $\mathbf{r}(a)$ be defined or that $\mathbf{r}(a)$ be equal to $\mathbf{L}$.

Figure 12.6

If $\mathbf{r}(t)$ approaches the vector $\mathbf{L}$ as $t \to a$, then the length of the vector $\mathbf{r}(t) - \mathbf{L}$ approaches 0. That is,

$$\|\mathbf{r}(t) - \mathbf{L}\| \to 0 \quad \text{as} \quad t \to a.$$

This is illustrated graphically in Figure 12.6. With this definition of the limit of a vector-valued function, you can develop vector versions of most of the limit theorems given in Chapter 2. For example, the limit of the sum of two vector-valued functions is the sum of their individual limits. Also, you can use the orientation of the curve $\mathbf{r}(t)$ to define one-sided limits of vector-valued functions. The next definition extends the notion of continuity to vector-valued functions.

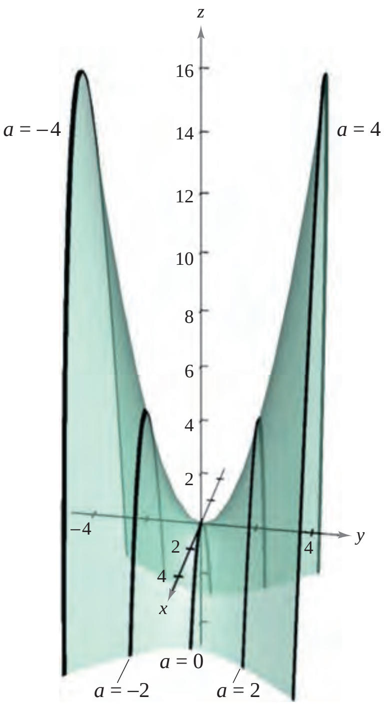
Definition of Continuity of a Vector-Valued Function

A vector-valued function $\mathbf{r}$ is **continuous at the point** given by $t = a$ when the limit of $\mathbf{r}(t)$ exists as $t \to a$ and

$$\lim_{t \to a} \mathbf{r}(t) = \mathbf{r}(a).$$

A vector-valued function $\mathbf{r}$ is **continuous on an interval** I when it is continuous at every point in the interval.

From this definition, it follows that a vector-valued function is continuous at $t = a$ if and only if each of its component functions is continuous at $t = a$.



For each value of a, the curve represented by the vector-valued function $\mathbf{r}(t) = t\mathbf{i} + a\mathbf{j} + (a^2 - t^2)\mathbf{k}$ is a parabola.
Figure 12.7

TECHNOLOGY Almost any type of three-dimensional sketch is difficult to do by hand, but sketching curves in space is especially difficult. The problem is trying to create the illusion of three dimensions. Graphing utilities use a variety of techniques to add "three-dimensionality" to graphs of space curves: one way is to show the curve on a surface, as in Figure 12.7.

EXAMPLE 5 Continuity of a Vector-Valued Function

Discuss the continuity of the vector-valued function

$$\mathbf{r}(t) = t\mathbf{i} + a\mathbf{j} + (a^2 - t^2)\mathbf{k} \qquad a \text{ is a constant.}$$

at $t = 0$.

Solution As t approaches 0, the limit is

$$\begin{aligned}\lim_{t \to 0} \mathbf{r}(t) &= \left[\lim_{t \to 0} t\right]\mathbf{i} + \left[\lim_{t \to 0} a\right]\mathbf{j} + \left[\lim_{t \to 0} (a^2 - t^2)\right]\mathbf{k} \\ &= 0\mathbf{i} + a\mathbf{j} + a^2\mathbf{k} \\ &= a\mathbf{j} + a^2\mathbf{k}.\end{aligned}$$

Because

$$\begin{aligned}\mathbf{r}(0) &= (0)\mathbf{i} + (a)\mathbf{j} + (a^2)\mathbf{k} \\ &= a\mathbf{j} + a^2\mathbf{k}\end{aligned}$$

you can conclude that $\mathbf{r}$ is continuous at $t = 0$. By similar reasoning, you can conclude that the vector-valued function $\mathbf{r}$ is continuous at all real-number values of t. ■

For each value of a, the curve represented by the vector-valued function in Example 5

$$\mathbf{r}(t) = t\mathbf{i} + a\mathbf{j} + (a^2 - t^2)\mathbf{k} \qquad a \text{ is a constant.}$$

is a parabola. You can think of each parabola as the intersection of the vertical plane $y = a$ and the hyperbolic paraboloid

$$y^2 - x^2 = z$$

as shown in Figure 12.7.

EXAMPLE 6 Continuity of a Vector-Valued Function

Determine the interval(s) on which the vector-valued function

$$\mathbf{r}(t) = t\mathbf{i} + \sqrt{t + 1}\,\mathbf{j} + (t^2 + 1)\mathbf{k}$$

is continuous.

Solution The component functions are $f(t) = t$, $g(t) = \sqrt{t + 1}$, and $h(t) = (t^2 + 1)$. Both f and h are continuous for all real-number values of t. The function g, however, is continuous only for $t \geq -1$. So, $\mathbf{r}$ is continuous on the interval $[-1, \infty)$. ■

12.1 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Finding the Domain **In Exercises 1–8, find the domain of the vector-valued function.**

1. $\mathbf{r}(t) = \dfrac{1}{t+1}\mathbf{i} + \dfrac{t}{2}\mathbf{j} - 3t\mathbf{k}$

2. $\mathbf{r}(t) = \sqrt{4-t^2}\,\mathbf{i} + t^2\mathbf{j} - 6t\mathbf{k}$

3. $\mathbf{r}(t) = \ln t\,\mathbf{i} - e^t\,\mathbf{j} - t\mathbf{k}$

4. $\mathbf{r}(t) = \sin t\,\mathbf{i} + 4\cos t\,\mathbf{j} + t\mathbf{k}$

5. $\mathbf{r}(t) = \mathbf{F}(t) + \mathbf{G}(t)$, where

$\mathbf{F}(t) = \cos t\,\mathbf{i} - \sin t\,\mathbf{j} + \sqrt{t}\,\mathbf{k}, \quad \mathbf{G}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j}$

6. $\mathbf{r}(t) = \mathbf{F}(t) - \mathbf{G}(t)$, where

$\mathbf{F}(t) = \ln t\,\mathbf{i} + 5t\mathbf{j} - 3t^2\mathbf{k}, \quad \mathbf{G}(t) = \mathbf{i} + 4t\mathbf{j} - 3t^2\mathbf{k}$

7. $\mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t)$, where

$\mathbf{F}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j}, \quad \mathbf{G}(t) = \sin t\,\mathbf{j} + \cos t\,\mathbf{k}$

8. $\mathbf{r}(t) = \mathbf{F}(t) \times \mathbf{G}(t)$, where

$\mathbf{F}(t) = t^3\mathbf{i} - t\mathbf{j} + t\mathbf{k}, \quad \mathbf{G}(t) = \sqrt[3]{t}\,\mathbf{i} + \dfrac{1}{t+1}\mathbf{j} + (t+2)\mathbf{k}$

Evaluating a Function **In Exercises 9–12, evaluate (if possible) the vector-valued function at each given value of *t*.**

9. $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - (t-1)\mathbf{j}$

(a) $\mathbf{r}(1)$ (b) $\mathbf{r}(0)$ (c) $\mathbf{r}(s+1)$

(d) $\mathbf{r}(2+\Delta t) - \mathbf{r}(2)$

10. $\mathbf{r}(t) = \cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$

(a) $\mathbf{r}(0)$ (b) $\mathbf{r}(\pi/4)$ (c) $\mathbf{r}(\theta - \pi)$

(d) $\mathbf{r}(\pi/6 + \Delta t) - \mathbf{r}(\pi/6)$

11. $\mathbf{r}(t) = \ln t\,\mathbf{i} + \dfrac{1}{t}\mathbf{j} + 3t\mathbf{k}$

(a) $\mathbf{r}(2)$ (b) $\mathbf{r}(-3)$ (c) $\mathbf{r}(t-4)$

(d) $\mathbf{r}(1+\Delta t) - \mathbf{r}(1)$

12. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + t^{3/2}\mathbf{j} + e^{-t/4}\mathbf{k}$

(a) $\mathbf{r}(0)$ (b) $\mathbf{r}(4)$ (c) $\mathbf{r}(c+2)$

(d) $\mathbf{r}(9+\Delta t) - \mathbf{r}(9)$

Writing a Vector-Valued Function **In Exercises 13–16, represent the line segment from *P* to *Q* by a vector-valued function and by a set of parametric equations.**

13. $P(0, 0, 0), Q(3, 1, 2)$

14. $P(0, 2, -1), Q(4, 7, 2)$

15. $P(-2, 5, -3), Q(-1, 4, 9)$

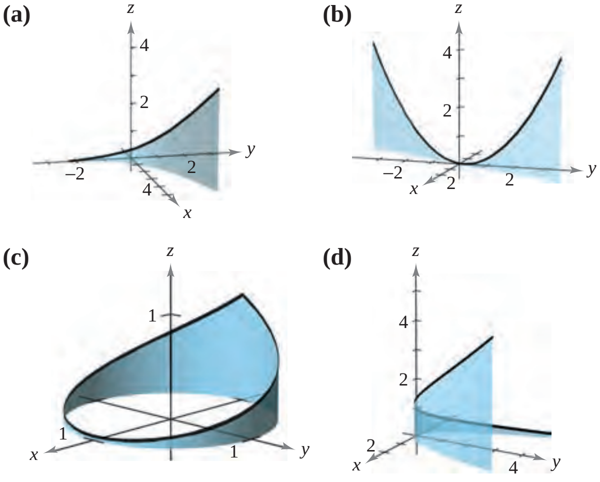
16. $P(1, -6, 8), Q(-3, -2, 5)$

Think About It **In Exercises 17 and 18, find $\mathbf{r}(t) \cdot \mathbf{u}(t)$. Is the result a vector-valued function? Explain.**

17. $\mathbf{r}(t) = (3t-1)\mathbf{i} + \frac{1}{4}t^3\mathbf{j} + 4\mathbf{k}, \quad \mathbf{u}(t) = t^2\mathbf{i} - 8\mathbf{j} + t^3\mathbf{k}$

18. $\mathbf{r}(t) = \langle 3\cos t, 2\sin t, t-2\rangle, \quad \mathbf{u}(t) = \langle 4\sin t, -6\cos t, t^2\rangle$

Matching **In Exercises 19–22, match the equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]**



19. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} + t^2\mathbf{k}, \quad -2 \le t \le 2$

20. $\mathbf{r}(t) = \cos(\pi t)\mathbf{i} + \sin(\pi t)\mathbf{j} + t^2\mathbf{k}, \quad -1 \le t \le 1$

21. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + e^{0.75t}\mathbf{k}, \quad -2 \le t \le 2$

22. $\mathbf{r}(t) = t\mathbf{i} + \ln t\,\mathbf{j} + \dfrac{2t}{3}\mathbf{k}, \quad 0.1 \le t \le 5$

Sketching a Curve **In Exercises 23–38, sketch the curve represented by the vector-valued function and give the orientation of the curve.**

23. $\mathbf{r}(t) = \dfrac{t}{4}\mathbf{i} + (t-1)\mathbf{j}$

24. $\mathbf{r}(t) = (5-t)\mathbf{i} + \sqrt{t}\,\mathbf{j}$

25. $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}$

26. $\mathbf{r}(t) = (t^2+t)\mathbf{i} + (t^2-t)\mathbf{j}$

27. $\mathbf{r}(\theta) = \cos\theta\mathbf{i} + 3\sin\theta\mathbf{j}$

28. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$

29. $\mathbf{r}(\theta) = 3\sec\theta\mathbf{i} + 2\tan\theta\mathbf{j}$

30. $\mathbf{r}(t) = 2\cos^3 t\,\mathbf{i} + 2\sin^3 t\,\mathbf{j}$

31. $\mathbf{r}(t) = (-t+1)\mathbf{i} + (4t+2)\mathbf{j} + (2t+3)\mathbf{k}$

32. $\mathbf{r}(t) = t\mathbf{i} + (2t-5)\mathbf{j} + 3t\mathbf{k}$

33. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + t\mathbf{k}$

34. $\mathbf{r}(t) = t\mathbf{i} + 3\cos t\,\mathbf{j} + 3\sin t\,\mathbf{k}$

35. $\mathbf{r}(t) = 2\sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} + e^{-t}\,\mathbf{k}$

36. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j} + \frac{3}{2}t\mathbf{k}$

37. $\mathbf{r}(t) = \left\langle t, t^2, \frac{2}{3}t^3\right\rangle$

38. $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t\rangle$

Identifying a Common Curve **In Exercises 39–42, use a computer algebra system to graph the vector-valued function and identify the common curve.**

39. $\mathbf{r}(t) = -\dfrac{1}{2}t^2\mathbf{i} + t\mathbf{j} - \dfrac{\sqrt{3}}{2}t^2\mathbf{k}$

40. $\mathbf{r}(t) = t\mathbf{i} - \frac{\sqrt{3}}{2}t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

41. $\mathbf{r}(t) = \sin t\mathbf{i} + \left(\frac{\sqrt{3}}{2}\cos t - \frac{1}{2}t\right)\mathbf{j} + \left(\frac{1}{2}\cos t + \frac{\sqrt{3}}{2}\right)\mathbf{k}$

42. $\mathbf{r}(t) = -\sqrt{2}\sin t\mathbf{i} + 2\cos t\mathbf{j} + \sqrt{2}\sin t\mathbf{k}$

Think About It **In Exercises 43 and 44, use a computer algebra system to graph the vector-valued function r(*t*). For each u(*t*), make a conjecture about the transformation (if any) of the graph of r(*t*). Use a computer algebra system to verify your conjecture.**

43. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + \frac{1}{2}t\mathbf{k}$

(a) $\mathbf{u}(t) = 2(\cos t - 1)\mathbf{i} + 2\sin t\mathbf{j} + \frac{1}{2}t\mathbf{k}$

(b) $\mathbf{u}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + 2t\mathbf{k}$

(c) $\mathbf{u}(t) = 2\cos(-t)\mathbf{i} + 2\sin(-t)\mathbf{j} + \frac{1}{2}(-t)\mathbf{k}$

(d) $\mathbf{u}(t) = \frac{1}{2}t\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$

(e) $\mathbf{u}(t) = 6\cos t\mathbf{i} + 6\sin t\mathbf{j} + \frac{1}{2}t\mathbf{k}$

44. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^3\mathbf{k}$

(a) $\mathbf{u}(t) = t\mathbf{i} + (t^2 - 2)\mathbf{j} + \frac{1}{2}t^3\mathbf{k}$

(b) $\mathbf{u}(t) = t^2\mathbf{i} + t\mathbf{j} + \frac{1}{2}t^3\mathbf{k}$

(c) $\mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \left(\frac{1}{2}t^3 + 4\right)\mathbf{k}$

(d) $\mathbf{u}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{8}t^3\mathbf{k}$

(e) $\mathbf{u}(t) = (-t)\mathbf{i} + (-t)^2\mathbf{j} + \frac{1}{2}(-t)^3\mathbf{k}$

Representing a Graph by a Vector-Valued Function **In Exercises 45–52, represent the plane curve by a vector-valued function. (There are many correct answers.)**

45. $y = x + 5$

46. $2x - 3y + 5 = 0$

47. $y = (x - 2)^2$

48. $y = 4 - x^2$

49. $x^2 + y^2 = 25$

50. $(x - 2)^2 + y^2 = 4$

51. $\frac{x^2}{16} - \frac{y^2}{4} = 1$

52. $\frac{x^2}{9} + \frac{y^2}{16} = 1$

Representing a Graph by a Vector-Valued Function **In Exercises 53–60, sketch the space curve represented by the intersection of the surfaces. Then represent the curve by a vector-valued function using the given parameter.**

	Surfaces	**Parameter**
53.	$z = x^2 + y^2, \quad x + y = 0$	$x = t$
54.	$z = x^2 + y^2, \quad z = 4$	$x = 2\cos t$
55.	$x^2 + y^2 = 4, \quad z = x^2$	$x = 2\sin t$
56.	$4x^2 + 4y^2 + z^2 = 16, \quad x = z^2$	$z = t$
57.	$x^2 + y^2 + z^2 = 4, \quad x + z = 2$	$x = 1 + \sin t$
58.	$x^2 + y^2 + z^2 = 10, \quad x + y = 4$	$x = 2 + \sin t$
59.	$x^2 + z^2 = 4, \quad y^2 + z^2 = 4$	$x = t$ (first octant)
60.	$x^2 + y^2 + z^2 = 16, \quad xy = 4$	$x = t$ (first octant)

61. Sketching a Curve Show that the vector-valued function $\mathbf{r}(t) = t\mathbf{i} + 2t\cos t\mathbf{j} + 2t\sin t\mathbf{k}$ lies on the cone $4x^2 = y^2 + z^2$. Sketch the curve.

62. Sketching a Curve Show that the vector-valued function $\mathbf{r}(t) = e^{-t}\cos t\mathbf{i} + e^{-t}\sin t\mathbf{j} + e^{-t}\mathbf{k}$ lies on the cone $z^2 = x^2 + y^2$. Sketch the curve.

Finding a Limit **In Exercises 63–68, find the limit (if it exists).**

63. $\lim_{t\to\pi}(t\mathbf{i} + \cos t\mathbf{j} + \sin t\mathbf{k})$

64. $\lim_{t\to 2}\left(3t\mathbf{i} + \frac{2}{t^2 - 1}\mathbf{j} + \frac{1}{t}\mathbf{k}\right)$

65. $\lim_{t\to 0}\left(t^2\mathbf{i} + 3t\mathbf{j} + \frac{1 - \cos t}{t}\mathbf{k}\right)$

66. $\lim_{t\to 1}\left(\sqrt{t}\,\mathbf{i} + \frac{\ln t}{t^2 - 1}\mathbf{j} + \frac{1}{t - 1}\mathbf{k}\right)$

67. $\lim_{t\to 0}\left(e^t\mathbf{i} + \frac{\sin t}{t}\mathbf{j} + e^{-t}\mathbf{k}\right)$

68. $\lim_{t\to\infty}\left(e^{-t}\mathbf{i} + \frac{1}{t}\mathbf{j} + \frac{t}{t^2 + 1}\mathbf{k}\right)$

Continuity of a Vector-Valued Function **In Exercises 69–74, determine the interval(s) on which the vector-valued function is continuous.**

69. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}$

70. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \sqrt{t - 1}\,\mathbf{j}$

71. $\mathbf{r}(t) = t\mathbf{i} + \arcsin t\mathbf{j} + (t - 1)\mathbf{k}$

72. $\mathbf{r}(t) = 2e^{-t}\mathbf{i} + e^{-t}\mathbf{j} + \ln(t - 1)\mathbf{k}$

73. $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan t\rangle$

74. $\mathbf{r}(t) = \langle 8, \sqrt{t}, \sqrt[3]{t}\rangle$

WRITING ABOUT CONCEPTS

Writing a Transformation **In Exercises 75–78, consider the vector-valued function**

$$\mathbf{r}(t) = t^2\mathbf{i} + (t - 3)\mathbf{j} + t\mathbf{k}.$$

Write a vector-valued function s(*t*) that is the specified transformation of r.

75. A vertical translation three units upward

76. A vertical translation four units downward

77. A horizontal translation two units in the direction of the negative x-axis

78. A horizontal translation five units in the direction of the positive y-axis

79. Continuity of a Vector-Valued Function State the definition of continuity of a vector-valued function. Give an example of a vector-valued function that is defined but not continuous at $t = 2$.

80. Comparing Functions Which of the following vector-valued functions represent the same graph?

(a) $\mathbf{r}(t) = (-3\cos t + 1)\mathbf{i} + (5\sin t + 2)\mathbf{j} + 4\mathbf{k}$

(b) $\mathbf{r}(t) = 4\mathbf{i} + (-3\cos t + 1)\mathbf{j} + (5\sin t + 2)\mathbf{k}$

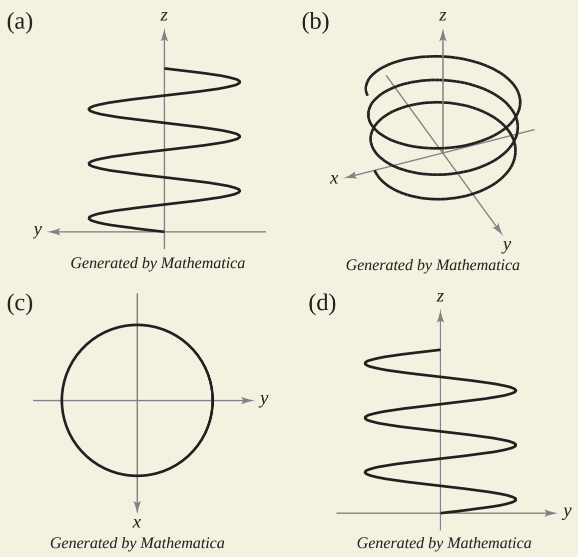
(c) $\mathbf{r}(t) = (3\cos t - 1)\mathbf{i} + (-5\sin t - 2)\mathbf{j} + 4\mathbf{k}$

(d) $\mathbf{r}(t) = (-3\cos 2t + 1)\mathbf{i} + (5\sin 2t + 2)\mathbf{j} + 4\mathbf{k}$

81. Playground Slide

The outer edge of a playground slide is in the shape of a helix of radius 1.5 meters. The slide has a height of 2 meters and makes one complete revolution from top to bottom. Find a vector-valued function for the helix. Use a computer algebra system to graph your function. (There are many correct answers.)

82. HOW DO YOU SEE IT? The four figures below are graphs of the vector-valued function $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + (t/4)\mathbf{k}$. Match each of the four graphs with the point in space from which the helix is viewed. The four points are $(0, 0, 20)$, $(20, 0, 0)$, $(-20, 0, 0)$, and $(10, 20, 10)$.



83. Proof Let $\mathbf{r}(t)$ and $\mathbf{u}(t)$ be vector-valued functions whose limits exist as $t \to c$. Prove that

$$\lim_{t\to c}[\mathbf{r}(t) \times \mathbf{u}(t)] = \lim_{t\to c}\mathbf{r}(t) \times \lim_{t\to c}\mathbf{u}(t).$$

84. Proof Let $\mathbf{r}(t)$ and $\mathbf{u}(t)$ be vector-valued functions whose limits exist as $t \to c$. Prove that

$$\lim_{t\to c}[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \lim_{t\to c}\mathbf{r}(t) \cdot \lim_{t\to c}\mathbf{u}(t).$$

85. Proof Prove that if $\mathbf{r}$ is a vector-valued function that is continuous at c, then $\|\mathbf{r}\|$ is continuous at c.

86. Verifying a Converse Verify that the converse of Exercise 85 is not true by finding a vector-valued function $\mathbf{r}$ such that $\|\mathbf{r}\|$ is continuous at c but $\mathbf{r}$ is not continuous at c.

Particle Motion **In Exercises 87 and 88, two particles travel along the space curves $\mathbf{r}(t)$ and $\mathbf{u}(t)$. A collision will occur at the point of intersection P when both particles are at P at the same time. Do the particles collide? Do their paths intersect?**

87. $\mathbf{r}(t) = t^2\mathbf{i} + (9t - 20)\mathbf{j} + t^2\mathbf{k}$

$\mathbf{u}(t) = (3t + 4)\mathbf{i} + t^2\mathbf{j} + (5t - 4)\mathbf{k}$

88. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$

$\mathbf{u}(t) = (-2t + 3)\mathbf{i} + 8t\mathbf{j} + (12t + 2)\mathbf{k}$

Think About It **In Exercises 89 and 90, two particles travel along the space curves $\mathbf{r}(t)$ and $\mathbf{u}(t)$.**

89. If $\mathbf{r}(t)$ and $\mathbf{u}(t)$ intersect, will the particles collide?

90. If the particles collide, do their paths $\mathbf{r}(t)$ and $\mathbf{u}(t)$ intersect?

True or False? **In Exercises 91–94, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

91. If f, g, and h are first-degree polynomial functions, then the curve given by $x = f(t)$, $y = g(t)$, and $z = h(t)$ is a line.

92. If the curve given by $x = f(t)$, $y = g(t)$, and $z = h(t)$ is a line, then f, g, and h are first-degree polynomial functions of t.

93. Two particles travel along the space curves $\mathbf{r}(t)$ and $\mathbf{u}(t)$. The intersection of their paths depends only on the curves traced out by $\mathbf{r}(t)$ and $\mathbf{u}(t)$, while collision depends on the parametrizations.

94. The vector-valued function $\mathbf{r}(t) = t^2\mathbf{i} + t\sin t\,\mathbf{j} + t\cos t\,\mathbf{k}$ lies on the paraboloid $x = y^2 + z^2$.

SECTION PROJECT

Witch of Agnesi

In Section 4.5, you studied a famous curve called the **Witch of Agnesi.** In this project, you will take a closer look at this function.

Consider a circle of radius a centered on the y-axis at $(0, a)$. Let A be a point on the horizontal line $y = 2a$, let O be the origin, and let B be the point where the segment OA intersects the circle. A point P is on the Witch of Agnesi when P lies on the horizontal line through B and on the vertical line through A.

(a) Show that the point A is traced out by the vector-valued function $\mathbf{r}_A(\theta) = 2a\cot\theta\,\mathbf{i} + 2a\mathbf{j}$ for $0 < \theta < \pi$, where θ is the angle that OA makes with the positive x-axis.

(b) Show that the point B is traced out by the vector-valued function $\mathbf{r}_B(\theta) = a\sin 2\theta\,\mathbf{i} + a(1 - \cos 2\theta)\mathbf{j}$ for $0 < \theta < \pi$.

(c) Combine the results of parts (a) and (b) to find the vector-valued function $\mathbf{r}(\theta)$ for the Witch of Agnesi. Use a graphing utility to graph this curve for $a = 1$.

(d) Describe the limits $\lim_{\theta\to 0^+}\mathbf{r}(\theta)$ and $\lim_{\theta\to\pi^-}\mathbf{r}(\theta)$.

(e) Eliminate the parameter θ and determine the rectangular equation of the Witch of Agnesi. Use a graphing utility to graph this function for $a = 1$ and compare your graph with that obtained in part (c).

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12.2 Differentiation and Integration of Vector-Valued Functions

- **Differentiate a vector-valued function.**
- **Integrate a vector-valued function.**

Differentiation of Vector-Valued Functions

In Sections 12.3–12.5, you will study several important applications involving the calculus of vector-valued functions. In preparation for that study, this section is devoted to the mechanics of differentiation and integration of vector-valued functions.

The definition of the derivative of a vector-valued function parallels the definition for real-valued functions.

REMARK In addition to $\mathbf{r}'(t)$, other notations for the derivative of a vector-valued function are

$$\frac{d}{dt}[\mathbf{r}(t)], \quad \frac{d\mathbf{r}}{dt}, \quad \text{and} \quad D_t[\mathbf{r}(t)].$$

Definition of the Derivative of a Vector-Valued Function

The **derivative of a vector-valued function r** is

$$\mathbf{r}'(t) = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

for all t for which the limit exists. If $\mathbf{r}'(t)$ exists, then $\mathbf{r}$ is **differentiable at *t*.** If $\mathbf{r}'(t)$ exists for all t in an open interval I, then $\mathbf{r}$ is **differentiable on the interval *I*.** Differentiability of vector-valued functions can be extended to closed intervals by considering one-sided limits.

Differentiation of vector-valued functions can be done on a *component-by-component basis.* To see why this is true, consider the function $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$. Applying the definition of the derivative produces the following.

$$\begin{aligned}
\mathbf{r}'(t) &= \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t} \\
&= \lim_{\Delta t \to 0} \frac{f(t + \Delta t)\mathbf{i} + g(t + \Delta t)\mathbf{j} - f(t)\mathbf{i} - g(t)\mathbf{j}}{\Delta t} \\
&= \lim_{\Delta t \to 0} \left\{\left[\frac{f(t + \Delta t) - f(t)}{\Delta t}\right]\mathbf{i} + \left[\frac{g(t + \Delta t) - g(t)}{\Delta t}\right]\mathbf{j}\right\} \\
&= \left\{\lim_{\Delta t \to 0}\left[\frac{f(t + \Delta t) - f(t)}{\Delta t}\right]\right\}\mathbf{i} + \left\{\lim_{\Delta t \to 0}\left[\frac{g(t + \Delta t) - g(t)}{\Delta t}\right]\right\}\mathbf{j} \\
&= f'(t)\mathbf{i} + g'(t)\mathbf{j}
\end{aligned}$$

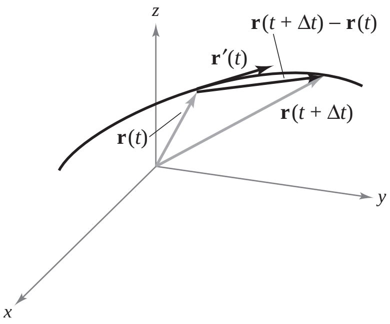


Figure 12.8

This important result is listed in the theorem shown below. Note that the derivative of the vector-valued function $\mathbf{r}$ is itself a vector-valued function. You can see from Figure 12.8 that $\mathbf{r}'(t)$ is a vector tangent to the curve given by $\mathbf{r}(t)$ and pointing in the direction of increasing t-values.

THEOREM 12.1 Differentiation of Vector-Valued Functions

1. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where f and g are differentiable functions of t, then

   $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j}$. Plane

2. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are differentiable functions of t, then

   $\mathbf{r}'(t) = f'(t)\mathbf{i} + g'(t)\mathbf{j} + h'(t)\mathbf{k}$. Space

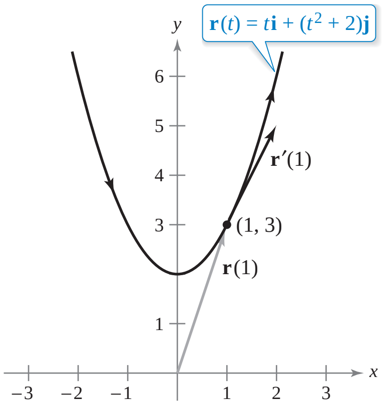


Figure 12.9

EXAMPLE 1 Differentiation of a Vector-Valued Function

See LarsonCalculus.com for an interactive version of this type of example.

For the vector-valued function

$$\mathbf{r}(t) = t\mathbf{i} + (t^2 + 2)\mathbf{j}$$

find $\mathbf{r}'(t)$. Then sketch the plane curve represented by $\mathbf{r}(t)$ and the graphs of $\mathbf{r}(1)$ and $\mathbf{r}'(1)$.

Solution Differentiate on a component-by-component basis to obtain

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}.$$ Derivative

From the position vector $\mathbf{r}(t)$, you can write the parametric equations $x = t$ and $y = t^2 + 2$. The corresponding rectangular equation is $y = x^2 + 2$. When $t = 1$,

$$\mathbf{r}(1) = \mathbf{i} + 3\mathbf{j}$$

and

$$\mathbf{r}'(1) = \mathbf{i} + 2\mathbf{j}.$$

In Figure 12.9, $\mathbf{r}(1)$ is drawn starting at the origin, and $\mathbf{r}'(1)$ is drawn starting at the terminal point of $\mathbf{r}(1)$.

Higher-order derivatives of vector-valued functions are obtained by successive differentiation of each component function.

EXAMPLE 2 Higher-Order Differentiation

For the vector-valued function

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$$

find each of the following.

a. $\mathbf{r}'(t)$

b. $\mathbf{r}''(t)$

c. $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$

d. $\mathbf{r}'(t) \times \mathbf{r}''(t)$

Solution

a. $\mathbf{r}'(t) = -\sin t\mathbf{i} + \cos t\mathbf{j} + 2\mathbf{k}$ First derivative

b. $\mathbf{r}''(t) = -\cos t\mathbf{i} - \sin t\mathbf{j} + 0\mathbf{k}$

$\quad = -\cos t\mathbf{i} - \sin t\mathbf{j}$ Second derivative

c. $\mathbf{r}'(t) \cdot \mathbf{r}''(t) = \sin t \cos t - \sin t \cos t = 0$ Dot product

d. $$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin t & \cos t & 2 \\ -\cos t & -\sin t & 0 \end{vmatrix}$$ Cross product

$$= \begin{vmatrix} \cos t & 2 \\ -\sin t & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -\sin t & 2 \\ -\cos t & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -\sin t & \cos t \\ -\cos t & -\sin t \end{vmatrix}\mathbf{k}$$

$$= 2\sin t\mathbf{i} - 2\cos t\mathbf{j} + \mathbf{k}$$

In Example 2(c), note that the dot product is a real-valued function, not a vector-valued function.

The parametrization of the curve represented by the vector-valued function

$$\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$$

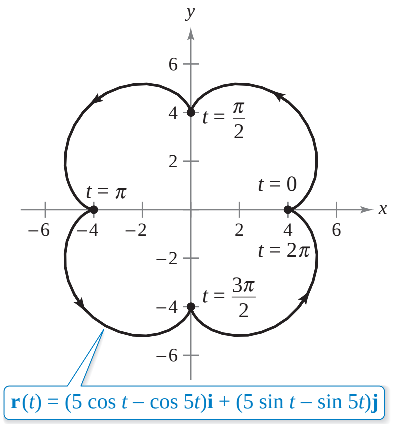
is **smooth on an open interval** I when f', g', and h' are continuous on I and $\mathbf{r}'(t) \neq \mathbf{0}$ for any value of t in the interval I.

EXAMPLE 3 Finding Intervals on Which a Curve Is Smooth

Find the intervals on which the epicycloid C given by

$$\mathbf{r}(t) = (5\cos t - \cos 5t)\mathbf{i} + (5\sin t - \sin 5t)\mathbf{j}, \quad 0 \le t \le 2\pi$$

is smooth.



The epicycloid is not smooth at the points where it intersects the axes.
Figure 12.10

Solution The derivative of $\mathbf{r}$ is

$$\mathbf{r}'(t) = (-5\sin t + 5\sin 5t)\mathbf{i} + (5\cos t - 5\cos 5t)\mathbf{j}.$$

In the interval $[0, 2\pi]$, the only values of t for which

$$\mathbf{r}'(t) = 0\mathbf{i} + 0\mathbf{j}$$

are $t = 0$, $\pi/2$, π, $3\pi/2$, and 2π. Therefore, you can conclude that C is smooth on the intervals

$$\left(0, \frac{\pi}{2}\right), \quad \left(\frac{\pi}{2}, \pi\right), \quad \left(\pi, \frac{3\pi}{2}\right), \quad \text{and} \quad \left(\frac{3\pi}{2}, 2\pi\right)$$

as shown in Figure 12.10. ■

In Figure 12.10, note that the curve is not smooth at points at which the curve makes abrupt changes in direction. Such points are called **cusps** or **nodes.**

Most of the differentiation rules in Chapter 3 have counterparts for vector-valued functions, and several of these are listed in the next theorem. Note that the theorem contains three versions of "product rules." Property 3 gives the derivative of the product of a real-valued function w and a vector-valued function $\mathbf{r}$, Property 4 gives the derivative of the dot product of two vector-valued functions, and Property 5 gives the derivative of the cross product of two vector-valued functions (in space).

REMARK Note that Property 5 applies only to three-dimensional vector-valued functions because the cross product is not defined for two-dimensional vectors.

THEOREM 12.2 Properties of the Derivative

Let $\mathbf{r}$ and $\mathbf{u}$ be differentiable vector-valued functions of t, let w be a differentiable real-valued function of t, and let c be a scalar.

1. $\dfrac{d}{dt}[c\mathbf{r}(t)] = c\mathbf{r}'(t)$
2. $\dfrac{d}{dt}[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$
3. $\dfrac{d}{dt}[w(t)\mathbf{r}(t)] = w(t)\mathbf{r}'(t) + w'(t)\mathbf{r}(t)$
4. $\dfrac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)$
5. $\dfrac{d}{dt}[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$
6. $\dfrac{d}{dt}[\mathbf{r}(w(t))] = \mathbf{r}'(w(t))w'(t)$
7. If $\mathbf{r}(t) \cdot \mathbf{r}(t) = c$, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

Proof To prove Property 4, let

$$\mathbf{r}(t) = f_1(t)\mathbf{i} + g_1(t)\mathbf{j} \quad \text{and} \quad \mathbf{u}(t) = f_2(t)\mathbf{i} + g_2(t)\mathbf{j}$$

where f_1, f_2, g_1, and g_2 are differentiable functions of t. Then

$$\mathbf{r}(t) \cdot \mathbf{u}(t) = f_1(t)f_2(t) + g_1(t)g_2(t)$$

and it follows that

$$\begin{aligned}\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)] &= f_1(t)f_2'(t) + f_1'(t)f_2(t) + g_1(t)g_2'(t) + g_1'(t)g_2(t)\\ &= [f_1(t)f_2'(t) + g_1(t)g_2'(t)] + [f_1'(t)f_2(t) + g_1'(t)g_2(t)]\\ &= \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t).\end{aligned}$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

Proofs of the other properties are left as exercises (see Exercises 67–71 and Exercise 74).

Exploration

Let $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}$. Sketch the graph of $\mathbf{r}(t)$. Explain why the graph is a circle of radius 1 centered at the origin. Calculate $\mathbf{r}(\pi/4)$ and $\mathbf{r}'(\pi/4)$. Position the vector $\mathbf{r}'(\pi/4)$ so that its initial point is at the terminal point of $\mathbf{r}(\pi/4)$. What do you observe? Show that $\mathbf{r}(t) \cdot \mathbf{r}(t)$ is constant and that $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$ for all t. How does this example relate to Property 7 of Theorem 12.2?

EXAMPLE 4 Using Properties of the Derivative

For $\mathbf{r}(t) = \frac{1}{t}\mathbf{i} - \mathbf{j} + \ln t\mathbf{k}$ and $\mathbf{u}(t) = t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k}$, find

a. $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)]$ and **b.** $\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{u}'(t)]$.

Solution

a. Because $\mathbf{r}'(t) = -\frac{1}{t^2}\mathbf{i} + \frac{1}{t}\mathbf{k}$ and $\mathbf{u}'(t) = 2t\mathbf{i} - 2\mathbf{j}$, you have

$$\begin{aligned}\frac{d}{dt}&[\mathbf{r}(t) \cdot \mathbf{u}(t)]\\ &= \mathbf{r}(t) \cdot \mathbf{u}'(t) + \mathbf{r}'(t) \cdot \mathbf{u}(t)\\ &= \left(\frac{1}{t}\mathbf{i} - \mathbf{j} + \ln t\mathbf{k}\right) \cdot (2t\mathbf{i} - 2\mathbf{j}) + \left(-\frac{1}{t^2}\mathbf{i} + \frac{1}{t}\mathbf{k}\right) \cdot (t^2\mathbf{i} - 2t\mathbf{j} + \mathbf{k})\\ &= 2 + 2 + (-1) + \frac{1}{t}\\ &= 3 + \frac{1}{t}.\end{aligned}$$

b. Because $\mathbf{u}'(t) = 2t\mathbf{i} - 2\mathbf{j}$ and $\mathbf{u}''(t) = 2\mathbf{i}$, you have

$$\begin{aligned}\frac{d}{dt}[\mathbf{u}(t) \times \mathbf{u}'(t)] &= [\mathbf{u}(t) \times \mathbf{u}''(t)] + [\mathbf{u}'(t) \times \mathbf{u}'(t)]\\ &= \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ t^2 & -2t & 1\\ 2 & 0 & 0\end{vmatrix} + \mathbf{0}\\ &= \begin{vmatrix}-2t & 1\\ 0 & 0\end{vmatrix}\mathbf{i} - \begin{vmatrix}t^2 & 1\\ 2 & 0\end{vmatrix}\mathbf{j} + \begin{vmatrix}t^2 & -2t\\ 2 & 0\end{vmatrix}\mathbf{k}\\ &= 0\mathbf{i} - (-2)\mathbf{j} + 4t\mathbf{k}\\ &= 2\mathbf{j} + 4t\mathbf{k}.\end{aligned}$$

Try reworking parts (a) and (b) in Example 4 by first forming the dot and cross products and then differentiating to see that you obtain the same results.

Integration of Vector-Valued Functions

The next definition is a consequence of the definition of the derivative of a vector-valued function.

Definition of Integration of Vector-Valued Functions

1. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j}$, where f and g are continuous on $[a, b]$, then the **indefinite integral (antiderivative)** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j} \qquad \text{Plane}$$

and its **definite integral** over the interval $a \le t \le b$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j}.$$

2. If $\mathbf{r}(t) = f(t)\mathbf{i} + g(t)\mathbf{j} + h(t)\mathbf{k}$, where f, g, and h are continuous on $[a, b]$, then the **indefinite integral (antiderivative)** of $\mathbf{r}$ is

$$\int \mathbf{r}(t)\,dt = \left[\int f(t)\,dt\right]\mathbf{i} + \left[\int g(t)\,dt\right]\mathbf{j} + \left[\int h(t)\,dt\right]\mathbf{k} \qquad \text{Space}$$

and its **definite integral** over the interval $a \le t \le b$ is

$$\int_a^b \mathbf{r}(t)\,dt = \left[\int_a^b f(t)\,dt\right]\mathbf{i} + \left[\int_a^b g(t)\,dt\right]\mathbf{j} + \left[\int_a^b h(t)\,dt\right]\mathbf{k}.$$

The antiderivative of a vector-valued function is a family of vector-valued functions all differing by a constant vector $\mathbf{C}$. For instance, if $\mathbf{r}(t)$ is a three-dimensional vector-valued function, then for the indefinite integral $\int \mathbf{r}(t)\,dt$, you obtain three constants of integration

$$\int f(t)\,dt = F(t) + C_1, \quad \int g(t)\,dt = G(t) + C_2, \quad \int h(t)\,dt = H(t) + C_3$$

where $F'(t) = f(t)$, $G'(t) = g(t)$, and $H'(t) = h(t)$. These three *scalar* constants produce one *vector* constant of integration

$$\begin{aligned}\int \mathbf{r}(t)\,dt &= [F(t) + C_1]\mathbf{i} + [G(t) + C_2]\mathbf{j} + [H(t) + C_3]\mathbf{k}\\ &= [F(t)\mathbf{i} + G(t)\mathbf{j} + H(t)\mathbf{k}] + [C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}]\\ &= \mathbf{R}(t) + \mathbf{C}\end{aligned}$$

where $\mathbf{R}'(t) = \mathbf{r}(t)$.

EXAMPLE 5 Integrating a Vector-Valued Function

Find the indefinite integral

$$\int (t\,\mathbf{i} + 3\mathbf{j})\,dt.$$

Solution Integrating on a component-by-component basis produces

$$\int (t\,\mathbf{i} + 3\mathbf{j})\,dt = \frac{t^2}{2}\mathbf{i} + 3t\mathbf{j} + \mathbf{C}.$$

Example 6 shows how to evaluate the definite integral of a vector-valued function.

EXAMPLE 6 Definite Integral of a Vector-Valued Function

Evaluate the integral

$$\int_0^1 \mathbf{r}(t)\,dt = \int_0^1 \left(\sqrt[3]{t}\,\mathbf{i} + \frac{1}{t+1}\mathbf{j} + e^{-t}\mathbf{k}\right) dt.$$

Solution

$$\begin{aligned}\int_0^1 \mathbf{r}(t)\,dt &= \left(\int_0^1 t^{1/3}\,dt\right)\mathbf{i} + \left(\int_0^1 \frac{1}{t+1}\,dt\right)\mathbf{j} + \left(\int_0^1 e^{-t}\,dt\right)\mathbf{k}\\ &= \left[\left(\frac{3}{4}\right)t^{4/3}\right]_0^1 \mathbf{i} + \Big[\ln|t+1|\Big]_0^1 \mathbf{j} + \Big[-e^{-t}\Big]_0^1 \mathbf{k}\\ &= \frac{3}{4}\mathbf{i} + (\ln 2)\mathbf{j} + \left(1 - \frac{1}{e}\right)\mathbf{k}\end{aligned}$$

As with real-valued functions, you can narrow the family of antiderivatives of a vector-valued function $\mathbf{r}'$ down to a single antiderivative by imposing an initial condition on the vector-valued function $\mathbf{r}$. This is demonstrated in the next example.

EXAMPLE 7 The Antiderivative of a Vector-Valued Function

Find the antiderivative of

$$\mathbf{r}'(t) = \cos 2t\mathbf{i} - 2\sin t\mathbf{j} + \frac{1}{1+t^2}\mathbf{k}$$

that satisfies the initial condition

$$\mathbf{r}(0) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

Solution

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{r}'(t)\,dt\\ &= \left(\int \cos 2t\,dt\right)\mathbf{i} + \left(\int -2\sin t\,dt\right)\mathbf{j} + \left(\int \frac{1}{1+t^2}\,dt\right)\mathbf{k}\\ &= \left(\frac{1}{2}\sin 2t + C_1\right)\mathbf{i} + (2\cos t + C_2)\mathbf{j} + (\arctan t + C_3)\mathbf{k}\end{aligned}$$

Letting $t = 0$, you can write

$$\mathbf{r}(0) = (0 + C_1)\mathbf{i} + (2 + C_2)\mathbf{j} + (0 + C_3)\mathbf{k}.$$

Using the fact that $\mathbf{r}(0) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$, you have

$$(0 + C_1)\mathbf{i} + (2 + C_2)\mathbf{j} + (0 + C_3)\mathbf{k} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}.$$

Equating corresponding components produces

$$C_1 = 3, \qquad 2 + C_2 = -2, \quad \text{and} \quad C_3 = 1.$$

So, the antiderivative that satisfies the initial condition is

$$\mathbf{r}(t) = \left(\frac{1}{2}\sin 2t + 3\right)\mathbf{i} + (2\cos t - 4)\mathbf{j} + (\arctan t + 1)\mathbf{k}.$$

12.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Differentiation of Vector-Valued Functions **In Exercises 1–6, find $\mathbf{r}'(t)$, $\mathbf{r}(t_0)$, and $\mathbf{r}'(t_0)$ for the given value of t_0. Then sketch the plane curve represented by the vector-valued function, and sketch the vectors $\mathbf{r}(t_0)$ and $\mathbf{r}'(t_0)$. Position the vectors such that the initial point of $\mathbf{r}(t_0)$ is at the origin and the initial point of $\mathbf{r}'(t_0)$ is at the terminal point of $\mathbf{r}(t_0)$. What is the relationship between $\mathbf{r}'(t_0)$ and the curve?**

1. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}, \quad t_0 = 2$
2. $\mathbf{r}(t) = (1 + t)\mathbf{i} + t^3\mathbf{j}, \quad t_0 = 1$
3. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j}, \quad t_0 = \frac{\pi}{2}$
4. $\mathbf{r}(t) = 3\sin t\mathbf{i} + 4\cos t\mathbf{j}, \quad t_0 = \frac{\pi}{2}$
5. $\mathbf{r}(t) = \langle e^t, e^{2t}\rangle, \quad t_0 = 0$
6. $\mathbf{r}(t) = \langle e^{-t}, e^t\rangle, \quad t_0 = 0$

Differentiation of Vector-Valued Functions **In Exercises 7 and 8, find $\mathbf{r}'(t)$, $\mathbf{r}(t_0)$, and $\mathbf{r}'(t_0)$ for the given value of t_0. Then sketch the space curve represented by the vector-valued function, and sketch the vectors $\mathbf{r}(t_0)$ and $\mathbf{r}'(t_0)$.**

7. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}, \quad t_0 = \frac{3\pi}{2}$
8. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{3}{2}\mathbf{k}, \quad t_0 = 2$

Finding a Derivative **In Exercises 9–20, find $\mathbf{r}'(t)$.**

9. $\mathbf{r}(t) = t^3\mathbf{i} - 3t\mathbf{j}$
10. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (1 - t^3)\mathbf{j}$
11. $\mathbf{r}(t) = \langle 2\cos t, 5\sin t\rangle$
12. $\mathbf{r}(t) = \langle t\cos t, -2\sin t\rangle$
13. $\mathbf{r}(t) = 6t\mathbf{i} - 7t^2\mathbf{j} + t^3\mathbf{k}$
14. $\mathbf{r}(t) = \frac{1}{t}\mathbf{i} + 16t\mathbf{j} + \frac{t^2}{2}\mathbf{k}$
15. $\mathbf{r}(t) = a\cos^3 t\mathbf{i} + a\sin^3 t\mathbf{j} + \mathbf{k}$
16. $\mathbf{r}(t) = 4\sqrt{t}\,\mathbf{i} + t^2\sqrt{t}\,\mathbf{j} + \ln t^2\mathbf{k}$
17. $\mathbf{r}(t) = e^{-t}\mathbf{i} + 4\mathbf{j} + 5te^t\mathbf{k}$
18. $\mathbf{r}(t) = \langle t^3, \cos 3t, \sin 3t\rangle$
19. $\mathbf{r}(t) = \langle t\sin t, t\cos t, t\rangle$
20. $\mathbf{r}(t) = \langle \arcsin t, \arccos t, 0\rangle$

Higher-Order Differentiation **In Exercises 21–24, find (a) $\mathbf{r}'(t)$, (b) $\mathbf{r}''(t)$, and (c) $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$.**

21. $\mathbf{r}(t) = t^3\mathbf{i} + \frac{1}{2}t^2\mathbf{j}$
22. $\mathbf{r}(t) = (t^2 + t)\mathbf{i} + (t^2 - t)\mathbf{j}$
23. $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}$
24. $\mathbf{r}(t) = 8\cos t\mathbf{i} + 3\sin t\mathbf{j}$

Higher-Order Differentiation **In Exercises 25–28, find (a) $\mathbf{r}'(t)$, (b) $\mathbf{r}''(t)$, (c) $\mathbf{r}'(t) \cdot \mathbf{r}''(t)$, and (d) $\mathbf{r}'(t) \times \mathbf{r}''(t)$.**

25. $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} - t\mathbf{j} + \frac{1}{6}t^3\mathbf{k}$
26. $\mathbf{r}(t) = t^3\mathbf{i} + (2t^2 + 3)\mathbf{j} + (3t - 5)\mathbf{k}$
27. $\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t\rangle$
28. $\mathbf{r}(t) = \langle e^{-t}, t^2, \tan t\rangle$

Finding Intervals on Which a Curve Is Smooth **In Exercises 29–38, find the open interval(s) on which the curve given by the vector-valued function is smooth.**

29. $\mathbf{r}(t) = t^2\mathbf{i} + t^3\mathbf{j}$
30. $\mathbf{r}(t) = \frac{1}{t - 1}\mathbf{i} + 3t\mathbf{j}$
31. $\mathbf{r}(\theta) = 2\cos^3\theta\mathbf{i} + 3\sin^3\theta\mathbf{j}$
32. $\mathbf{r}(\theta) = (\theta + \sin\theta)\mathbf{i} + (1 - \cos\theta)\mathbf{j}$
33. $\mathbf{r}(\theta) = (\theta - 2\sin\theta)\mathbf{i} + (1 - 2\cos\theta)\mathbf{j}$
34. $\mathbf{r}(t) = \frac{2t}{8 + t^3}\mathbf{i} + \frac{2t^2}{8 + t^3}\mathbf{j}$
35. $\mathbf{r}(t) = (t - 1)\mathbf{i} + \frac{1}{t}\mathbf{j} - t^2\mathbf{k}$
36. $\mathbf{r}(t) = e^t\mathbf{i} - e^{-t}\mathbf{j} + 3t\mathbf{k}$
37. $\mathbf{r}(t) = t\mathbf{i} - 3t\mathbf{j} + \tan t\mathbf{k}$
38. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + (t^2 - 1)\mathbf{j} + \frac{1}{4}t\mathbf{k}$

Using Properties of the Derivative **In Exercises 39 and 40, use the properties of the derivative to find the following.**

(a) $\mathbf{r}'(t)$ (b) $\frac{d}{dt}[3\mathbf{r}(t) - \mathbf{u}(t)]$ (c) $\frac{d}{dt}(5t)\mathbf{u}(t)$

(d) $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)]$ (e) $\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{u}(t)]$ (f) $\frac{d}{dt}\mathbf{r}(2t)$

39. $\mathbf{r}(t) = t\mathbf{i} + 3t\mathbf{j} + t^2\mathbf{k}, \quad \mathbf{u}(t) = 4t\mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$
40. $\mathbf{r}(t) = t\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$

    $\mathbf{u}(t) = \frac{1}{t}\mathbf{i} + 2\sin t\mathbf{j} + 2\cos t\mathbf{k}$

Using Two Methods **In Exercises 41 and 42, find (a) $\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)]$ and (b) $\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{u}(t)]$ in two different ways.**

(i) Find the product first, then differentiate.
(ii) Apply the properties of Theorem 12.2.

41. $\mathbf{r}(t) = t\mathbf{i} + 2t^2\mathbf{j} + t^3\mathbf{k}, \quad \mathbf{u}(t) = t^4\mathbf{k}$
42. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + t\mathbf{k}, \quad \mathbf{u}(t) = \mathbf{j} + t\mathbf{k}$

Finding an Indefinite Integral **In Exercises 43–50, find the indefinite integral.**

43. $\int (2t\mathbf{i} + \mathbf{j} + \mathbf{k})\,dt$
44. $\int \left(4t^3\mathbf{i} + 6t\mathbf{j} - 4\sqrt{t}\,\mathbf{k}\right)dt$
45. $\int \left(\frac{1}{t}\mathbf{i} + \mathbf{j} - t^{3/2}\mathbf{k}\right)dt$
46. $\int \left(\ln t\mathbf{i} + \frac{1}{t}\mathbf{j} + \mathbf{k}\right)dt$
47. $\int \left[(2t - 1)\mathbf{i} + 4t^3\mathbf{j} + 3\sqrt{t}\,\mathbf{k}\right]dt$
48. $\int (e^t\mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k})\,dt$
49. $\int \left(\sec^2 t\mathbf{i} + \frac{1}{1 + t^2}\mathbf{j}\right)dt$
50. $\int (e^{-t}\sin t\mathbf{i} + e^{-t}\cos t\mathbf{j})\,dt$

Evaluating a Definite Integral In Exercises 51–56, evaluate the definite integral.

51. $\int_0^1 (8t\mathbf{i} + t\mathbf{j} - \mathbf{k})\,dt$

52. $\int_{-1}^1 \left(t\mathbf{i} + t^3\mathbf{j} + \sqrt[3]{t}\,\mathbf{k}\right)dt$

53. $\int_0^{\pi/2} [(a\cos t)\mathbf{i} + (a\sin t)\mathbf{j} + \mathbf{k}]\,dt$

54. $\int_0^{\pi/4} [(\sec t\tan t)\mathbf{i} + (\tan t)\mathbf{j} + (2\sin t\cos t)\mathbf{k}]\,dt$

55. $\int_0^2 (t\mathbf{i} + e^t\mathbf{j} - te^t\mathbf{k})\,dt$

56. $\int_0^3 \|t\mathbf{i} + t^2\mathbf{j}\|\,dt$

Finding an Antiderivative In Exercises 57–62, find $\mathbf{r}(t)$ that satisifies the initial condition(s).

57. $\mathbf{r}'(t) = 4e^{2t}\mathbf{i} + 3e^t\mathbf{j}, \quad \mathbf{r}(0) = 2\mathbf{i}$

58. $\mathbf{r}'(t) = 3t^2\mathbf{j} + 6\sqrt{t}\,\mathbf{k}, \quad \mathbf{r}(0) = \mathbf{i} + 2\mathbf{j}$

59. $\mathbf{r}''(t) = -32\mathbf{j}, \quad \mathbf{r}'(0) = 600\sqrt{3}\mathbf{i} + 600\mathbf{j}, \quad \mathbf{r}(0) = \mathbf{0}$

60. $\mathbf{r}''(t) = -4\cos t\mathbf{j} - 3\sin t\mathbf{k}, \quad \mathbf{r}'(0) = 3\mathbf{k}, \quad \mathbf{r}(0) = 4\mathbf{j}$

61. $\mathbf{r}'(t) = te^{-t^2}\mathbf{i} - e^{-t}\mathbf{j} + \mathbf{k}, \quad \mathbf{r}(0) = \frac{1}{2}\mathbf{i} - \mathbf{j} + \mathbf{k}$

62. $\mathbf{r}'(t) = \frac{1}{1+t^2}\mathbf{i} + \frac{1}{t^2}\mathbf{j} + \frac{1}{t}\mathbf{k}, \quad \mathbf{r}(1) = 2\mathbf{i}$

WRITING ABOUT CONCEPTS

63. **Differentiation** State the definition of the derivative of a vector-valued function. Describe how to find the derivative of a vector-valued function and give its geometric interpretation.

64. **Integration** How do you find the integral of a vector-valued function?

65. **Using a Derivative** The three components of the derivative of the vector-valued function **u** are positive at $t = t_0$. Describe the behavior of **u** at $t = t_0$.

66. **Using a Derivative** The z-component of the derivative of the vector-valued function **u** is 0 for t in the domain of the function. What does this imply about the graph of **u**?

Proof In Exercises 67–74, prove the property. In each case, assume **r**, **u**, and **v** are differentiable vector-valued functions of t in space, w is a differentiable real-valued function of t, and c is a scalar.

67. $\frac{d}{dt}[c\mathbf{r}(t)] = c\mathbf{r}'(t)$

68. $\frac{d}{dt}[\mathbf{r}(t) \pm \mathbf{u}(t)] = \mathbf{r}'(t) \pm \mathbf{u}'(t)$

69. $\frac{d}{dt}[w(t)\mathbf{r}(t)] = w(t)\mathbf{r}'(t) + w'(t)\mathbf{r}(t)$

70. $\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{u}(t)] = \mathbf{r}(t) \times \mathbf{u}'(t) + \mathbf{r}'(t) \times \mathbf{u}(t)$

71. $\frac{d}{dt}[\mathbf{r}(w(t))] = \mathbf{r}'(w(t))w'(t)$

72. $\frac{d}{dt}[\mathbf{r}(t) \times \mathbf{r}'(t)] = \mathbf{r}(t) \times \mathbf{r}''(t)$

73. $\frac{d}{dt}\{\mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)]\} = \mathbf{r}'(t) \cdot [\mathbf{u}(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}'(t) \times \mathbf{v}(t)] + \mathbf{r}(t) \cdot [\mathbf{u}(t) \times \mathbf{v}'(t)]$

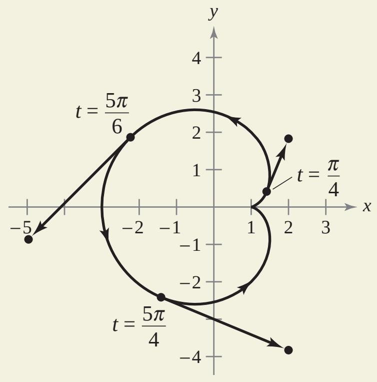
74. If $\mathbf{r}(t) \cdot \mathbf{r}(t)$ is a constant, then $\mathbf{r}(t) \cdot \mathbf{r}'(t) = 0$.

75. **Particle Motion** A particle moves in the xy-plane along the curve represented by the vector-valued function $\mathbf{r}(t) = (t - \sin t)\mathbf{i} + (1 - \cos t)\mathbf{j}$.
   (a) Use a graphing utility to graph **r**. Describe the curve.
   (b) Find the minimum and maximum values of $\|\mathbf{r}'\|$ and $\|\mathbf{r}''\|$.

76. **Particle Motion** A particle moves in the yz-plane along the curve represented by the vector-valued function $\mathbf{r}(t) = (2\cos t)\mathbf{j} + (3\sin t)\mathbf{k}$.
   (a) Describe the curve.
   (b) Find the minimum and maximum values of $\|\mathbf{r}'\|$ and $\|\mathbf{r}''\|$.

77. **Perpendicular Vectors** Consider the vector-valued function $\mathbf{r}(t) = (e^t\sin t)\mathbf{i} + (e^t\cos t)\mathbf{j}$. Show that $\mathbf{r}(t)$ and $\mathbf{r}''(t)$ are always perpendicular to each other.

78. **HOW DO YOU SEE IT?** The graph shows a vector-valued function $\mathbf{r}(t)$ for $0 \le t \le 2\pi$ and its derivative $\mathbf{r}'(t)$ for several values of t.



   (a) For each derivative shown in the graph, determine whether each component is positive or negative.
   (b) Is the curve smooth on the interval $[0, 2\pi]$? Explain.

True or False? In Exercises 79–82, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

79. If a particle moves along a sphere centered at the origin, then its derivative vector is always tangent to the sphere.

80. The definite integral of a vector-valued function is a real number.

81. $\frac{d}{dt}[\|\mathbf{r}(t)\|] = \|\mathbf{r}'(t)\|$

82. If **r** and **u** are differentiable vector-valued functions of t, then
$$\frac{d}{dt}[\mathbf{r}(t) \cdot \mathbf{u}(t)] = \mathbf{r}'(t) \cdot \mathbf{u}'(t).$$

12.3 Velocity and Acceleration

- **Describe the velocity and acceleration associated with a vector-valued function.**
- **Use a vector-valued function to analyze projectile motion.**

Velocity and Acceleration

You are now ready to combine your study of parametric equations, curves, vectors, and vector-valued functions to form a model for motion along a curve. You will begin by looking at the motion of an object in the plane. (The motion of an object in space can be developed similarly.)

As an object moves along a curve in the plane, the coordinates x and y of its center of mass are each functions of time t. Rather than using the letters f and g to represent these two functions, it is convenient to write $x = x(t)$ and $y = y(t)$. So, the position vector $\mathbf{r}(t)$ takes the form

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}. \qquad \text{Position vector}$$

The beauty of this vector model for representing motion is that you can use the first and second derivatives of the vector-valued function $\mathbf{r}$ to find the object's velocity and acceleration. (Recall from the preceding chapter that velocity and acceleration are both vector quantities having magnitude and direction.) To find the velocity and acceleration vectors at a given time t, consider a point $Q(x(t + \Delta t), y(t + \Delta t))$ that is approaching the point $P(x(t), y(t))$ along the curve C given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, as shown in Figure 12.11. As $\Delta t \to 0$, the direction of the vector $\overrightarrow{PQ}$ (denoted by $\Delta\mathbf{r}$) approaches the *direction of motion* at time t.

$$\Delta\mathbf{r} = \mathbf{r}(t + \Delta t) - \mathbf{r}(t)$$

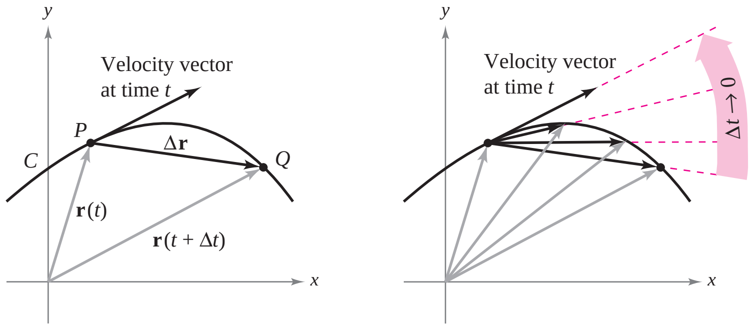
$$\frac{\Delta\mathbf{r}}{\Delta t} = \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

$$\lim_{\Delta t \to 0} \frac{\Delta\mathbf{r}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\mathbf{r}(t + \Delta t) - \mathbf{r}(t)}{\Delta t}$$

If this limit exists, it is defined as the **velocity vector** or **tangent vector** to the curve at point P. Note that this is the same limit used to define $\mathbf{r}'(t)$. So, the direction of $\mathbf{r}'(t)$ gives the direction of motion at time t. Moreover, the magnitude of the vector $\mathbf{r}'(t)$

$$\|\mathbf{r}'(t)\| = \|x'(t)\mathbf{i} + y'(t)\mathbf{j}\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

gives the **speed** of the object at time t. Similarly, you can use $\mathbf{r}''(t)$ to find acceleration, as indicated in the definitions at the top of the next page.



As $\Delta t \to 0$, $\dfrac{\Delta\mathbf{r}}{\Delta t}$ approaches the velocity vector.

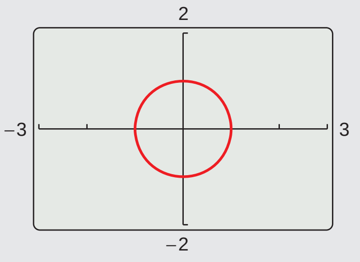
Figure 12.11

Exploration

Exploring Velocity Consider the circle given by

$$\mathbf{r}(t) = (\cos \omega t)\mathbf{i} + (\sin \omega t)\mathbf{j}.$$

(The symbol ω is the Greek letter omega.) Use a graphing utility in *parametric* mode to graph this circle for several values of ω. How does ω affect the velocity of the terminal point as it traces out the curve? For a given value of ω, does the speed appear constant? Does the acceleration appear constant? Explain your reasoning.



Definitions of Velocity and Acceleration

If x and y are twice-differentiable functions of t, and $\mathbf{r}$ is a vector-valued function given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, then the velocity vector, acceleration vector, and speed at time t are as follows.

$$\textbf{Velocity} = \mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j}$$
$$\textbf{Acceleration} = \mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j}$$
$$\textbf{Speed} = \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2}$$

For motion along a space curve, the definitions are similar. That is, for $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$, you have

$$\textbf{Velocity} = \mathbf{v}(t) = \mathbf{r}'(t) = x'(t)\mathbf{i} + y'(t)\mathbf{j} + z'(t)\mathbf{k}$$
$$\textbf{Acceleration} = \mathbf{a}(t) = \mathbf{r}''(t) = x''(t)\mathbf{i} + y''(t)\mathbf{j} + z''(t)\mathbf{k}$$
$$\textbf{Speed} = \|\mathbf{v}(t)\| = \|\mathbf{r}'(t)\| = \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}.$$

REMARK In Example 1, note that the velocity and acceleration vectors are orthogonal at any point in time. This is characteristic of motion at a constant speed. (See Exercise 53.)

EXAMPLE 1 Velocity and Acceleration Along a Plane Curve

Find the velocity vector, speed, and acceleration vector of a particle that moves along the plane curve C described by

$$\mathbf{r}(t) = 2\sin\frac{t}{2}\mathbf{i} + 2\cos\frac{t}{2}\mathbf{j}. \qquad \text{Position vector}$$

Solution

The velocity vector is

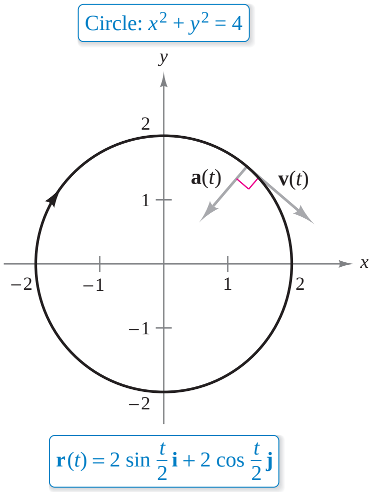
$$\mathbf{v}(t) = \mathbf{r}'(t) = \cos\frac{t}{2}\mathbf{i} - \sin\frac{t}{2}\mathbf{j}. \qquad \text{Velocity vector}$$

The speed (at any time) is

$$\|\mathbf{r}'(t)\| = \sqrt{\cos^2\frac{t}{2} + \sin^2\frac{t}{2}} = 1. \qquad \text{Speed}$$

The acceleration vector is

$$\mathbf{a}(t) = \mathbf{r}''(t) = -\frac{1}{2}\sin\frac{t}{2}\mathbf{i} - \frac{1}{2}\cos\frac{t}{2}\mathbf{j}. \qquad \text{Acceleration vector}$$



The particle moves around the circle at a constant speed.
Figure 12.12

The parametric equations for the curve in Example 1 are

$$x = 2\sin\frac{t}{2} \quad \text{and} \quad y = 2\cos\frac{t}{2}.$$

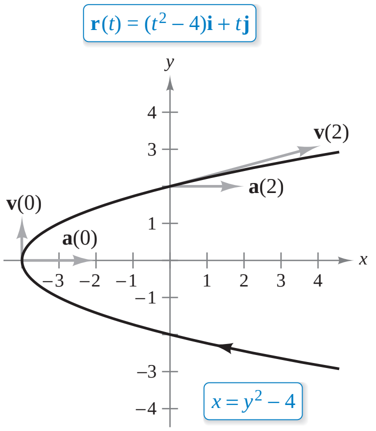
By eliminating the parameter t, you obtain the rectangular equation

$$x^2 + y^2 = 4. \qquad \text{Rectangular equation}$$

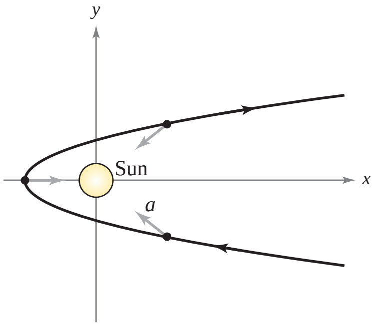
So, the curve is a circle of radius 2 centered at the origin, as shown in Figure 12.12. Because the velocity vector

$$\mathbf{v}(t) = \cos\frac{t}{2}\mathbf{i} - \sin\frac{t}{2}\mathbf{j}$$

has a constant magnitude but a changing direction as t increases, the particle moves around the circle at a constant speed.



At each point on the curve, the acceleration vector points to the right.
Figure 12.13



At each point in the comet's orbit, the acceleration vector points toward the sun.
Figure 12.14

EXAMPLE 2 Velocity and Acceleration Vectors in the Plane

Sketch the path of an object moving along the plane curve given by

$$\mathbf{r}(t) = (t^2 - 4)\mathbf{i} + t\,\mathbf{j} \qquad \text{Position vector}$$

and find the velocity and acceleration vectors when $t = 0$ and $t = 2$.

Solution Using the parametric equations $x = t^2 - 4$ and $y = t$, you can determine that the curve is a parabola given by

$$x = y^2 - 4 \qquad \text{Rectangular equation}$$

as shown in Figure 12.13. The velocity vector (at any time) is

$$\mathbf{v}(t) = \mathbf{r}'(t) = 2t\,\mathbf{i} + \mathbf{j} \qquad \text{Velocity vector}$$

and the acceleration vector (at any time) is

$$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{i}. \qquad \text{Acceleration vector}$$

When $t = 0$, the velocity and acceleration vectors are

$$\mathbf{v}(0) = 2(0)\mathbf{i} + \mathbf{j} = \mathbf{j} \quad \text{and} \quad \mathbf{a}(0) = 2\mathbf{i}.$$

When $t = 2$, the velocity and acceleration vectors are

$$\mathbf{v}(2) = 2(2)\mathbf{i} + \mathbf{j} = 4\mathbf{i} + \mathbf{j} \quad \text{and} \quad \mathbf{a}(2) = 2\mathbf{i}.$$

For the object moving along the path shown in Figure 12.13, note that the acceleration vector is constant (it has a magnitude of 2 and points to the right). This implies that the speed of the object is decreasing as the object moves toward the vertex of the parabola, and the speed is increasing as the object moves away from the vertex of the parabola.

This type of motion is *not* characteristic of comets that travel on parabolic paths through our solar system. For such comets, the acceleration vector always points to the origin (the sun), which implies that the comet's speed increases as it approaches the vertex of the path and decreases as it moves away from the vertex. (See Figure 12.14.)

EXAMPLE 3 Velocity and Acceleration Vectors in Space

See LarsonCalculus.com for an interactive version of this type of example.

Sketch the path of an object moving along the space curve C given by

$$\mathbf{r}(t) = t\,\mathbf{i} + t^3\mathbf{j} + 3t\mathbf{k}, \quad t \geq 0 \qquad \text{Position vector}$$

and find the velocity and acceleration vectors when $t = 1$.

Solution Using the parametric equations $x = t$ and $y = t^3$, you can determine that the path of the object lies on the cubic cylinder given by

$$y = x^3. \qquad \text{Rectangular equation}$$

Moreover, because $z = 3t$, the object starts at $(0, 0, 0)$ and moves upward as t increases, as shown in Figure 12.15. Because $\mathbf{r}(t) = t\,\mathbf{i} + t^3\mathbf{j} + 3t\mathbf{k}$, you have

$$\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + 3t^2\mathbf{j} + 3\mathbf{k} \qquad \text{Velocity vector}$$

and

$$\mathbf{a}(t) = \mathbf{r}''(t) = 6t\mathbf{j}. \qquad \text{Acceleration vector}$$

When $t = 1$, the velocity and acceleration vectors are

$$\mathbf{v}(1) = \mathbf{r}'(1) = \mathbf{i} + 3\mathbf{j} + 3\mathbf{k} \quad \text{and} \quad \mathbf{a}(1) = \mathbf{r}''(1) = 6\mathbf{j}.$$

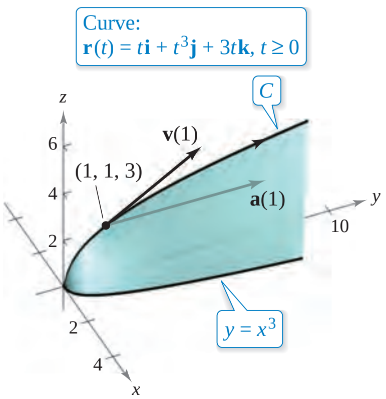


Figure 12.15

So far in this section, you have concentrated on finding the velocity and acceleration by differentiating the position vector. Many practical applications involve the reverse problem—finding the position vector for a given velocity or acceleration. This is demonstrated in the next example.

EXAMPLE 4 Finding a Position Vector by Integration

An object starts from rest at the point $(1, 2, 0)$ and moves with an acceleration of

$$\mathbf{a}(t) = \mathbf{j} + 2\mathbf{k} \qquad \text{Acceleration vector}$$

where $\|\mathbf{a}(t)\|$ is measured in feet per second per second. Find the location of the object after $t = 2$ seconds.

Solution From the description of the object's motion, you can deduce the following *initial conditions.* Because the object starts from rest, you have

$$\mathbf{v}(0) = \mathbf{0}.$$

Moreover, because the object starts at the point $(x, y, z) = (1, 2, 0)$, you have

$$\mathbf{r}(0) = x(0)\mathbf{i} + y(0)\mathbf{j} + z(0)\mathbf{k} = 1\mathbf{i} + 2\mathbf{j} + 0\mathbf{k} = \mathbf{i} + 2\mathbf{j}.$$

To find the position vector, you should integrate twice, each time using one of the initial conditions to solve for the constant of integration. The velocity vector is

$$\begin{aligned}\mathbf{v}(t) &= \int \mathbf{a}(t)\,dt \\ &= \int (\mathbf{j} + 2\mathbf{k})\,dt \\ &= t\mathbf{j} + 2t\mathbf{k} + \mathbf{C}\end{aligned}$$

where $\mathbf{C} = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k}$. Letting $t = 0$ and applying the initial condition $\mathbf{v}(0) = \mathbf{0}$, you obtain

$$\mathbf{v}(0) = C_1\mathbf{i} + C_2\mathbf{j} + C_3\mathbf{k} = \mathbf{0} \quad \Rightarrow \quad C_1 = C_2 = C_3 = 0.$$

So, the *velocity* at any time t is

$$\mathbf{v}(t) = t\mathbf{j} + 2t\mathbf{k}. \qquad \text{Velocity vector}$$

Integrating once more produces

$$\begin{aligned}\mathbf{r}(t) &= \int \mathbf{v}(t)\,dt \\ &= \int (t\mathbf{j} + 2t\mathbf{k})\,dt \\ &= \frac{t^2}{2}\mathbf{j} + t^2\mathbf{k} + \mathbf{C}\end{aligned}$$

where $\mathbf{C} = C_4\mathbf{i} + C_5\mathbf{j} + C_6\mathbf{k}$. Letting $t = 0$ and applying the initial condition $\mathbf{r}(0) = \mathbf{i} + 2\mathbf{j}$, you have

$$\mathbf{r}(0) = C_4\mathbf{i} + C_5\mathbf{j} + C_6\mathbf{k} = \mathbf{i} + 2\mathbf{j} \quad \Rightarrow \quad C_4 = 1,\ C_5 = 2,\ C_6 = 0.$$

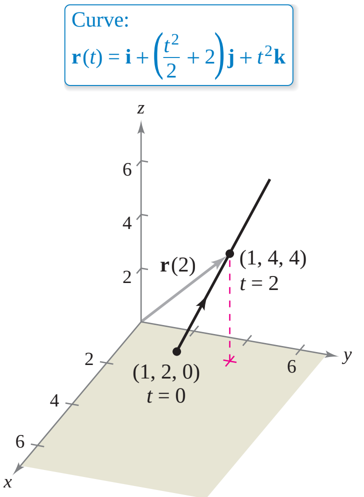
So, the *position* vector is

$$\mathbf{r}(t) = \mathbf{i} + \left(\frac{t^2}{2} + 2\right)\mathbf{j} + t^2\mathbf{k}. \qquad \text{Position vector}$$

The location of the object after $t = 2$ seconds is given by

$$\mathbf{r}(2) = \mathbf{i} + 4\mathbf{j} + 4\mathbf{k}$$

as shown in Figure 12.16.



The object takes 2 seconds to move from point $(1, 2, 0)$ to point $(1, 4, 4)$ along the curve.
Figure 12.16

Projectile Motion

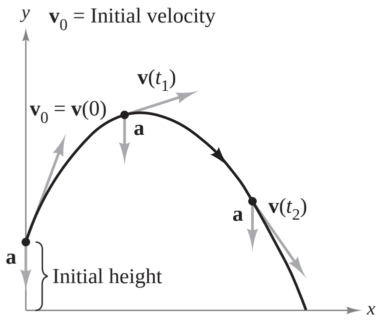


Figure 12.17

You now have the machinery to derive the parametric equations for the path of a projectile. Assume that gravity is the only force acting on the projectile after it is launched. So, the motion occurs in a vertical plane, which can be represented by the *xy*-coordinate system with the origin as a point on Earth's surface, as shown in Figure 12.17. For a projectile of mass m, the force due to gravity is

$$\mathbf{F} = -mg\mathbf{j}$$ Force due to gravity

where the acceleration due to gravity is $g = 32$ feet per second per second, or 9.81 meters per second per second. By **Newton's Second Law of Motion,** this same force produces an acceleration $\mathbf{a} = \mathbf{a}(t)$ and satisfies the equation $\mathbf{F} = m\mathbf{a}$. Consequently, the acceleration of the projectile is given by $m\mathbf{a} = -mg\mathbf{j}$, which implies that

$$\mathbf{a} = -g\mathbf{j}.$$ Acceleration of projectile

EXAMPLE 5 Derivation of the Position Vector for a Projectile

A projectile of mass m is launched from an initial position $\mathbf{r}_0$ with an initial velocity $\mathbf{v}_0$. Find its position vector as a function of time.

Solution Begin with the acceleration $\mathbf{a}(t) = -g\mathbf{j}$ and integrate twice.

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int -g\mathbf{j}\,dt = -gt\mathbf{j} + \mathbf{C}_1$$

$$\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int (-gt\mathbf{j} + \mathbf{C}_1)\,dt = -\frac{1}{2}gt^2\mathbf{j} + \mathbf{C}_1 t + \mathbf{C}_2$$

You can use the facts that $\mathbf{v}(0) = \mathbf{v}_0$ and $\mathbf{r}(0) = \mathbf{r}_0$ to solve for the constant vectors $\mathbf{C}_1$ and $\mathbf{C}_2$. Doing this produces

$$\mathbf{C}_1 = \mathbf{v}_0 \quad \text{and} \quad \mathbf{C}_2 = \mathbf{r}_0.$$

Therefore, the position vector is

$$\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0 + \mathbf{r}_0.$$ Position vector

In many projectile problems, the constant vectors $\mathbf{r}_0$ and $\mathbf{v}_0$ are not given explicitly. Often you are given the initial height h, the initial speed v_0, and the angle θ at which the projectile is launched, as shown in Figure 12.18. From the given height, you can deduce that $\mathbf{r}_0 = h\mathbf{j}$. Because the speed gives the magnitude of the initial velocity, it follows that $v_0 = \|\mathbf{v}_0\|$ and you can write

$$\begin{aligned}\mathbf{v}_0 &= x\mathbf{i} + y\mathbf{j} \\ &= (\|\mathbf{v}_0\| \cos\theta)\mathbf{i} + (\|\mathbf{v}_0\| \sin\theta)\mathbf{j} \\ &= v_0 \cos\theta\,\mathbf{i} + v_0 \sin\theta\,\mathbf{j}.\end{aligned}$$

So, the position vector can be written in the form

$$\mathbf{r}(t) = -\frac{1}{2}gt^2\mathbf{j} + t\mathbf{v}_0 + \mathbf{r}_0$$ Position vector

$$= -\frac{1}{2}gt^2\mathbf{j} + tv_0\cos\theta\,\mathbf{i} + tv_0\sin\theta\,\mathbf{j} + h\mathbf{j}$$

$$= (v_0\cos\theta)t\mathbf{i} + \left[h + (v_0\sin\theta)t - \frac{1}{2}gt^2\right]\mathbf{j}.$$

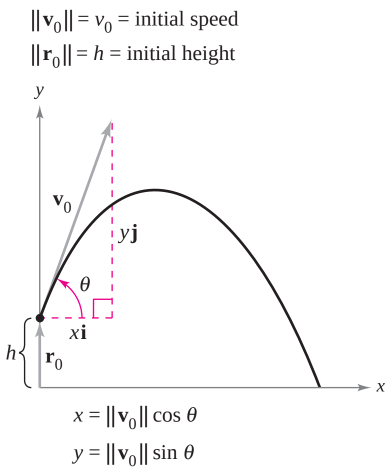


Figure 12.18

THEOREM 12.3 Position Vector for a Projectile

Neglecting air resistance, the path of a projectile launched from an initial height h with initial speed v_0 and angle of elevation θ is described by the vector function

$$\mathbf{r}(t) = (v_0 \cos \theta)t\mathbf{i} + \left[h + (v_0 \sin \theta)t - \tfrac{1}{2}gt^2\right]\mathbf{j}$$

where g is the acceleration due to gravity.

EXAMPLE 6 Describing the Path of a Baseball

A baseball is hit 3 feet above ground level at 100 feet per second and at an angle of 45° with respect to the ground, as shown in Figure 12.19. Find the maximum height reached by the baseball. Will it clear a 10-foot-high fence located 300 feet from home plate?

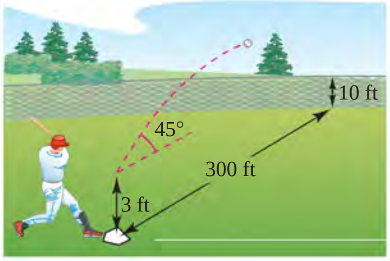


Figure 12.19

Solution You are given

$$h = 3, \quad v_0 = 100, \quad \text{and} \quad \theta = 45°.$$

So, using Theorem 12.3 with $g = 32$ feet per second per second produces

$$\begin{aligned}\mathbf{r}(t) &= \left(100 \cos \frac{\pi}{4}\right)t\mathbf{i} + \left[3 + \left(100 \sin \frac{\pi}{4}\right)t - 16t^2\right]\mathbf{j} \\ &= \left(50\sqrt{2}t\right)\mathbf{i} + \left(3 + 50\sqrt{2}t - 16t^2\right)\mathbf{j}.\end{aligned}$$

The velocity vector is

$$\mathbf{v}(t) = \mathbf{r}'(t) = 50\sqrt{2}\mathbf{i} + \left(50\sqrt{2} - 32t\right)\mathbf{j}.$$

The maximum height occurs when

$$y'(t) = 50\sqrt{2} - 32t$$

is equal to 0, which implies that

$$t = \frac{25\sqrt{2}}{16} \approx 2.21 \text{ seconds.}$$

So, the maximum height reached by the ball is

$$\begin{aligned}y &= 3 + 50\sqrt{2}\left(\frac{25\sqrt{2}}{16}\right) - 16\left(\frac{25\sqrt{2}}{16}\right)^2 \\ &= \frac{649}{8} \\ &\approx 81 \text{ feet.}\end{aligned}$$

Maximum height when $t \approx 2.21$ seconds

The ball is 300 feet from where it was hit when

$$300 = x(t) \quad \Rightarrow \quad 300 = 50\sqrt{2}t.$$

Solving this equation for t produces $t = 3\sqrt{2} \approx 4.24$ seconds. At this time, the height of the ball is

$$\begin{aligned}y &= 3 + 50\sqrt{2}\left(3\sqrt{2}\right) - 16\left(3\sqrt{2}\right)^2 \\ &= 303 - 288 \\ &= 15 \text{ feet.}\end{aligned}$$

Height when $t \approx 4.24$ seconds

Therefore, the ball clears the 10-foot fence for a home run.

12.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding Velocity and Acceleration Along a Plane Curve

In Exercises 1–8, the position vector r describes the path of an object moving in the xy-plane.

(a) Find the velocity vector, speed, and acceleration vector of the object.

(b) Evaluate the velocity vector and acceleration vector of the object at the given point.

(c) Sketch a graph of the path, and sketch the velocity and acceleration vectors at the given point.

Position Vector	**Point**
1. $\mathbf{r}(t) = 3t\mathbf{i} + (t - 1)\mathbf{j}$	$(3, 0)$
2. $\mathbf{r}(t) = t\mathbf{i} + (-t^2 + 4)\mathbf{j}$	$(1, 3)$
3. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j}$	$(4, 2)$
4. $\mathbf{r}(t) = \left(\frac{1}{4}t^3 + 1\right)\mathbf{i} + t\mathbf{j}$	$(3, 2)$
5. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$	$\left(\sqrt{2}, \sqrt{2}\right)$
6. $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$	$(3, 0)$
7. $\mathbf{r}(t) = \langle t - \sin t, 1 - \cos t\rangle$	$(\pi, 2)$
8. $\mathbf{r}(t) = \langle e^{-t}, e^t\rangle$	$(1, 1)$

Finding Velocity and Acceleration Vectors

In Exercises 9–18, the position vector r describes the path of an object moving in space.

(a) Find the velocity vector, speed, and acceleration vector of the object.

(b) Evaluate the velocity vector and acceleration vector of the object at the given value of t.

Position Vector	**Time**
9. $\mathbf{r}(t) = t\mathbf{i} + 5t\mathbf{j} + 3t\mathbf{k}$	$t = 1$
10. $\mathbf{r}(t) = 4t\mathbf{i} + 4t\mathbf{j} + 2t\mathbf{k}$	$t = 3$
11. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$	$t = 4$
12. $\mathbf{r}(t) = 3t\mathbf{i} + t\mathbf{j} + \frac{1}{4}t^2\mathbf{k}$	$t = 2$
13. $\mathbf{r}(t) = t\mathbf{i} + t\mathbf{j} + \sqrt{9 - t^2}\,\mathbf{k}$	$t = 0$
14. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + 2t^{3/2}\mathbf{k}$	$t = 4$
15. $\mathbf{r}(t) = \langle 4t, 3\cos t, 3\sin t\rangle$	$t = \pi$
16. $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, t^2\rangle$	$t = \frac{\pi}{4}$
17. $\mathbf{r}(t) = \langle e^t\cos t, e^t\sin t, e^t\rangle$	$t = 0$
18. $\mathbf{r}(t) = \left\langle \ln t, \frac{1}{t}, t^4\right\rangle$	$t = 2$

Finding a Position Vector by Integration

In Exercises 19–24, use the given acceleration vector to find the velocity and position vectors. Then find the position at time $t = 2$.

19. $\mathbf{a}(t) = \mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \mathbf{v}(0) = \mathbf{0}, \quad \mathbf{r}(0) = \mathbf{0}$

20. $\mathbf{a}(t) = 2\mathbf{i} + 3\mathbf{k}, \quad \mathbf{v}(0) = 4\mathbf{j}, \quad \mathbf{r}(0) = \mathbf{0}$

21. $\mathbf{a}(t) = t\mathbf{j} + t\mathbf{k}, \quad \mathbf{v}(1) = 5\mathbf{j}, \quad \mathbf{r}(1) = \mathbf{0}$

22. $\mathbf{a}(t) = -32\mathbf{k}, \quad \mathbf{v}(0) = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}, \quad \mathbf{r}(0) = 5\mathbf{j} + 2\mathbf{k}$

23. $\mathbf{a}(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}, \quad \mathbf{v}(0) = \mathbf{j} + \mathbf{k}, \quad \mathbf{r}(0) = \mathbf{i}$

24. $\mathbf{a}(t) = e^t\mathbf{i} - 8\mathbf{k}, \quad \mathbf{v}(0) = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}, \quad \mathbf{r}(0) = \mathbf{0}$

Projectile Motion

In Exercises 25–38, use the model for projectile motion, assuming there is no air resistance.

25. A baseball is hit from a height of 2.5 feet above the ground with an initial velocity of 140 feet per second and at an angle of 22° above the horizontal. Find the maximum height reached by the baseball. Determine whether it will clear a 10-foot-high fence located 375 feet from home plate.

26. Determine the maximum height and range of a projectile fired at a height of 3 feet above the ground with an initial velocity of 900 feet per second and at an angle of 45° above the horizontal.

27. A baseball, hit 3 feet above the ground, leaves the bat at an angle of 45° and is caught by an outfielder 3 feet above the ground and 300 feet from home plate. What is the initial speed of the ball, and how high does it rise?

28. A baseball player at second base throws a ball 90 feet to the player at first base. The ball is released at a point 5 feet above the ground with an initial velocity of 50 miles per hour and at an angle of 15° above the horizontal. At what height does the player at first base catch the ball?

29. Eliminate the parameter t from the position vector for the motion of a projectile to show that the rectangular equation is

$$y = -\frac{16\sec^2\theta}{v_0^2}x^2 + (\tan\theta)x + h.$$

30. The path of a ball is given by the rectangular equation

$$y = x - 0.005x^2.$$

Use the result of Exercise 29 to find the position vector. Then find the speed and direction of the ball at the point at which it has traveled 60 feet horizontally.

31. The Rogers Centre in Toronto, Ontario, has a center field fence that is 10 feet high and 400 feet from home plate. A ball is hit 3 feet above the ground and leaves the bat at a speed of 100 miles per hour.

(a) The ball leaves the bat at an angle of $\theta = \theta_0$ with the horizontal. Write the vector-valued function for the path of the ball.

(b) Use a graphing utility to graph the vector-valued function for $\theta_0 = 10°$, $\theta_0 = 15°$, $\theta_0 = 20°$, and $\theta_0 = 25°$. Use the graphs to approximate the minimum angle required for the hit to be a home run.

(c) Determine analytically the minimum angle required for the hit to be a home run.

32. Football

The quarterback of a football team releases a pass at a height of 7 feet above the playing field, and the football is caught by a receiver 30 yards directly downfield at a height of 4 feet. The pass is released at an angle of 35° with the horizontal.

(a) Find the speed of the football when it is released.

(b) Find the maximum height of the football.

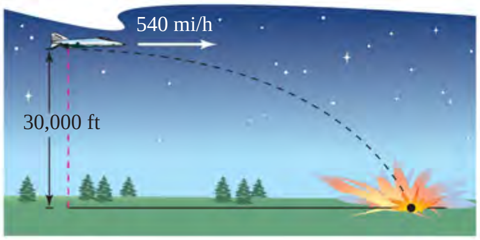
(c) Find the time the receiver has to reach the proper position after the quarterback releases the football.

33. A bale ejector consists of two variable-speed belts at the end of a baler. Its purpose is to toss bales into a trailing wagon. In loading the back of a wagon, a bale must be thrown to a position 8 feet above and 16 feet behind the ejector.

(a) Find the minimum initial speed of the bale and the corresponding angle at which it must be ejected from the baler.

(b) The ejector has a fixed angle of 45°. Find the initial speed required.

34. A bomber is flying at an altitude of 30,000 feet at a speed of 540 miles per hour (see figure). When should the bomb be released for it to hit the target? (Give your answer in terms of the angle of depression from the plane to the target.) What is the speed of the bomb at the time of impact?



35. A shot fired from a gun with a muzzle velocity of 1200 feet per second is to hit a target 3000 feet away. Determine the minimum angle of elevation of the gun.

36. A projectile is fired from ground level at an angle of 12° with the horizontal. The projectile is to have a range of 200 feet. Find the minimum initial velocity necessary.

37. Use a graphing utility to graph the paths of a projectile for the given values of θ and v_0. For each case, use the graph to approximate the maximum height and range of the projectile. (Assume that the projectile is launched from ground level.)

(a) $\theta = 10°,\ v_0 = 66$ ft/sec

(b) $\theta = 10°,\ v_0 = 146$ ft/sec

(c) $\theta = 45°,\ v_0 = 66$ ft/sec

(d) $\theta = 45°,\ v_0 = 146$ ft/sec

(e) $\theta = 60°,\ v_0 = 66$ ft/sec

(f) $\theta = 60°,\ v_0 = 146$ ft/sec

38. Find the angles at which an object must be thrown to obtain (a) the maximum range and (b) the maximum height.

Projectile Motion **In Exercises 39 and 40, use the model for projectile motion, assuming there is no air resistance. [$g = -9.8$ meters per second per second]**

39. Determine the maximum height and range of a projectile fired at a height of 1.5 meters above the ground with an initial velocity of 100 meters per second and at an angle of 30° above the horizontal.

40. A projectile is fired from ground level at an angle of 8° with the horizontal. The projectile is to have a range of 50 meters. Find the minimum initial velocity necessary.

41. Shot-Put Throw The path of a shot thrown at an angle θ is

$$\mathbf{r}(t) = (v_0 \cos \theta)t\,\mathbf{i} + \left[h + (v_0 \sin \theta)t - \frac{1}{2}gt^2\right]\mathbf{j}$$

where v_0 is the initial speed, h is the initial height, t is the time in seconds, and g is the acceleration due to gravity. Verify that the shot will remain in the air for a total of

$$t = \frac{v_0 \sin \theta + \sqrt{v_0^2 \sin^2 \theta + 2gh}}{g} \text{ seconds}$$

and will travel a horizontal distance of

$$\frac{v_0^2 \cos \theta}{g}\left(\sin \theta + \sqrt{\sin^2 \theta + \frac{2gh}{v_0^2}}\right) \text{ feet.}$$

42. Shot-Put Throw

A shot is thrown from a height of $h = 6$ feet with an initial speed of $v_0 = 45$ feet per second and at an angle of $\theta = 42.5°$ with the horizontal. Use the result of Exercise 41 to find the total time of travel and the total horizontal distance traveled.

Cycloidal Motion **In Exercises 43 and 44, consider the motion of a point (or particle) on the circumference of a rolling circle. As the circle rolls, it generates the cycloid**

$$\mathbf{r}(t) = b(\omega t - \sin \omega t)\mathbf{i} + b(1 - \cos \omega t)\mathbf{j}$$

where ω is the constant angular velocity of the circle and b is the radius of the circle.

43. Find the velocity and acceleration vectors of the particle. Use the results to determine the times at which the speed of the particle will be (a) zero and (b) maximized.

44. Find the maximum speed of a point on the circumference of an automobile tire of radius 1 foot when the automobile is traveling at 60 miles per hour. Compare this speed with the speed of the automobile.

Nicholas Moore/Shutterstock.com; Jamie Roach/Shutterstock.com

Circular Motion **In Exercises 45–48, consider a particle moving on a circular path of radius b described by $\mathbf{r}(t) = b\cos\omega t\,\mathbf{i} + b\sin\omega t\,\mathbf{j}$, where $\omega = du/dt$ is the constant angular velocity.**

45. Find the velocity vector and show that it is orthogonal to $\mathbf{r}(t)$.

46. (a) Show that the speed of the particle is $b\omega$.

(b) Use a graphing utility in *parametric* mode to graph the circle for $b = 6$. Try different values of ω. Does the graphing utility draw the circle faster for greater values of ω?

47. Find the acceleration vector and show that its direction is always toward the center of the circle.

48. Show that the magnitude of the acceleration vector is $b\omega^2$.

Circular Motion **In Exercises 49 and 50, use the results of Exercises 45–48.**

49. A stone weighing 1 pound is attached to a two-foot string and is whirled horizontally (see figure). The string will break under a force of 10 pounds. Find the maximum speed the stone can attain without breaking the string. $\left(\text{Use } \mathbf{F} = m\mathbf{a}, \text{ where } m = \frac{1}{32}.\right)$

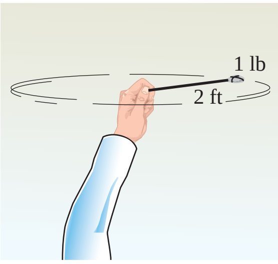


Figure for 49

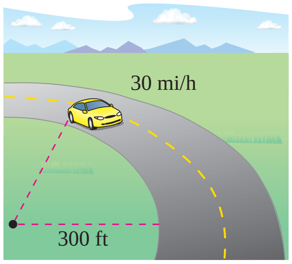


Figure for 50

50. A 3400-pound automobile is negotiating a circular interchange of radius 300 feet at 30 miles per hour (see figure). Assuming the roadway is level, find the force between the tires and the road such that the car stays on the circular path and does not skid. (Use $\mathbf{F} = m\mathbf{a}$, where $m = 3400/32$.) Find the angle at which the roadway should be banked so that no lateral frictional force is exerted on the tires of the automobile.

WRITING ABOUT CONCEPTS

51. Velocity and Speed In your own words, explain the difference between the velocity of an object and its speed.

52. Particle Motion Consider a particle that is moving on the path $\mathbf{r}_1(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$.

(a) Discuss any changes in the position, velocity, or acceleration of the particle when its position is given by the vector-valued function $\mathbf{r}_2(t) = \mathbf{r}_1(2t)$.

(b) Generalize the results for the vector-valued function $\mathbf{r}_3(t) = \mathbf{r}_1(\omega t)$.

53. Proof Prove that when an object is traveling at a constant speed, its velocity and acceleration vectors are orthogonal.

54. Proof Prove that an object moving in a straight line at a constant speed has an acceleration of 0.

55. Investigation A particle moves on an elliptical path given by the vector-valued function $\mathbf{r}(t) = 6\cos t\,\mathbf{i} + 3\sin t\,\mathbf{j}$.

(a) Find $\mathbf{v}(t)$, $\|\mathbf{v}(t)\|$, and $\mathbf{a}(t)$.

(b) Use a graphing utility to complete the table.

t	0	$\frac{\pi}{4}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
Speed					

(c) Graph the elliptical path and the velocity and acceleration vectors at the values of t given in the table in part (b).

(d) Use the results of parts (b) and (c) to describe the geometric relationship between the velocity and acceleration vectors when the speed of the particle is increasing, and when it is decreasing.

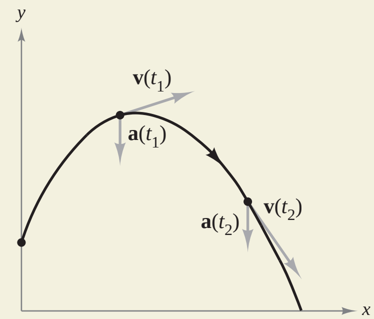
56. Particle Motion Consider a particle moving on an elliptical path described by $\mathbf{r}(t) = a\cos\omega t\,\mathbf{i} + b\sin\omega t\,\mathbf{j}$, where $\omega = d\theta/dt$ is the constant angular velocity.

(a) Find the velocity vector. What is the speed of the particle?

(b) Find the acceleration vector and show that its direction is always toward the center of the ellipse.

57. Path of an Object When $t = 0$, an object is at the point $(0, 1)$ and has a velocity vector $\mathbf{v}(0) = -\mathbf{i}$. It moves with an acceleration of $\mathbf{a}(t) = \sin t\,\mathbf{i} - \cos t\,\mathbf{j}$. Show that the path of the object is a circle.

58. HOW DO YOU SEE IT? The graph shows the path of a projectile and the velocity and acceleration vectors at times t_1 and t_2. Classify the angle between the velocity vector and the acceleration vector at times t_1 and t_2. Is the speed increasing or decreasing at times t_1 and t_2? Explain your reasoning.



True or False? **In Exercises 59–62, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

59. The acceleration of an object is the derivative of the speed.

60. The velocity of an object is the derivative of the position.

61. The velocity vector points in the direction of motion.

62. If a particle moves along a straight line, then the velocity and acceleration vectors are orthogonal.

12.4 Tangent Vectors and Normal Vectors

- **Find a unit tangent vector and a principal unit normal vector at a point on a space curve.**
- **Find the tangential and normal components of acceleration.**

Tangent Vectors and Normal Vectors

In the preceding section, you learned that the velocity vector points in the direction of motion. This observation leads to the next definition, which applies to any smooth curve—not just to those for which the parameter represents time.

Definition of Unit Tangent Vector

Let C be a smooth curve represented by $\mathbf{r}$ on an open interval I. The **unit tangent vector** $\mathbf{T}(t)$ at t is defined as

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}, \quad \mathbf{r}'(t) \neq \mathbf{0}.$$

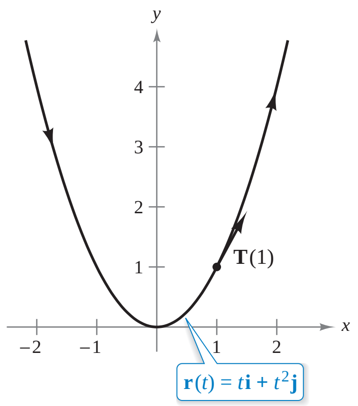
Recall that a curve is *smooth* on an interval when $\mathbf{r}'$ is continuous and nonzero on the interval. So, "smoothness" is sufficient to guarantee that a curve has a unit tangent vector.

EXAMPLE 1 Finding the Unit Tangent Vector

Find the unit tangent vector to the curve given by

$$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$$

when $t = 1$.



The direction of the unit tangent vector depends on the orientation of the curve.
Figure 12.20

Solution The derivative of $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = \mathbf{i} + 2t\mathbf{j}. \qquad \text{Derivative of } \mathbf{r}(t)$$

So, the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|} \qquad \text{Definition of } \mathbf{T}(t)$$

$$= \frac{1}{\sqrt{1 + 4t^2}}(\mathbf{i} + 2t\mathbf{j}). \qquad \text{Substitute for } \mathbf{r}'(t).$$

When $t = 1$, the unit tangent vector is

$$\mathbf{T}(1) = \frac{1}{\sqrt{5}}(\mathbf{i} + 2\mathbf{j})$$

as shown in Figure 12.20.

In Example 1, note that the direction of the unit tangent vector depends on the orientation of the curve. For the parabola described by

$$\mathbf{r}(t) = -(t - 2)\mathbf{i} + (t - 2)^2\mathbf{j}$$

$\mathbf{T}(1)$ would still represent the unit tangent vector at the point $(1, 1)$, but it would point in the opposite direction. Try verifying this.

The **tangent line to a curve** at a point is the line that passes through the point and is parallel to the unit tangent vector. In Example 2, the unit tangent vector is used to find the tangent line at a point on a helix.

EXAMPLE 2 Finding the Tangent Line at a Point on a Curve

Find $\mathbf{T}(t)$ and then find a set of parametric equations for the tangent line to the helix given by

$$\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}$$

at the point $\left(\sqrt{2}, \sqrt{2}, \frac{\pi}{4}\right)$.

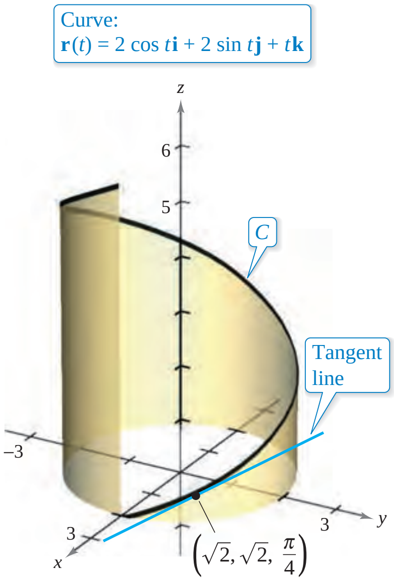
Solution The derivative of $\mathbf{r}(t)$ is

$$\mathbf{r}'(t) = -2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}$$

which implies that $\|\mathbf{r}'(t)\| = \sqrt{4\sin^2 t + 4\cos^2 t + 1} = \sqrt{5}$. Therefore, the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

$$= \frac{1}{\sqrt{5}}(-2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}).$$ Unit tangent vector



The tangent line to a curve at a point is determined by the unit tangent vector at the point.
Figure 12.21

At the point $\left(\sqrt{2}, \sqrt{2}, \pi/4\right)$, $t = \pi/4$ and the unit tangent vector is

$$\mathbf{T}\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{5}}\left(-2\frac{\sqrt{2}}{2}\mathbf{i} + 2\frac{\sqrt{2}}{2}\mathbf{j} + \mathbf{k}\right)$$

$$= \frac{1}{\sqrt{5}}\left(-\sqrt{2}\,\mathbf{i} + \sqrt{2}\,\mathbf{j} + \mathbf{k}\right).$$

Using the direction numbers $a = -\sqrt{2}$, $b = \sqrt{2}$, and $c = 1$, and the point $(x_1, y_1, z_1) = \left(\sqrt{2}, \sqrt{2}, \pi/4\right)$, you can obtain the parametric equations (given with parameter s) listed below.

$$x = x_1 + as = \sqrt{2} - \sqrt{2}s$$

$$y = y_1 + bs = \sqrt{2} + \sqrt{2}s$$

$$z = z_1 + cs = \frac{\pi}{4} + s$$

This tangent line is shown in Figure 12.21.

In Example 2, there are infinitely many vectors that are orthogonal to the tangent vector $\mathbf{T}(t)$. One of these is the vector $\mathbf{T}'(t)$. This follows from Property 7 of Theorem 12.2. That is,

$$\mathbf{T}(t) \cdot \mathbf{T}(t) = \|\mathbf{T}(t)\|^2 = 1 \quad \Longrightarrow \quad \mathbf{T}(t) \cdot \mathbf{T}'(t) = 0.$$

By normalizing the vector $\mathbf{T}'(t)$, you obtain a special vector called the **principal unit normal vector,** as indicated in the next definition.

Definition of Principal Unit Normal Vector

Let C be a smooth curve represented by $\mathbf{r}$ on an open interval I. If $\mathbf{T}'(t) \neq \mathbf{0}$, then the **principal unit normal vector** at t is defined as

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}.$$

EXAMPLE 3 Finding the Principal Unit Normal Vector

Find $\mathbf{N}(t)$ and $\mathbf{N}(1)$ for the curve represented by $\mathbf{r}(t) = 3t\mathbf{i} + 2t^2\mathbf{j}$.

Solution By differentiating, you obtain

$$\mathbf{r}'(t) = 3\mathbf{i} + 4t\mathbf{j}$$

which implies that

$$\|\mathbf{r}'(t)\| = \sqrt{9 + 16t^2}.$$

So, the unit tangent vector is

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

$$= \frac{1}{\sqrt{9 + 16t^2}}(3\mathbf{i} + 4t\mathbf{j}).$$ Unit tangent vector

Using Theorem 12.2, differentiate $\mathbf{T}(t)$ with respect to t to obtain

$$\mathbf{T}'(t) = \frac{1}{\sqrt{9 + 16t^2}}(4\mathbf{j}) - \frac{16t}{(9 + 16t^2)^{3/2}}(3\mathbf{i} + 4t\mathbf{j})$$

$$= \frac{12}{(9 + 16t^2)^{3/2}}(-4t\mathbf{i} + 3\mathbf{j})$$

which implies that

$$\|\mathbf{T}'(t)\| = 12\sqrt{\frac{9 + 16t^2}{(9 + 16t^2)^3}} = \frac{12}{9 + 16t^2}.$$

Therefore, the principal unit normal vector is

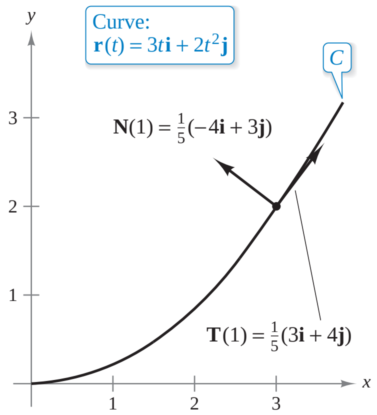
$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$

$$= \frac{1}{\sqrt{9 + 16t^2}}(-4t\mathbf{i} + 3\mathbf{j}).$$ Principal unit normal vector

When $t = 1$, the principal unit normal vector is

$$\mathbf{N}(1) = \frac{1}{5}(-4\mathbf{i} + 3\mathbf{j})$$

as shown in Figure 12.22.



The principal unit normal vector points toward the concave side of the curve.
Figure 12.22

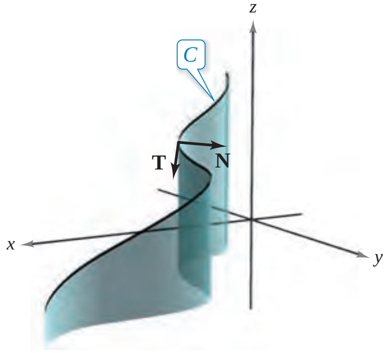
The principal unit normal vector can be difficult to evaluate algebraically. For plane curves, you can simplify the algebra by finding

$$\mathbf{T}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$ Unit tangent vector

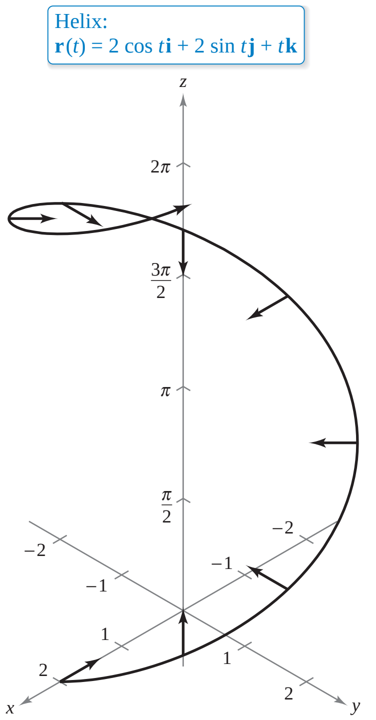
and observing that $\mathbf{N}(t)$ must be either

$$\mathbf{N}_1(t) = y(t)\mathbf{i} - x(t)\mathbf{j} \quad \text{or} \quad \mathbf{N}_2(t) = -y(t)\mathbf{i} + x(t)\mathbf{j}.$$

Because $\sqrt{[x(t)]^2 + [y(t)]^2} = 1$, it follows that both $\mathbf{N}_1(t)$ and $\mathbf{N}_2(t)$ are unit normal vectors. The *principal* unit normal vector $\mathbf{N}$ is the one that points toward the concave side of the curve, as shown in Figure 12.22 (see Exercise 76). This also holds for curves in space. That is, for an object moving along a curve C in space, the vector $\mathbf{T}(t)$ points in the direction the object is moving, whereas the vector $\mathbf{N}(t)$ is orthogonal to $\mathbf{T}(t)$ and points in the direction in which the object is turning, as shown in Figure 12.23.



At any point on a curve, a unit normal vector is orthogonal to the unit tangent vector. The *principal* unit normal vector points in the direction in which the curve is turning.
Figure 12.23



$\mathbf{N}(t)$ is horizontal and points toward the z-axis.
Figure 12.24

EXAMPLE 4 Finding the Principal Unit Normal Vector

Find the principal unit normal vector for the helix $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}$.

Solution From Example 2, you know that the unit tangent vector is

$$\mathbf{T}(t) = \frac{1}{\sqrt{5}}(-2\sin t\mathbf{i} + 2\cos t\mathbf{j} + \mathbf{k}).$$ Unit tangent vector

So, $\mathbf{T}'(t)$ is given by

$$\mathbf{T}'(t) = \frac{1}{\sqrt{5}}(-2\cos t\mathbf{i} - 2\sin t\mathbf{j}).$$

Because $\|\mathbf{T}'(t)\| = 2/\sqrt{5}$, it follows that the principal unit normal vector is

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$
$$= \frac{1}{2}(-2\cos t\mathbf{i} - 2\sin t\mathbf{j})$$
$$= -\cos t\mathbf{i} - \sin t\mathbf{j}.$$ Principal unit normal vector

Note that this vector is horizontal and points toward the z-axis, as shown in Figure 12.24.

Tangential and Normal Components of Acceleration

In the preceding section, you considered the problem of describing the motion of an object along a curve. You saw that for an object traveling at a *constant speed*, the velocity and acceleration vectors are perpendicular. This seems reasonable, because the speed would not be constant if any acceleration were acting in the direction of motion. You can verify this observation by noting that

$$\mathbf{r}''(t) \cdot \mathbf{r}'(t) = 0$$

when $\|\mathbf{r}'(t)\|$ is a constant. (See Property 7 of Theorem 12.2.)

For an object traveling at a *variable speed*, however, the velocity and acceleration vectors are not necessarily perpendicular. For instance, you saw that the acceleration vector for a projectile always points down, regardless of the direction of motion.

In general, part of the acceleration (the tangential component) acts in the line of motion, and part of it (the normal component) acts perpendicular to the line of motion. In order to determine these two components, you can use the unit vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$, which serve in much the same way as do $\mathbf{i}$ and $\mathbf{j}$ in representing vectors in the plane. The next theorem states that the acceleration vector lies in the plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

THEOREM 12.4 Acceleration Vector

If $\mathbf{r}(t)$ is the position vector for a smooth curve C and $\mathbf{N}(t)$ exists, then the acceleration vector $\mathbf{a}(t)$ lies in the plane determined by $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

Proof To simplify the notation, write $\mathbf{T}$ for $\mathbf{T}(t)$, $\mathbf{T}'$ for $\mathbf{T}'(t)$, and so on. Because $\mathbf{T} = \mathbf{r}'/\|\mathbf{r}'\| = \mathbf{v}/\|\mathbf{v}\|$, it follows that

$$\mathbf{v} = \|\mathbf{v}\|\mathbf{T}.$$

By differentiating, you obtain

$$\begin{aligned}
\mathbf{a} &= \mathbf{v}' && \text{Product Rule}\\
&= \frac{d}{dt}[\|\mathbf{v}\|]\mathbf{T} + \|\mathbf{v}\|\mathbf{T}'\\
&= \frac{d}{dt}[\|\mathbf{v}\|]\mathbf{T} + \|\mathbf{v}\|\mathbf{T}'\left(\frac{\|\mathbf{T}'\|}{\|\mathbf{T}'\|}\right)\\
&= \frac{d}{dt}[\|\mathbf{v}\|]\mathbf{T} + \|\mathbf{v}\|\,\|\mathbf{T}'\|\,\mathbf{N}. && \mathbf{N} = \mathbf{T}'/\|\mathbf{T}'\|
\end{aligned}$$

Because $\mathbf{a}$ is written as a linear combination of $\mathbf{T}$ and $\mathbf{N}$, it follows that $\mathbf{a}$ lies in the plane determined by $\mathbf{T}$ and $\mathbf{N}$.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

The coefficients of $\mathbf{T}$ and $\mathbf{N}$ in the proof of Theorem 12.4 are called the **tangential and normal components of acceleration** and are denoted by

$$a_{\mathbf{T}} = \frac{d}{dt}[\|\mathbf{v}\|]$$

and $a_{\mathbf{N}} = \|\mathbf{v}\|\,\|\mathbf{T}'\|$. So, you can write

$$\mathbf{a}(t) = a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t).$$

The next theorem lists some convenient formulas for $a_{\mathbf{N}}$ and $a_{\mathbf{T}}$.

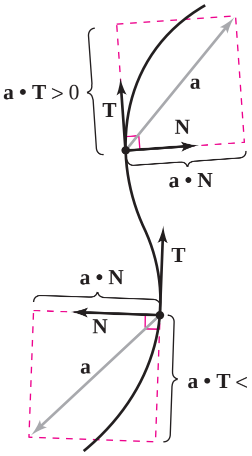
THEOREM 12.5 Tangential and Normal Components of Acceleration

If $\mathbf{r}(t)$ is the position vector for a smooth curve C [for which $\mathbf{N}(t)$ exists], then the tangential and normal components of acceleration are as follows.

$$a_{\mathbf{T}} = \frac{d}{dt}[\|\mathbf{v}\|] = \mathbf{a} \cdot \mathbf{T} = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|}$$

$$a_{\mathbf{N}} = \|\mathbf{v}\|\,\|\mathbf{T}'\| = \mathbf{a} \cdot \mathbf{N} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2}$$

Note that $a_{\mathbf{N}} \geq 0$. The normal component of acceleration is also called the **centripetal component of acceleration.**



The tangential and normal components of acceleration are obtained by projecting **a** onto **T** and **N**.
Figure 12.25

Proof Note that a lies in the plane of $\mathbf{T}$ and $\mathbf{N}$. So, you can use Figure 12.25 to conclude that, for any time t, the components of the projection of the acceleration vector onto $\mathbf{T}$ and onto $\mathbf{N}$ are given by $a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T}$ and $a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N}$, respectively. Moreover, because $\mathbf{a} = \mathbf{v}'$ and $\mathbf{T} = \mathbf{v}/\|\mathbf{v}\|$, you have

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \mathbf{T} \cdot \mathbf{a} = \frac{\mathbf{v}}{\|\mathbf{v}\|} \cdot \mathbf{a} = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|}.$$

In Exercises 78 and 79, you are asked to prove the other parts of the theorem.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 5 Tangential and Normal Components of Acceleration

See LarsonCalculus.com for an interactive version of this type of example.

Find the tangential and normal components of acceleration for the position vector given by $\mathbf{r}(t) = 3t\mathbf{i} - t\mathbf{j} + t^2\mathbf{k}$.

Solution Begin by finding the velocity, speed, and acceleration.

$$\mathbf{v}(t) = \mathbf{r}'(t) = 3\mathbf{i} - \mathbf{j} + 2t\mathbf{k} \qquad \text{Velocity vector}$$

$$\|\mathbf{v}(t)\| = \sqrt{9 + 1 + 4t^2} = \sqrt{10 + 4t^2} \qquad \text{Speed}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = 2\mathbf{k} \qquad \text{Acceleration vector}$$

By Theorem 12.5, the tangential component of acceleration is

$$a_{\mathbf{T}} = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|} = \frac{4t}{\sqrt{10 + 4t^2}} \qquad \text{Tangential component of acceleration}$$

and because

$$\mathbf{v} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -1 & 2t \\ 0 & 0 & 2 \end{vmatrix} = -2\mathbf{i} - 6\mathbf{j}$$

the normal component of acceleration is

$$a_{\mathbf{N}} = \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|} = \frac{\sqrt{4 + 36}}{\sqrt{10 + 4t^2}} = \frac{2\sqrt{10}}{\sqrt{10 + 4t^2}}. \qquad \text{Normal component of acceleration}$$

In Example 5, you could have used the alternative formula for $a_{\mathbf{N}}$ as follows.

$$a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2} = \sqrt{(2)^2 - \frac{16t^2}{10 + 4t^2}} = \frac{2\sqrt{10}}{\sqrt{10 + 4t^2}}$$

EXAMPLE 6 Finding $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ for a Circular Helix

Find the tangential and normal components of acceleration for the helix given by

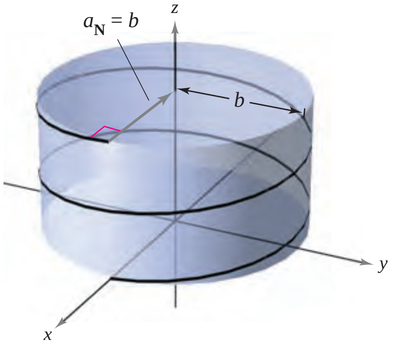
$$\mathbf{r}(t) = b\cos t\mathbf{i} + b\sin t\mathbf{j} + ct\mathbf{k}, \quad b > 0.$$

Solution

$$\mathbf{v}(t) = \mathbf{r}'(t) = -b\sin t\mathbf{i} + b\cos t\mathbf{j} + c\mathbf{k} \qquad \text{Velocity vector}$$

$$\|\mathbf{v}(t)\| = \sqrt{b^2\sin^2 t + b^2\cos^2 t + c^2} = \sqrt{b^2 + c^2} \qquad \text{Speed}$$

$$\mathbf{a}(t) = \mathbf{r}''(t) = -b\cos t\mathbf{i} - b\sin t\mathbf{j} \qquad \text{Acceleration vector}$$



The normal component of acceleration is equal to the radius of the cylinder around which the helix is spiraling.
Figure 12.26

By Theorem 12.5, the tangential component of acceleration is

$$a_{\mathbf{T}} = \frac{\mathbf{v}\cdot\mathbf{a}}{\|\mathbf{v}\|} = \frac{b^2\sin t\cos t - b^2\sin t\cos t + 0}{\sqrt{b^2 + c^2}} = 0. \qquad \text{Tangential component of acceleration}$$

Moreover, because

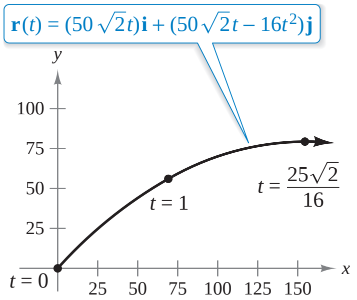
$$\|\mathbf{a}\| = \sqrt{b^2\cos^2 t + b^2\sin^2 t} = b$$

you can use the alternative formula for the normal component of acceleration to obtain

$$a_{\mathbf{N}} = \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2} = \sqrt{b^2 - 0^2} = b. \qquad \text{Normal component of acceleration}$$

Note that the normal component of acceleration is equal to the magnitude of the acceleration. In other words, because the speed is constant, the acceleration is perpendicular to the velocity. See Figure 12.26.

EXAMPLE 7 Projectile Motion



The path of a projectile
Figure 12.27

The position vector for the projectile shown in Figure 12.27 is

$$\mathbf{r}(t) = (50\sqrt{2}t)\mathbf{i} + (50\sqrt{2}t - 16t^2)\mathbf{j}. \qquad \text{Position vector}$$

Find the tangential components of acceleration when $t = 0$, 1, and $25\sqrt{2}/16$.

Solution

$$\mathbf{v}(t) = 50\sqrt{2}\mathbf{i} + (50\sqrt{2} - 32t)\mathbf{j} \qquad \text{Velocity vector}$$

$$\|\mathbf{v}(t)\| = 2\sqrt{50^2 - 16(50)\sqrt{2}t + 16^2t^2} \qquad \text{Speed}$$

$$\mathbf{a}(t) = -32\mathbf{j} \qquad \text{Acceleration vector}$$

The tangential component of acceleration is

$$a_{\mathbf{T}}(t) = \frac{\mathbf{v}(t)\cdot\mathbf{a}(t)}{\|\mathbf{v}(t)\|} = \frac{-32(50\sqrt{2} - 32t)}{2\sqrt{50^2 - 16(50)\sqrt{2}t + 16^2t^2}}. \qquad \text{Tangential component of acceleration}$$

At the specified times, you have

$$a_{\mathbf{T}}(0) = \frac{-32(50\sqrt{2})}{100} = -16\sqrt{2} \approx -22.6$$

$$a_{\mathbf{T}}(1) = \frac{-32(50\sqrt{2} - 32)}{2\sqrt{50^2 - 16(50)\sqrt{2} + 16^2}} \approx -15.4$$

$$a_{\mathbf{T}}\left(\frac{25\sqrt{2}}{16}\right) = \frac{-32(50\sqrt{2} - 50\sqrt{2})}{50\sqrt{2}} = 0.$$

You can see from Figure 12.27 that at the maximum height, when $t = 25\sqrt{2}/16$, the tangential component is 0. This is reasonable because the direction of motion is horizontal at the point and the tangential component of the acceleration is equal to the horizontal component of the acceleration.

12.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding the Unit Tangent Vector In Exercises 1–6, find the unit tangent vector to the curve at the specified value of the parameter.

1. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j}, \quad t = 1$
2. $\mathbf{r}(t) = t^3\mathbf{i} + 2t^2\mathbf{j}, \quad t = 1$
3. $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j}, \quad t = \frac{\pi}{4}$
4. $\mathbf{r}(t) = 6\cos t\mathbf{i} + 2\sin t\mathbf{j}, \quad t = \frac{\pi}{3}$
5. $\mathbf{r}(t) = 3t\mathbf{i} - \ln t\mathbf{j}, \quad t = e$
6. $\mathbf{r}(t) = e^t\cos t\mathbf{i} + e^t\mathbf{j}, \quad t = 0$

Finding a Tangent Line In Exercises 7–12, find the unit tangent vector $\mathbf{T}(t)$ and find a set of parametric equations for the line tangent to the space curve at point P.

7. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + t\mathbf{k}, \quad P(0, 0, 0)$
8. $\mathbf{r}(t) = t^2\mathbf{i} + t\mathbf{j} + \frac{4}{3}\mathbf{k}, \quad P\left(1, 1, \frac{4}{3}\right)$
9. $\mathbf{r}(t) = 3\cos t\mathbf{i} + 3\sin t\mathbf{j} + t\mathbf{k}, \quad P(3, 0, 0)$
10. $\mathbf{r}(t) = \langle t, t, \sqrt{4 - t^2}\rangle, \quad P\left(1, 1, \sqrt{3}\right)$
11. $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 4\rangle, \quad P\left(\sqrt{2}, \sqrt{2}, 4\right)$
12. $\mathbf{r}(t) = \langle 2\sin t, 2\cos t, 4\sin^2 t\rangle, \quad P\left(1, \sqrt{3}, 1\right)$

Finding the Principal Unit Normal Vector In Exercises 13–20, find the principal unit normal vector to the curve at the specified value of the parameter.

13. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{2}t^2\mathbf{j}, \quad t = 2$
14. $\mathbf{r}(t) = t\mathbf{i} + \frac{6}{t}\mathbf{j}, \quad t = 3$
15. $\mathbf{r}(t) = \ln t\mathbf{i} + (t + 1)\mathbf{j}, \quad t = 2$
16. $\mathbf{r}(t) = \pi\cos t\mathbf{i} + \pi\sin t\mathbf{j}, \quad t = \frac{\pi}{6}$
17. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \ln t\mathbf{k}, \quad t = 1$
18. $\mathbf{r}(t) = \sqrt{2}t\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}, \quad t = 0$
19. $\mathbf{r}(t) = 6\cos t\mathbf{i} + 6\sin t\mathbf{j} + \mathbf{k}, \quad t = \frac{3\pi}{4}$
20. $\mathbf{r}(t) = \cos 3t\mathbf{i} + 2\sin 3t\mathbf{j} + \mathbf{k}, \quad t = \pi$

Finding Tangential and Normal Components of Acceleration In Exercises 21–28, find $\mathbf{T}(t)$, $\mathbf{N}(t)$, $a_{\mathbf{T}}$, and $a_{\mathbf{N}}$ at the given time t for the plane curve $\mathbf{r}(t)$.

21. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}, \quad t = 1$
22. $\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{j}, \quad t = 1$
23. $\mathbf{r}(t) = (t - t^3)\mathbf{i} + 2t^2\mathbf{j}, \quad t = 1$
24. $\mathbf{r}(t) = (t^3 - 4t)\mathbf{i} + (t^2 - 1)\mathbf{j}, \quad t = 0$
25. $\mathbf{r}(t) = e^t\mathbf{i} + e^{-2t}\mathbf{j}, \quad t = 0$
26. $\mathbf{r}(t) = e^t\mathbf{i} + e^{-t}\mathbf{j} + t\mathbf{k}, \quad t = 0$
27. $\mathbf{r}(t) = e^t\cos t\mathbf{i} + e^t\sin t\mathbf{j}, \quad t = \frac{\pi}{2}$
28. $\mathbf{r}(t) = 4\cos 3t\mathbf{i} + 4\sin 3t\mathbf{j}, \quad t = \pi$

Circular Motion In Exercises 29–32, consider an object moving according to the position vector

$$\mathbf{r}(t) = a\cos\omega t\,\mathbf{i} + a\sin\omega t\,\mathbf{j}.$$

29. Find $\mathbf{T}(t)$, $\mathbf{N}(t)$, $a_{\mathbf{T}}$, and $a_{\mathbf{N}}$.
30. Determine the directions of $\mathbf{T}$ and $\mathbf{N}$ relative to the position vector $\mathbf{r}$.
31. Determine the speed of the object at any time t and explain its value relative to the value of $a_{\mathbf{T}}$.
32. When the angular velocity ω is halved, by what factor is $a_{\mathbf{N}}$ changed?

Sketching a Graph and Vectors In Exercises 33–36, sketch the graph of the plane curve given by the vector-valued function, and, at the point on the curve determined by $\mathbf{r}(t_0)$, sketch the vectors $\mathbf{T}$ and $\mathbf{N}$. Note that $\mathbf{N}$ points toward the concave side of the curve.

Vector-Valued Function	Time
33. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}$	$t_0 = 2$
34. $\mathbf{r}(t) = t^3\mathbf{i} + t\mathbf{j}$	$t_0 = 1$
35. $\mathbf{r}(\mathbf{t}) = (2t + 1)\mathbf{i} - t^2\mathbf{j}$	$t_0 = 2$
36. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j}$	$t_0 = \frac{\pi}{4}$

Finding Vectors In Exercises 37–42, find $\mathbf{T}(t)$, $\mathbf{N}(t)$, $a_{\mathbf{T}}$, and $a_{\mathbf{N}}$ at the given time t for the space curve $\mathbf{r}(t)$. [*Hint:* Find $\mathbf{a}(t)$, $\mathbf{T}(t)$, $a_{\mathbf{T}}$, and $a_{\mathbf{N}}$. Solve for $\mathbf{N}$ in the equation $\mathbf{a}(t) = a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}$.]

Vector-Valued Function	Time
37. $\mathbf{r}(t) = t\mathbf{i} + 2t\mathbf{j} - 3t\mathbf{k}$	$t = 1$
38. $\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} + 2t\mathbf{k}$	$t = \frac{\pi}{3}$
39. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}$	$t = 1$
40. $\mathbf{r}(t) = (2t - 1)\mathbf{i} + t^2\mathbf{j} - 4t\mathbf{k}$	$t = 2$
41. $\mathbf{r}(t) = e^t\sin t\mathbf{i} + e^t\cos t\mathbf{j} + e^t\mathbf{k}$	$t = 0$
42. $\mathbf{r}(t) = e^t\mathbf{i} + 2t\mathbf{j} + e^{-t}\mathbf{k}$	$t = 0$

WRITING ABOUT CONCEPTS

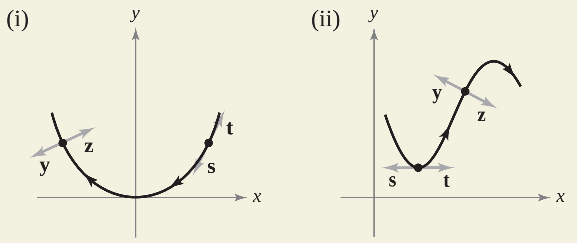
43. **Definitions** Define the unit tangent vector, the principal unit normal vector, and the tangential and normal components of acceleration.
44. **Unit Tangent Vector** How is the unit tangent vector related to the orientation of a curve? Explain.
45. **Acceleration** Describe the motion of a particle when the normal component of acceleration is 0.
46. **Acceleration** Describe the motion of a particle when the tangential component of acceleration is 0.

47. Finding Vectors An object moves along the path given by

$\mathbf{r}(t) = 3t\mathbf{i} + 4t\mathbf{j}$.

Find $\mathbf{v}(t)$, $\mathbf{a}(t)$, $\mathbf{T}(t)$, and $\mathbf{N}(t)$ (if it exists). What is the form of the path? Is the speed of the object constant or changing?

48. HOW DO YOU SEE IT? The figures show the paths of two particles.



(a) Which vector, **s** or **t**, represents the unit tangent vector?

(b) Which vector, **y** or **z**, represents the principal unit normal vector? Explain.

49. Cycloidal Motion The figure shows the path of a particle modeled by the vector-valued function

$\mathbf{r}(t) = \langle \pi t - \sin \pi t, 1 - \cos \pi t \rangle$.

The figure also shows the vectors $\mathbf{v}(t)/\|\mathbf{v}(t)\|$ and $\mathbf{a}(t)/\|\mathbf{a}(t)\|$ at the indicated values of t.

(a) Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ at $t = \frac{1}{2}$, $t = 1$, and $t = \frac{3}{2}$.

(b) Determine whether the speed of the particle is increasing or decreasing at each of the indicated values of t. Give reasons for your answers.

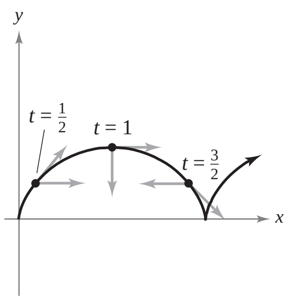


Figure for 49

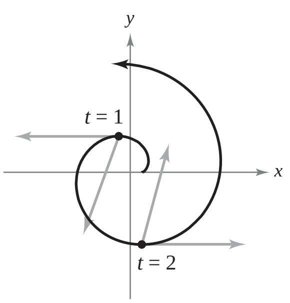


Figure for 50

50. Motion Along an Involute of a Circle The figure shows a particle moving along a path modeled by

$\mathbf{r}(t) = \langle \cos \pi t + \pi t \sin \pi t, \sin \pi t - \pi t \cos \pi t \rangle$.

The figure also shows the vectors $\mathbf{v}(t)$ and $\mathbf{a}(t)$ for $t = 1$ and $t = 2$.

(a) Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ at $t = 1$ and $t = 2$.

(b) Determine whether the speed of the particle is increasing or decreasing at each of the indicated values of t. Give reasons for your answers.

Finding a Binormal Vector **In Exercises 51–56, find the vectors T and N, and the binormal vector B = T × N, for the vector-valued function r(*t*) at the given value of *t*.**

51. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + \frac{t}{2}\mathbf{k}, \quad t_0 = \frac{\pi}{2}$

52. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^3}{3}\mathbf{k}, \quad t_0 = 1$

53. $\mathbf{r}(t) = \mathbf{i} + \sin t\mathbf{j} + \cos t\mathbf{k}, \quad t_0 = \frac{\pi}{4}$

54. $\mathbf{r}(t) = 2e^t\mathbf{i} + e^t\cos t\mathbf{j} + e^t\sin t\mathbf{k}, \quad t_0 = 0$

55. $\mathbf{r}(t) = 4\sin t\mathbf{i} + 4\cos t\mathbf{j} + 2t\mathbf{k}, \quad t_0 = \frac{\pi}{3}$

56. $\mathbf{r}(t) = 3\cos 2t\mathbf{i} + 3\sin 2t\mathbf{j} + t\mathbf{k}, \quad t_0 = \frac{\pi}{4}$

Alternative Formula for the Principal Unit Normal Vector **In Exercises 57–60, use the vector-valued function r(*t*) to find the principal unit normal vector N(*t*) using the alternative formula**

$$\mathbf{N} = \frac{(\mathbf{v}\cdot\mathbf{v})\mathbf{a} - (\mathbf{v}\cdot\mathbf{a})\mathbf{v}}{\|(\mathbf{v}\cdot\mathbf{v})\mathbf{a} - (\mathbf{v}\cdot\mathbf{a})\mathbf{v}\|}.$$

57. $\mathbf{r}(t) = 3t\mathbf{i} + 2t^2\mathbf{j}$

58. $\mathbf{r}(t) = 3\cos 2t\mathbf{i} + 3\sin 2t\mathbf{j}$

59. $\mathbf{r}(t) = 2t\mathbf{i} + 4t\mathbf{j} + t^2\mathbf{k}$

60. $\mathbf{r}(t) = 5\cos t\mathbf{i} + 5\sin t\mathbf{j} + 3t\mathbf{k}$

61. Projectile Motion Find the tangential and normal components of acceleration for a projectile fired at an angle θ with the horizontal at an initial speed of v_0. What are the components when the projectile is at its maximum height?

62. Projectile Motion Use your results from Exercise 61 to find the tangential and normal components of acceleration for a projectile fired at an angle of 45° with the horizontal at an initial speed of 150 feet per second. What are the components when the projectile is at its maximum height?

63. Projectile Motion A projectile is launched with an initial velocity of 120 feet per second at a height of 5 feet and at an angle of 30° with the horizontal.

(a) Determine the vector-valued function for the path of the projectile.

(b) Use a graphing utility to graph the path and approximate the maximum height and range of the projectile.

(c) Find $\mathbf{v}(t)$, $\|\mathbf{v}(t)\|$, and $\mathbf{a}(t)$.

(d) Use a graphing utility to complete the table.

t	0.5	1.0	1.5	2.0	2.5	3.0
Speed						

(e) Use a graphing utility to graph the scalar functions $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$. How is the speed of the projectile changing when $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ have opposite signs?

64. Projectile Motion A projectile is launched with an initial velocity of 220 feet per second at a height of 4 feet and at an angle of 45° with the horizontal.

(a) Determine the vector-valued function for the path of the projectile.

(b) Use a graphing utility to graph the path and approximate the maximum height and range of the projectile.

(c) Find $\mathbf{v}(t)$, $\|\mathbf{v}(t)\|$, and $\mathbf{a}(t)$.

(d) Use a graphing utility to complete the table.

t	0.5	1.0	1.5	2.0	2.5	3.0
Speed						

65. Air Traffic Control

Because of a storm, ground controllers instruct the pilot of a plane flying at an altitude of 4 miles to make a 90° turn and climb to an altitude of 4.2 miles. The model for the path of the plane during this maneuver is

$$\mathbf{r}(t) = \langle 10 \cos 10\pi t, 10 \sin 10\pi t, 4 + 4t \rangle, \quad 0 \le t \le \frac{1}{20}$$

where t is the time in hours and $\mathbf{r}$ is the distance in miles.

(a) Determine the speed of the plane.

(b) Calculate $a_\mathbf{T}$ and $a_\mathbf{N}$. Why is one of these equal to 0?

66. Projectile Motion A plane flying at an altitude of 36,000 feet at a speed of 600 miles per hour releases a bomb. Find the tangential and normal components of acceleration acting on the bomb.

67. Centripetal Acceleration An object is spinning at a constant speed on the end of a string, according to the position vector given in Exercises 29–32.

(a) When the angular velocity ω is doubled, how is the centripetal component of acceleration changed?

(b) When the angular velocity is unchanged but the length of the string is halved, how is the centripetal component of acceleration changed?

68. Centripetal Force An object of mass m moves at a constant speed v in a circular path of radius r. The force required to produce the centripetal component of acceleration is called the *centripetal force* and is given by $F = mv^2/r$. Newton's Law of Universal Gravitation is given by $F = GMm/d^2$, where d is the distance between the centers of the two bodies of masses M and m, and G is a gravitational constant. Use this law to show that the speed required for circular motion is $v = \sqrt{GM/r}$.

Orbital Speed In Exercises 69–72, use the result of Exercise 68 to find the speed necessary for the given circular orbit around Earth. Let $GM = 9.56 \times 10^4$ cubic miles per second per second, and assume the radius of Earth is 4000 miles.

69. The orbit of the International Space Station 255 miles above the surface of Earth

70. The orbit of the Hubble telescope 360 miles above the surface of Earth

71. The orbit of a heat capacity mapping satellite 385 miles above the surface of Earth

72. The orbit of a communications satellite r miles above the surface of Earth that is in geosynchronous orbit. [The satellite completes one orbit per sidereal day (approximately 23 hours, 56 minutes), and therefore appears to remain stationary above a point on Earth.]

True or False? In Exercises 73 and 74, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

73. If a car's speedometer is constant, then the car cannot be accelerating.

74. If $a_\mathbf{N} = 0$ for a moving object, then the object is moving in a straight line.

75. Motion of a Particle A particle moves along a path modeled by

$$\mathbf{r}(t) = \cosh(bt)\mathbf{i} + \sinh(bt)\mathbf{j}$$

where b is a positive constant.

(a) Show that the path of the particle is a hyperbola.

(b) Show that $\mathbf{a}(t) = b^2\,\mathbf{r}(t)$.

76. Proof Prove that the principal unit normal vector $\mathbf{N}$ points toward the concave side of a plane curve.

77. Proof Prove that the vector $\mathbf{T}'(t)$ is 0 for an object moving in a straight line.

78. Proof Prove that $a_\mathbf{N} = \dfrac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}$.

79. Proof Prove that $a_\mathbf{N} = \sqrt{\|\mathbf{a}\|^2 - a_\mathbf{T}^2}$.

PUTNAM EXAM CHALLENGE

80. A particle of unit mass moves on a straight line under the action of a force which is a function $f(v)$ of the velocity v of the particle, but the form of this function is not known. A motion is observed, and the distance x covered in time t is found to be connected with t by the formula $x = at + bt^2 + ct^3$, where a, b, and c have numerical values determined by observation of the motion. Find the function $f(v)$ for the range of v covered by the experiment.

Elena Aliaga/Shutterstock.com

12.5 Arc Length and Curvature

- **Find the arc length of a space curve.**
- **Use the arc length parameter to describe a plane curve or space curve.**
- **Find the curvature of a curve at a point on the curve.**
- **Use a vector-valued function to find frictional force.**

Arc Length

In Section 10.3, you saw that the arc length of a smooth *plane* curve C given by the parametric equations $x = x(t)$ and $y = y(t)$, $a \le t \le b$, is

$$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2}\, dt.$$

In vector form, where C is given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, you can rewrite this equation for arc length as

$$s = \int_a^b \|\mathbf{r}'(t)\|\, dt.$$

The formula for the arc length of a plane curve has a natural extension to a smooth curve in *space*, as stated in the next theorem.

THEOREM 12.6 Arc Length of a Space Curve

If C is a smooth curve given by $\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$ on an interval $[a, b]$, then the arc length of C on the interval is

$$s = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt = \int_a^b \|\mathbf{r}'(t)\|\, dt.$$

Exploration

Arc Length Formula The formula for the arc length of a space curve is given in terms of the parametric equations used to represent the curve. Does this mean that the arc length of the curve depends on the parameter being used? Would you want this to be true? Explain your reasoning.

Here is a different parametric representation of the curve in Example 1.

$$\mathbf{r}(t) = t^2\mathbf{i} + \frac{4}{3}t^3\mathbf{j} + \frac{1}{2}t^4\mathbf{k}$$

Find the arc length from $t = 0$ to $t = \sqrt{2}$ and compare the result with that found in Example 1.

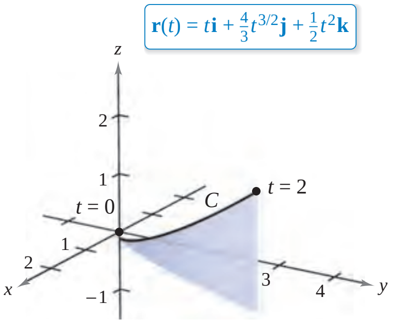
EXAMPLE 1 Finding the Arc Length of a Curve in Space

See LarsonCalculus.com for an interactive version of this type of example.

Find the arc length of the curve given by

$$\mathbf{r}(t) = t\mathbf{i} + \frac{4}{3}t^{3/2}\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$$

from $t = 0$ to $t = 2$, as shown in Figure 12.28.



As t increases from 0 to 2, the vector $\mathbf{r}(t)$ traces out a curve.
Figure 12.28

Solution Using $x(t) = t$, $y(t) = \frac{4}{3}t^{3/2}$, and $z(t) = \frac{1}{2}t^2$, you obtain $x'(t) = 1$, $y'(t) = 2t^{1/2}$, and $z'(t) = t$. So, the arc length from $t = 0$ to $t = 2$ is given by

$$s = \int_0^2 \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\, dt \qquad \text{Formula for arc length}$$

$$= \int_0^2 \sqrt{1 + 4t + t^2}\, dt$$

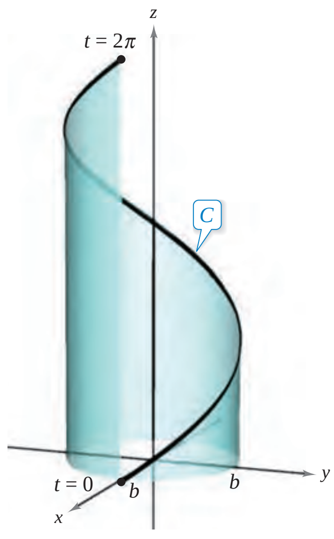
$$= \int_0^2 \sqrt{(t+2)^2 - 3}\, dt \qquad \text{Integration tables (Appendix B), Formula 26}$$

$$= \left[\frac{t+2}{2}\sqrt{(t+2)^2 - 3} - \frac{3}{2}\ln\left|(t+2) + \sqrt{(t+2)^2 - 3}\right|\right]_0^2$$

$$= 2\sqrt{13} - \frac{3}{2}\ln\left(4 + \sqrt{13}\right) - 1 + \frac{3}{2}\ln 3$$

$$\approx 4.816.$$

Curve:
$\mathbf{r}(t) = b\cos t\mathbf{i} + b\sin t\mathbf{j} + \sqrt{1-b^2}\,t\mathbf{k}$



One turn of a helix
Figure 12.29

EXAMPLE 2 Finding the Arc Length of a Helix

Find the length of one turn of the helix given by

$$\mathbf{r}(t) = b\cos t\mathbf{i} + b\sin t\mathbf{j} + \sqrt{1-b^2}\,t\mathbf{k}$$

as shown in Figure 12.29.

Solution Begin by finding the derivative.

$$\mathbf{r}'(t) = -b\sin t\mathbf{i} + b\cos t\mathbf{j} + \sqrt{1-b^2}\mathbf{k} \qquad \text{Derivative}$$

Now, using the formula for arc length, you can find the length of one turn of the helix by integrating $\|\mathbf{r}'(t)\|$ from 0 to 2π.

$$\begin{aligned} s &= \int_0^{2\pi} \|\mathbf{r}'(t)\|\,dt \qquad \text{Formula for arc length} \\ &= \int_0^{2\pi} \sqrt{b^2(\sin^2 t + \cos^2 t) + (1-b^2)}\,dt \\ &= \int_0^{2\pi} dt \\ &= t\Big]_0^{2\pi} \\ &= 2\pi \end{aligned}$$

So, the length is 2π units.

Arc Length Parameter

You have seen that curves can be represented by vector-valued functions in different ways, depending on the choice of parameter. For *motion* along a curve, the convenient parameter is time t. For studying the *geometric properties* of a curve, however, the convenient parameter is often arc length s.

$s(t) = \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\,du$

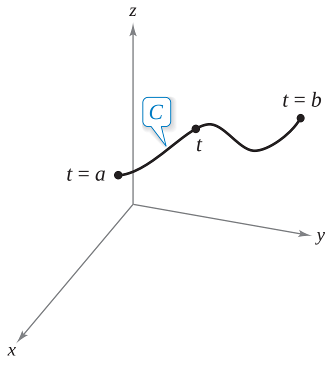


Figure 12.30

Definition of Arc Length Function

Let C be a smooth curve given by $\mathbf{r}(t)$ defined on the closed interval $[a, b]$. For $a \le t \le b$, the **arc length function** is

$$s(t) = \int_a^t \|\mathbf{r}'(u)\|\,du = \int_a^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\,du.$$

The arc length s is called the **arc length parameter.** (See Figure 12.30.)

Note that the arc length function s is *nonnegative*. It measures the distance along C from the initial point $(x(a), y(a), z(a))$ to the point $(x(t), y(t), z(t))$.

Using the definition of the arc length function and the Second Fundamental Theorem of Calculus, you can conclude that

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\|. \qquad \text{Derivative of arc length function}$$

In differential form, you can write

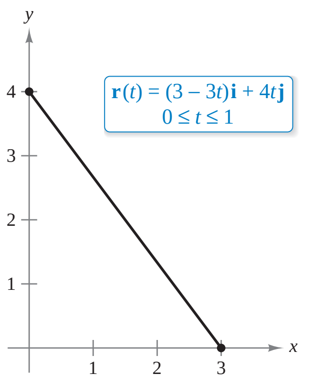
$$ds = \|\mathbf{r}'(t)\|\,dt.$$

EXAMPLE 3 Finding the Arc Length Function for a Line

Find the arc length function $s(t)$ for the line segment given by

$$\mathbf{r}(t) = (3 - 3t)\mathbf{i} + 4t\mathbf{j}, \quad 0 \le t \le 1$$

and write $\mathbf{r}$ as a function of the parameter s. (See Figure 12.31.)



The line segment from (3, 0) to (0, 4) can be parametrized using the arc length parameter s.
Figure 12.31

Solution Because $\mathbf{r}'(t) = -3\mathbf{i} + 4\mathbf{j}$ and

$$\|\mathbf{r}'(t)\| = \sqrt{(-3)^2 + 4^2} = 5$$

you have

$$\begin{aligned} s(t) &= \int_0^t \|\mathbf{r}'(u)\|\, du \\ &= \int_0^t 5\, du \\ &= 5t. \end{aligned}$$

Using $s = 5t$ (or $t = s/5$), you can rewrite $\mathbf{r}$ using the arc length parameter as follows.

$$\mathbf{r}(s) = \left(3 - \frac{3}{5}s\right)\mathbf{i} + \frac{4}{5}s\,\mathbf{j}, \quad 0 \le s \le 5$$

One of the advantages of writing a vector-valued function in terms of the arc length parameter is that $\|\mathbf{r}'(s)\| = 1$. For instance, in Example 3, you have

$$\|\mathbf{r}'(s)\| = \sqrt{\left(-\frac{3}{5}\right)^2 + \left(\frac{4}{5}\right)^2} = 1.$$

So, for a smooth curve C represented by $\mathbf{r}(s)$, where s is the arc length parameter, the arc length between a and b is

$$\begin{aligned} \text{Length of arc} &= \int_a^b \|\mathbf{r}'(s)\|\, ds \\ &= \int_a^b ds \\ &= b - a \\ &= \text{length of interval.} \end{aligned}$$

Furthermore, if t is *any* parameter such that $\|\mathbf{r}'(t)\| = 1$, then t must be the arc length parameter. These results are summarized in the next theorem, which is stated without proof.

THEOREM 12.7 Arc Length Parameter

If C is a smooth curve given by

$$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} \qquad \text{Plane curve}$$

or

$$\mathbf{r}(s) = x(s)\mathbf{i} + y(s)\mathbf{j} + z(s)\mathbf{k} \qquad \text{Space curve}$$

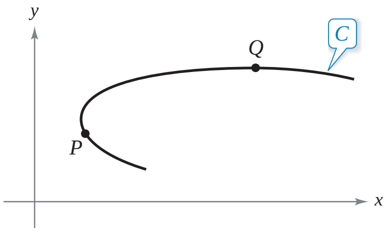
where s is the arc length parameter, then

$$\|\mathbf{r}'(s)\| = 1.$$

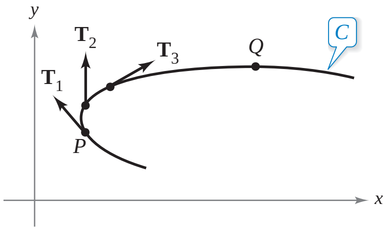
Moreover, if t is *any* parameter for the vector-valued function $\mathbf{r}$ such that $\|\mathbf{r}'(t)\| = 1$, then t must be the arc length parameter.

Curvature

An important use of the arc length parameter is to find **curvature**—the measure of how sharply a curve bends. For instance, in Figure 12.32, the curve bends more sharply at P than at Q, and you can say that the curvature is greater at P than at Q. You can calculate curvature by calculating the magnitude of the rate of change of the unit tangent vector $\mathbf{T}$ with respect to the arc length s, as shown in Figure 12.33.



Curvature at P is greater than at Q.
Figure 12.32



The magnitude of the rate of change of $\mathbf{T}$ with respect to the arc length is the curvature of a curve.
Figure 12.33

Definition of Curvature

Let C be a smooth curve (in the plane *or* in space) given by $\mathbf{r}(s)$, where s is the arc length parameter. The **curvature** K at s is

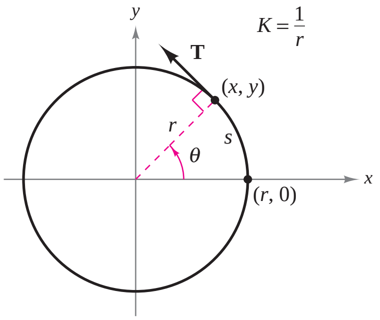
$$K = \left\| \frac{d\mathbf{T}}{ds} \right\| = \|\mathbf{T}'(s)\|.$$

A circle has the same curvature at any point. Moreover, the curvature and the radius of the circle are inversely related. That is, a circle with a large radius has a small curvature, and a circle with a small radius has a large curvature. This inverse relationship is made explicit in the next example.

EXAMPLE 4 Finding the Curvature of a Circle

Show that the curvature of a circle of radius r is

$$K = \frac{1}{r}.$$



The curvature of a circle is constant.
Figure 12.34

Solution Without loss of generality, you can consider the circle to be centered at the origin. Let (x, y) be any point on the circle and let s be the length of the arc from $(r, 0)$ to (x, y), as shown in Figure 12.34. By letting θ be the central angle of the circle, you can represent the circle by

$$\mathbf{r}(\theta) = r\cos\theta\,\mathbf{i} + r\sin\theta\,\mathbf{j}. \qquad \theta \text{ is the parameter.}$$

Using the formula for the length of a circular arc $s = r\theta$, you can rewrite $\mathbf{r}(\theta)$ in terms of the arc length parameter as follows.

$$\mathbf{r}(s) = r\cos\frac{s}{r}\,\mathbf{i} + r\sin\frac{s}{r}\,\mathbf{j} \qquad \text{Arc length } s \text{ is the parameter.}$$

So, $\mathbf{r}'(s) = -\sin\frac{s}{r}\,\mathbf{i} + \cos\frac{s}{r}\,\mathbf{j}$, and it follows that $\|\mathbf{r}'(s)\| = 1$, which implies that the unit tangent vector is

$$\mathbf{T}(s) = \frac{\mathbf{r}'(s)}{\|\mathbf{r}'(s)\|} = -\sin\frac{s}{r}\,\mathbf{i} + \cos\frac{s}{r}\,\mathbf{j}$$

and the curvature is

$$K = \|\mathbf{T}'(s)\| = \left\| -\frac{1}{r}\cos\frac{s}{r}\,\mathbf{i} - \frac{1}{r}\sin\frac{s}{r}\,\mathbf{j} \right\| = \frac{1}{r}$$

at every point on the circle.

Because a straight line doesn't curve, you would expect its curvature to be 0. Try checking this by finding the curvature of the line given by

$$\mathbf{r}(s) = \left(3 - \frac{3}{5}s\right)\mathbf{i} + \frac{4}{5}s\mathbf{j}.$$

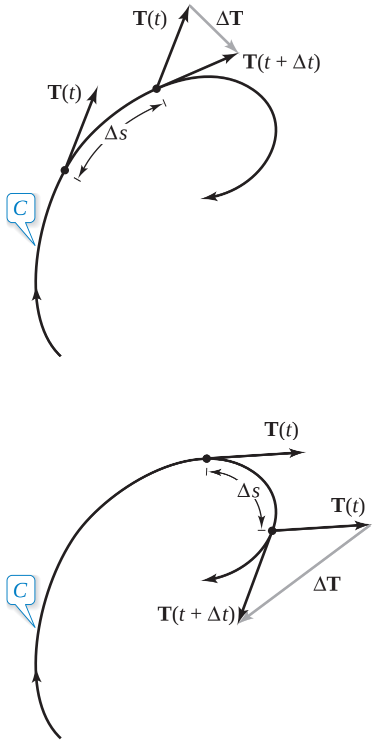


Figure 12.35

In Example 4, the curvature was found by applying the definition directly. This requires that the curve be written in terms of the arc length parameter s. The next theorem gives two other formulas for finding the curvature of a curve written in terms of an arbitrary parameter t. The proof of this theorem is left as an exercise [see Exercise 84, parts (a) and (b)].

THEOREM 12.8 Formulas for Curvature

If C is a smooth curve given by $\mathbf{r}(t)$, then the curvature K of C at t is

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}.$$

Because $\|\mathbf{r}'(t)\| = ds/dt$, the first formula implies that curvature is the ratio of the rate of change in the tangent vector $\mathbf{T}$ to the rate of change in arc length. To see that this is reasonable, let Δt be a "small number." Then,

$$\frac{\mathbf{T}'(t)}{ds/dt} \approx \frac{[\mathbf{T}(t + \Delta t) - \mathbf{T}(t)]/\Delta t}{[s(t + \Delta t) - s(t)]/\Delta t} = \frac{\mathbf{T}(t + \Delta t) - \mathbf{T}(t)}{s(t + \Delta t) - s(t)} = \frac{\Delta \mathbf{T}}{\Delta s}.$$

In other words, for a given Δs, the greater the length of $\Delta \mathbf{T}$, the more the curve bends at t, as shown in Figure 12.35.

EXAMPLE 5 Finding the Curvature of a Space Curve

Find the curvature of the curve given by

$$\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} - \frac{1}{3}t^3\mathbf{k}.$$

Solution It is not apparent whether this parameter represents arc length, so you should use the formula $K = \|\mathbf{T}'(t)\|/\|\mathbf{r}'(t)\|$.

$$\mathbf{r}'(t) = 2\mathbf{i} + 2t\mathbf{j} - t^2\mathbf{k}$$

$$\|\mathbf{r}'(t)\| = \sqrt{4 + 4t^2 + t^4} \quad \text{Length of } \mathbf{r}'(t)$$

$$= t^2 + 2$$

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$

$$= \frac{2\mathbf{i} + 2t\mathbf{j} - t^2\mathbf{k}}{t^2 + 2}$$

$$\mathbf{T}'(t) = \frac{(t^2 + 2)(2\mathbf{j} - 2t\mathbf{k}) - (2t)(2\mathbf{i} + 2t\mathbf{j} - t^2\mathbf{k})}{(t^2 + 2)^2}$$

$$= \frac{-4t\mathbf{i} + (4 - 2t^2)\mathbf{j} - 4t\mathbf{k}}{(t^2 + 2)^2}$$

$$\|\mathbf{T}'(t)\| = \frac{\sqrt{16t^2 + 16 - 16t^2 + 4t^4 + 16t^2}}{(t^2 + 2)^2}$$

$$= \frac{2(t^2 + 2)}{(t^2 + 2)^2}$$

$$= \frac{2}{t^2 + 2} \quad \text{Length of } \mathbf{T}'(t)$$

Therefore,

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{2}{(t^2 + 2)^2}. \quad \text{Curvature}$$

The next theorem presents a formula for calculating the curvature of a plane curve given by $y = f(x)$.

THEOREM 12.9 Curvature in Rectangular Coordinates

If C is the graph of a twice-differentiable function given by $y = f(x)$, then the curvature K at the point (x, y) is

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}}.$$

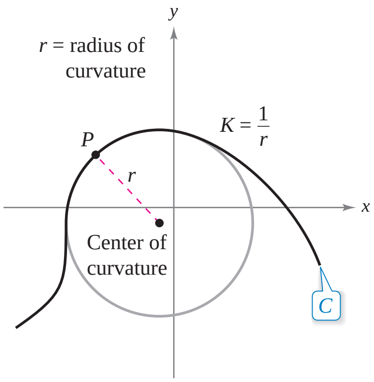
Proof By representing the curve C by $\mathbf{r}(x) = x\mathbf{i} + f(x)\mathbf{j} + 0\mathbf{k}$ (where x is the parameter), you obtain $\mathbf{r}'(x) = \mathbf{i} + f'(x)\mathbf{j}$,

$$\|\mathbf{r}'(x)\| = \sqrt{1 + [f'(x)]^2}$$

and $\mathbf{r}''(x) = f''(x)\mathbf{j}$. Because $\mathbf{r}'(x) \times \mathbf{r}''(x) = f''(x)\mathbf{k}$, it follows that the curvature is

$$\begin{aligned} K &= \frac{\|\mathbf{r}'(x) \times \mathbf{r}''(x)\|}{\|\mathbf{r}'(x)\|^3} \\ &= \frac{|f''(x)|}{\{1 + [f'(x)]^2\}^{3/2}} \\ &= \frac{|y''|}{[1 + (y')^2]^{3/2}}. \end{aligned}$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.



The circle of curvature
Figure 12.36

Let C be a curve with curvature K at point P. The circle passing through point P with radius $r = 1/K$ is called the **circle of curvature** when the circle lies on the concave side of the curve and shares a common tangent line with the curve at point P. The radius is called the **radius of curvature** at P, and the center of the circle is called the **center of curvature.**

The circle of curvature gives you a nice way to estimate the curvature K at a point P on a curve graphically. Using a compass, you can sketch a circle that lies against the concave side of the curve at point P, as shown in Figure 12.36. If the circle has a radius of r, then you can estimate the curvature to be $K = 1/r$.

EXAMPLE 6 Finding Curvature in Rectangular Coordinates

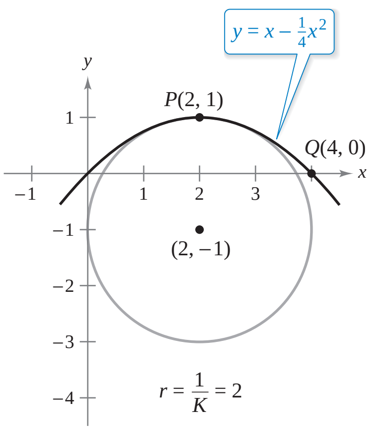
Find the curvature of the parabola given by $y = x - \frac{1}{4}x^2$ at $x = 2$. Sketch the circle of curvature at $(2, 1)$.

Solution The curvature at $x = 2$ is as follows.

$$y' = 1 - \frac{x}{2} \qquad y' = 0$$

$$y'' = -\frac{1}{2} \qquad y'' = -\frac{1}{2}$$

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}} \qquad K = \frac{1}{2}$$



The circle of curvature
Figure 12.37

Because the curvature at $P(2, 1)$ is $\frac{1}{2}$, it follows that the radius of the circle of curvature at that point is 2. So, the center of curvature is $(2, -1)$, as shown in Figure 12.37. [In the figure, note that the curve has the greatest curvature at P. Try showing that the curvature at $Q(4, 0)$ is $1/2^{5/2} \approx 0.177$.]

The amount of thrust felt by passengers in a car that is turning depends on two things—the speed of the car and the sharpness of the turn.
Figure 12.38

Arc length and curvature are closely related to the tangential and normal components of acceleration. The tangential component of acceleration is the rate of change of the speed, which in turn is the rate of change of the arc length. This component is negative as a moving object slows down and positive as it speeds up—regardless of whether the object is turning or traveling in a straight line. So, the tangential component is solely a function of the arc length and is independent of the curvature.

On the other hand, the normal component of acceleration is a function of *both* speed and curvature. This component measures the acceleration acting perpendicular to the direction of motion. To see why the normal component is affected by both speed and curvature, imagine that you are driving a car around a turn, as shown in Figure 12.38. When your speed is high and the turn is sharp, you feel yourself thrown against the car door. By lowering your speed *or* taking a more gentle turn, you are able to lessen this sideways thrust.

The next theorem explicitly states the relationships among speed, curvature, and the components of acceleration.

REMARK Note that Theorem 12.10 gives additional formulas for $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$.

THEOREM 12.10 Acceleration, Speed, and Curvature

If $\mathbf{r}(t)$ is the position vector for a smooth curve C, then the acceleration vector is given by

$$\mathbf{a}(t) = \frac{d^2s}{dt^2}\mathbf{T} + K\left(\frac{ds}{dt}\right)^2\mathbf{N}$$

where K is the curvature of C and ds/dt is the speed.

Proof For the position vector $\mathbf{r}(t)$, you have

$$\begin{aligned}\mathbf{a}(t) &= a_{\mathbf{T}}\mathbf{T} + a_{\mathbf{N}}\mathbf{N}\\ &= \frac{d}{dt}[\|\mathbf{v}\|]\mathbf{T} + \|\mathbf{v}\|\,\|\mathbf{T}'\|\mathbf{N}\\ &= \frac{d^2s}{dt^2}\mathbf{T} + \frac{ds}{dt}(\|\mathbf{v}\|K)\mathbf{N}\\ &= \frac{d^2s}{dt^2}\mathbf{T} + K\left(\frac{ds}{dt}\right)^2\mathbf{N}.\end{aligned}$$

See LarsonCalculus.com for Bruce Edwards's video of this proof.

EXAMPLE 7 Tangential and Normal Components of Acceleration

Find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ for the curve given by

$$\mathbf{r}(t) = 2t\mathbf{i} + t^2\mathbf{j} - \tfrac{1}{3}t^3\mathbf{k}.$$

Solution From Example 5, you know that

$$\frac{ds}{dt} = \|\mathbf{r}'(t)\| = t^2 + 2 \quad \text{and} \quad K = \frac{2}{(t^2+2)^2}.$$

Therefore,

$$a_{\mathbf{T}} = \frac{d^2s}{dt^2} = 2t \qquad \text{Tangential component}$$

and

$$a_{\mathbf{N}} = K\left(\frac{ds}{dt}\right)^2 = \frac{2}{(t^2+2)^2}(t^2+2)^2 = 2. \qquad \text{Normal component}$$

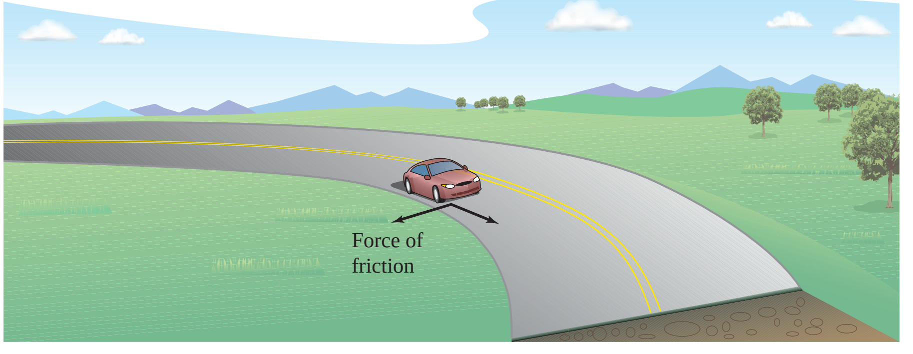
Application

There are many applications in physics and engineering dynamics that involve the relationships among speed, arc length, curvature, and acceleration. One such application concerns frictional force.

A moving object with mass m is in contact with a stationary object. The total force required to produce an acceleration **a** along a given path is

$$\begin{aligned}\mathbf{F} &= m\mathbf{a}\\ &= m\left(\frac{d^2s}{dt^2}\right)\mathbf{T} + mK\left(\frac{ds}{dt}\right)^2\mathbf{N}\\ &= ma_{\mathbf{T}}\mathbf{T} + ma_{\mathbf{N}}\mathbf{N}.\end{aligned}$$

The portion of this total force that is supplied by the stationary object is called the **force of friction.** For example, when a car moving with constant speed is rounding a turn, the roadway exerts a frictional force that keeps the car from sliding off the road. If the car is not sliding, the frictional force is perpendicular to the direction of motion and has magnitude equal to the normal component of acceleration, as shown in Figure 12.39. The potential frictional force of a road around a turn can be increased by banking the roadway.



The force of friction is perpendicular to the direction of motion.
Figure 12.39

EXAMPLE 8 Frictional Force

A 360-kilogram go-cart is driven at a speed of 60 kilometers per hour around a circular racetrack of radius 12 meters, as shown in Figure 12.40. To keep the cart from skidding off course, what frictional force must the track surface exert on the tires?

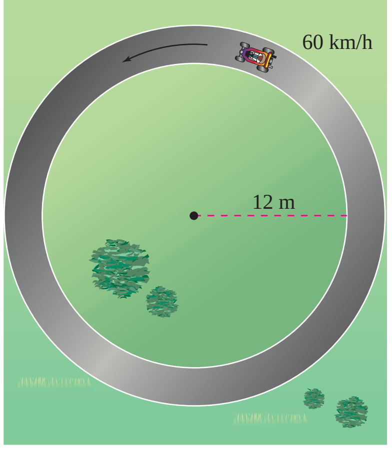


Figure 12.40

Solution The frictional force must equal the mass times the normal component of acceleration. For this circular path, you know that the curvature is

$$K = \frac{1}{12}. \qquad \text{Curvature of circular racetrack}$$

Therefore, the frictional force is

$$\begin{aligned}ma_{\mathbf{N}} &= mK\left(\frac{ds}{dt}\right)^2\\ &= (360 \text{ kg})\left(\frac{1}{12 \text{ m}}\right)\left(\frac{60{,}000 \text{ m}}{3600 \text{ sec}}\right)^2\\ &\approx 8333 \text{ (kg)(m)/sec}^2.\end{aligned}$$

SUMMARY OF VELOCITY, ACCELERATION, AND CURVATURE

Unless noted otherwise, let C be a curve (in the plane or in space) given by the position vector

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$$ Curve in the plane

or

$$\mathbf{r}(t) = x(t)\mathbf{i} + y(t)\mathbf{j} + z(t)\mathbf{k}$$ Curve in space

where x, y, and z are twice-differentiable functions of t.

Velocity vector, speed, and acceleration vector

$$\mathbf{v}(t) = \mathbf{r}'(t)$$ Velocity vector

$$\|\mathbf{v}(t)\| = \frac{ds}{dt} = \|\mathbf{r}'(t)\|$$ Speed

$$\mathbf{a}(t) = \mathbf{r}''(t)$$ Acceleration vector

$$= a_{\mathbf{T}}\mathbf{T}(t) + a_{\mathbf{N}}\mathbf{N}(t)$$

$$= \frac{d^2s}{dt^2}\mathbf{T}(t) + K\left(\frac{ds}{dt}\right)^2\mathbf{N}(t)$$ K is curvature and $\frac{ds}{dt}$ is speed.

Unit tangent vector and principal unit normal vector

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{\|\mathbf{r}'(t)\|}$$ Unit tangent vector

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{\|\mathbf{T}'(t)\|}$$ Principal unit normal vector

Components of acceleration

$$a_{\mathbf{T}} = \mathbf{a} \cdot \mathbf{T} = \frac{\mathbf{v} \cdot \mathbf{a}}{\|\mathbf{v}\|} = \frac{d^2s}{dt^2}$$ Tangential component of acceleration

$$a_{\mathbf{N}} = \mathbf{a} \cdot \mathbf{N}$$ Normal component of acceleration

$$= \frac{\|\mathbf{v} \times \mathbf{a}\|}{\|\mathbf{v}\|}$$

$$= \sqrt{\|\mathbf{a}\|^2 - a_{\mathbf{T}}^2}$$

$$= K\left(\frac{ds}{dt}\right)^2$$ K is curvature and $\frac{ds}{dt}$ is speed.

Formulas for curvature in the plane

$$K = \frac{|y''|}{[1 + (y')^2]^{3/2}}$$ C given by $y = f(x)$

$$K = \frac{|x'y'' - y'x''|}{[(x')^2 + (y')^2]^{3/2}}$$ C given by $x = x(t), y = y(t)$

Formulas for curvature in the plane or in space

$$K = \|\mathbf{T}'(s)\| = \|\mathbf{r}''(s)\|$$ s is arc length parameter.

$$K = \frac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|} = \frac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$$ t is general parameter.

$$K = \frac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{v}(t)\|^2}$$

Cross product formulas apply only to curves in space.

12.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding the Arc Length of a Plane Curve **In Exercises 1–6, sketch the plane curve and find its length over the given interval.**

1. $\mathbf{r}(t) = 3t\mathbf{i} - t\mathbf{j}, \quad [0, 3]$
2. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}, \quad [0, 4]$
3. $\mathbf{r}(t) = t^3\mathbf{i} + t^2\mathbf{j}, \quad [0, 1]$
4. $\mathbf{r}(t) = (t+1)\mathbf{i} + t^2\mathbf{j}, \quad [0, 6]$
5. $\mathbf{r}(t) = a\cos^3 t\,\mathbf{i} + a\sin^3 t\,\mathbf{j}, \quad [0, 2\pi]$
6. $\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j}, \quad [0, 2\pi]$

7. **Projectile Motion** A baseball is hit 3 feet above the ground at 100 feet per second and at an angle of 45° with respect to the ground.
   (a) Find the vector-valued function for the path of the baseball.
   (b) Find the maximum height.
   (c) Find the range.
   (d) Find the arc length of the trajectory.

8. **Projectile Motion** Repeat Exercise 7 for a baseball that is hit 4 feet above the ground at 80 feet per second and at an angle of 30° with respect to the ground.

Finding the Arc Length of a Curve in Space **In Exercises 9–14, sketch the space curve and find its length over the given interval.**

	Vector-Valued Function	Interval
9.	$\mathbf{r}(t) = -t\mathbf{i} + 4t\mathbf{j} + 3t\mathbf{k}$	$[0, 1]$
10.	$\mathbf{r}(t) = \mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k}$	$[0, 2]$
11.	$\mathbf{r}(t) = \langle 4t, -\cos t, \sin t\rangle$	$\left[0, \frac{3\pi}{2}\right]$
12.	$\mathbf{r}(t) = \langle 2\sin t, 5t, 2\cos t\rangle$	$[0, \pi]$
13.	$\mathbf{r}(t) = a\cos t\,\mathbf{i} + a\sin t\,\mathbf{j} + bt\,\mathbf{k}$	$[0, 2\pi]$
14.	$\mathbf{r}(t) = \langle \cos t + t\sin t, \sin t - t\cos t, t^2\rangle$	$\left[0, \frac{\pi}{2}\right]$

15. **Investigation** Consider the graph of the vector-valued function $\mathbf{r}(t) = t\mathbf{i} + (4 - t^2)\mathbf{j} + t^3\mathbf{k}$ on the interval $[0, 2]$.
   (a) Approximate the length of the curve by finding the length of the line segment connecting its endpoints.
   (b) Approximate the length of the curve by summing the lengths of the line segments connecting the terminal points of the vectors $\mathbf{r}(0)$, $\mathbf{r}(0.5)$, $\mathbf{r}(1)$, $\mathbf{r}(1.5)$, and $\mathbf{r}(2)$.
   (c) Describe how you could obtain a more accurate approximation by continuing the processes in parts (a) and (b).
   (d) Use the integration capabilities of a graphing utility to approximate the length of the curve. Compare this result with the answers in parts (a) and (b).

16. **Investigation** Repeat Exercise 15 for the vector-valued function $\mathbf{r}(t) = 6\cos(\pi t/4)\mathbf{i} + 2\sin(\pi t/4)\mathbf{j} + t\mathbf{k}$.

17. **Investigation** Consider the helix represented by the vector-valued function $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$.
   (a) Write the length of the arc s on the helix as a function of t by evaluating the integral
   $$s = \int_0^t \sqrt{[x'(u)]^2 + [y'(u)]^2 + [z'(u)]^2}\,du.$$
   (b) Solve for t in the relationship derived in part (a), and substitute the result into the original set of parametric equations. This yields a parametrization of the curve in terms of the arc length parameter s.
   (c) Find the coordinates of the point on the helix for arc lengths $s = \sqrt{5}$ and $s = 4$.
   (d) Verify that $\|\mathbf{r}'(s)\| = 1$.

18. **Investigation** Repeat Exercise 17 for the curve represented by the vector-valued function
   $$\mathbf{r}(t) = \left\langle 4(\sin t - t\cos t), 4(\cos t + t\sin t), \tfrac{3}{2}t^2\right\rangle.$$

Finding Curvature **In Exercises 19–22, find the curvature K of the curve, where s is the arc length parameter.**

19. $\mathbf{r}(s) = \left(1 + \frac{\sqrt{2}}{2}s\right)\mathbf{i} + \left(1 - \frac{\sqrt{2}}{2}s\right)\mathbf{j}$
20. $\mathbf{r}(s) = (3 + s)\mathbf{i} + \mathbf{j}$
21. Helix in Exercise 17: $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, t\rangle$
22. Curve in Exercise 18:
   $\mathbf{r}(t) = \left\langle 4(\sin t - t\cos t), 4(\cos t + t\sin t), \tfrac{3}{2}t^2\right\rangle$

Finding Curvature **In Exercises 23–28, find the curvature K of the plane curve at the given value of the parameter.**

23. $\mathbf{r}(t) = 4t\mathbf{i} - 2t\mathbf{j}, \quad t = 1$
24. $\mathbf{r}(t) = t^2\mathbf{i} + \mathbf{j}, \quad t = 2$
25. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{t}\mathbf{j}, \quad t = 1$
26. $\mathbf{r}(t) = t\mathbf{i} + \frac{1}{9}t^3\mathbf{j}, \quad t = 2$
27. $\mathbf{r}(t) = \langle t, \sin t\rangle, \quad t = \frac{\pi}{2}$
28. $\mathbf{r}(t) = \langle 5\cos t, 4\sin t\rangle, \quad t = \frac{\pi}{3}$

Finding Curvature **In Exercises 29–36, find the curvature K of the curve.**

29. $\mathbf{r}(t) = 4\cos 2\pi t\,\mathbf{i} + 4\sin 2\pi t\,\mathbf{j}$
30. $\mathbf{r}(t) = 2\cos \pi t\,\mathbf{i} + \sin \pi t\,\mathbf{j}$
31. $\mathbf{r}(t) = a\cos \omega t\,\mathbf{i} + a\sin \omega t\,\mathbf{j}$
32. $\mathbf{r}(t) = a\cos \omega t\,\mathbf{i} + b\sin \omega t\,\mathbf{j}$
33. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{t^2}{2}\mathbf{k}$
34. $\mathbf{r}(t) = 2t^2\mathbf{i} + t\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$
35. $\mathbf{r}(t) = 4t\mathbf{i} + 3\cos t\,\mathbf{j} + 3\sin t\,\mathbf{k}$
36. $\mathbf{r}(t) = e^{2t}\mathbf{i} + e^{2t}\cos t\,\mathbf{j} + e^{2t}\sin t\,\mathbf{k}$

Finding Curvature **In Exercises 37–40, find the curvature *K* of the curve at the point *P*.**

37. $\mathbf{r}(t) = 3t\mathbf{i} + 2t^2\mathbf{j}, \quad P(-3, 2)$

38. $\mathbf{r}(t) = e^t\mathbf{i} + 4t\mathbf{j}, \quad P(1, 0)$

39. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \dfrac{t^3}{4}\mathbf{k}, \quad P(2, 4, 2)$

40. $\mathbf{r}(t) = e^t \cos t\mathbf{i} + e^t \sin t\mathbf{j} + e^t\mathbf{k}, \quad P(1, 0, 1)$

Finding Curvature in Rectangular Coordinates **In Exercises 41–48, find the curvature and radius of curvature of the plane curve at the given value of *x*.**

41. $y = 3x - 2, \quad x = a$

42. $y = 2x + \dfrac{4}{x}, \quad x = 1$

43. $y = 2x^2 + 3, \quad x = -1$

44. $y = \frac{3}{4}\sqrt{16 - x^2}, \quad x = 0$

45. $y = \cos 2x, \quad x = 2\pi$

46. $y = e^{3x}, \quad x = 0$

47. $y = x^3, \quad x = 2$

48. $y = x^n, \quad x = 1, \quad n \geq 2$

Maximum Curvature **In Exercises 49–54, (a) find the point on the curve at which the curvature *K* is a maximum, and (b) find the limit of *K* as $x \to \infty$.**

49. $y = (x - 1)^2 + 3$

50. $y = x^3$

51. $y = x^{2/3}$

52. $y = \dfrac{1}{x}$

53. $y = \ln x$

54. $y = e^x$

Curvature **In Exercises 55–58, find all points on the graph of the function such that the curvature is zero.**

55. $y = 1 - x^3$

56. $y = (x - 1)^3 + 3$

57. $y = \cos x$

58. $y = \sin x$

WRITING ABOUT CONCEPTS

59. **Arc Length** Give the formula for the arc length of a smooth curve in space.

60. **Curvature** Give the formulas for curvature in the plane and in space.

61. **Curvature** Describe the graph of a vector-valued function for which the curvature is 0 for all values of *t* in its domain.

62. **Curvature** Given a twice-differentiable function $y = f(x)$, determine its curvature at a relative extremum. Can the curvature ever be greater than it is at a relative extremum? Why or why not?

63. **Investigation** Consider the function $f(x) = x^4 - x^2$.

(a) Use a computer algebra system to find the curvature *K* of the curve as a function of *x*.

(b) Use the result of part (a) to find the circles of curvature to the graph of *f* when $x = 0$ and $x = 1$. Use a computer algebra system to graph the function and the two circles of curvature.

(c) Graph the function $K(x)$ and compare it with the graph of $f(x)$. For example, do the extrema of *f* and *K* occur at the same critical numbers? Explain your reasoning.

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64. **Motion of a Particle** A particle moves along the plane curve *C* described by $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j}$.

(a) Find the length of *C* on the interval $0 \leq t \leq 2$.

(b) Find the curvature *K* of the plane curve at $t = 0$, $t = 1$, and $t = 2$.

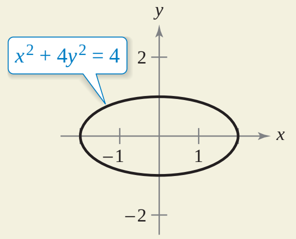
(c) Describe the curvature of *C* as *t* changes from $t = 0$ to $t = 2$.

65. **Investigation** Find all *a* and *b* such that the two curves given by

$$y_1 = ax(b - x) \quad \text{and} \quad y_2 = \frac{x}{x + 2}$$

intersect at only one point and have a common tangent line and equal curvature at that point. Sketch a graph for each set of values of *a* and *b*.

66. **HOW DO YOU SEE IT?** Using the graph of the ellipse, at what point(s) is the curvature the least and the greatest?



67. **Sphere and Paraboloid** A sphere of radius 4 is dropped into the paraboloid given by $z = x^2 + y^2$.

(a) How close will the sphere come to the vertex of the paraboloid?

(b) What is the radius of the largest sphere that will touch the vertex?

68. **Speed**

The smaller the curvature of a bend in a road, the faster a car can travel. Assume that the maximum speed around a turn is inversely proportional to the square root of the curvature. A car moving on the path $y = \frac{1}{3}x^3$, where *x* and *y* are measured in miles, can safely go 30 miles per hour at $\left(1, \frac{1}{3}\right)$. How fast can it go at $\left(\frac{3}{2}, \frac{9}{8}\right)$?

69. **Center of Curvature** Let *C* be a curve given by $y = f(x)$. Let *K* be the curvature $(K \neq 0)$ at the point $P(x_0, y_0)$ and let

$$z = \frac{1 + f'(x_0)^2}{f''(x_0)}.$$

Show that the coordinates (α, β) of the center of curvature at *P* are $(\alpha, \beta) = (x_0 - f'(x_0)z, y_0 + z)$.

70. Center of Curvature Use the result of Exercise 69 to find the center of curvature for the curve at the given point.

(a) $y = e^x, (0, 1)$ (b) $y = \frac{x^2}{2}, \left(1, \frac{1}{2}\right)$ (c) $y = x^2, (0, 0)$

71. Curvature A curve C is given by the polar equation $r = f(\theta)$. Show that the curvature K at the point (r, θ) is

$$K = \frac{|2(r')^2 - rr'' + r^2|}{[(r')^2 + r^2]^{3/2}}.$$

[*Hint:* Represent the curve by $\mathbf{r}(\theta) = r\cos\theta\mathbf{i} + r\sin\theta\mathbf{j}$.]

72. Curvature Use the result of Exercise 71 to find the curvature of each polar curve.

(a) $r = 1 + \sin\theta$ (b) $r = \theta$

(c) $r = a\sin\theta$ (d) $r = e^{\theta}$

73. Curvature Given the polar curve $r = e^{a\theta}$, $a > 0$, find the curvature K and determine the limit of K as (a) $\theta \to \infty$ and (b) $a \to \infty$.

74. Curvature at the Pole Show that the formula for the curvature of a polar curve $r = f(\theta)$ given in Exercise 71 reduces to $K = 2/|r'|$ for the curvature *at the pole.*

Curvature at the Pole In Exercises 75 and 76, use the result of Exercise 74 to find the curvature of the rose curve at the pole.

75. $r = 4\sin 2\theta$ **76.** $r = 6\cos 3\theta$

77. Proof For a smooth curve given by the parametric equations $x = f(t)$ and $y = g(t)$, prove that the curvature is given by

$$K = \frac{|f'(t)g''(t) - g'(t)f''(t)|}{\{[f'(t)]^2 + [g'(t)]^2\}^{3/2}}.$$

78. Horizontal Asymptotes Use the result of Exercise 77 to find the curvature K of the curve represented by the parametric equations $x(t) = t^3$ and $y(t) = \frac{1}{2}t^2$. Use a graphing utility to graph K and determine any horizontal asymptotes. Interpret the asymptotes in the context of the problem.

79. Curvature of a Cycloid Use the result of Exercise 77 to find the curvature K of the cycloid represented by the parametric equations

$$x(\theta) = a(\theta - \sin\theta) \quad \text{and} \quad y(\theta) = a(1 - \cos\theta).$$

What are the minimum and maximum values of K?

80. Tangential and Normal Components of Acceleration Use Theorem 12.10 to find $a_{\mathbf{T}}$ and $a_{\mathbf{N}}$ for each curve given by the vector-valued function.

(a) $\mathbf{r}(t) = 3t^2\mathbf{i} + (3t - t^3)\mathbf{j}$

(b) $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{1}{2}t^2\mathbf{k}$

81. Frictional Force A 5500-pound vehicle is driven at a speed of 30 miles per hour on a circular interchange of radius 100 feet. To keep the vehicle from skidding off course, what frictional force must the road surface exert on the tires?

82. Frictional Force A 6400-pound vehicle is driven at a speed of 35 miles per hour on a circular interchange of radius 250 feet. To keep the vehicle from skidding off course, what frictional force must the road surface exert on the tires?

83. Curvature Verify that the curvature at any point (x, y) on the graph of $y = \cosh x$ is $1/y^2$.

84. Formulas for Curvature Use the definition of curvature in space, $K = \|\mathbf{T}'(s)\| = \|\mathbf{r}''(s)\|$, to verify each formula.

(a) $K = \dfrac{\|\mathbf{T}'(t)\|}{\|\mathbf{r}'(t)\|}$

(b) $K = \dfrac{\|\mathbf{r}'(t) \times \mathbf{r}''(t)\|}{\|\mathbf{r}'(t)\|^3}$

(c) $K = \dfrac{\mathbf{a}(t) \cdot \mathbf{N}(t)}{\|\mathbf{v}(t)\|^2}$

True or False? In Exercises 85–88, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.

85. The arc length of a space curve depends on the parametrization.

86. The curvature of a circle is the same as its radius.

87. The curvature of a line is 0.

88. The normal component of acceleration is a function of both speed and curvature.

Kepler's Laws In Exercises 89–96, you are asked to verify Kepler's Laws of Planetary Motion. For these exercises, assume that each planet moves in an orbit given by the vector-valued function r. Let $r = \|\mathbf{r}\|$, let G represent the universal gravitational constant, let M represent the mass of the sun, and let m represent the mass of the planet.

89. Prove that $\mathbf{r} \cdot \mathbf{r}' = r\dfrac{dr}{dt}$.

90. Using Newton's Second Law of Motion, $\mathbf{F} = m\mathbf{a}$, and Newton's Second Law of Gravitation

$$\mathbf{F} = -\frac{GmM}{r^3}\mathbf{r}$$

show that $\mathbf{a}$ and $\mathbf{r}$ are parallel, and that $\mathbf{r}(t) \times \mathbf{r}'(t) = \mathbf{L}$ is a constant vector. So, $\mathbf{r}(t)$ moves *in a fixed plane,* orthogonal to $\mathbf{L}$.

91. Prove that $\dfrac{d}{dt}\left[\dfrac{\mathbf{r}}{r}\right] = \dfrac{1}{r^3}\{[\mathbf{r} \times \mathbf{r}'] \times \mathbf{r}\}$.

92. Show that $\dfrac{\mathbf{r}'}{GM} \times \mathbf{L} - \dfrac{\mathbf{r}}{r} = \mathbf{e}$ is a constant vector.

93. Prove Kepler's First Law: Each planet moves in an elliptical orbit with the sun as a focus.

94. Assume that the elliptical orbit

$$r = \frac{ed}{1 + e\cos\theta}$$

is in the xy-plane, with $\mathbf{L}$ along the z-axis. Prove that

$$\|\mathbf{L}\| = r^2\frac{d\theta}{dt}.$$

95. Prove Kepler's Second Law: Each ray from the sun to a planet sweeps out equal areas of the ellipse in equal times.

96. Prove Kepler's Third Law: The square of the period of a planet's orbit is proportional to the cube of the mean distance between the planet and the sun.

Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Domain and Continuity In Exercises 1–4, (a) find the domain of **r**, and (b) determine the values (if any) of t for which the function is continuous.

1. $\mathbf{r}(t) = \tan t\,\mathbf{i} + \mathbf{j} + t\,\mathbf{k}$
2. $\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \dfrac{1}{t-4}\mathbf{j} + \mathbf{k}$
3. $\mathbf{r}(t) = \ln t\,\mathbf{i} + t\,\mathbf{j} + t\,\mathbf{k}$
4. $\mathbf{r}(t) = (2t+1)\mathbf{i} + t^2\mathbf{j} + t\,\mathbf{k}$

Evaluating a Function In Exercises 5 and 6, evaluate (if possible) the vector-valued function at each given value of t.

5. $\mathbf{r}(t) = (2t+1)\mathbf{i} + t^2\mathbf{j} - \sqrt{t+2}\,\mathbf{k}$
   (a) $\mathbf{r}(0)$ (b) $\mathbf{r}(-2)$ (c) $\mathbf{r}(c-1)$
   (d) $\mathbf{r}(1+\Delta t) - \mathbf{r}(1)$
6. $\mathbf{r}(t) = 3\cos t\,\mathbf{i} + (1-\sin t)\mathbf{j} - t\,\mathbf{k}$
   (a) $\mathbf{r}(0)$ (b) $\mathbf{r}\left(\dfrac{\pi}{2}\right)$ (c) $\mathbf{r}(s-\pi)$
   (d) $\mathbf{r}(\pi+\Delta t) - \mathbf{r}(\pi)$

Writing a Vector-Valued Function In Exercises 7 and 8, represent the line segment from P to Q by a vector-valued function and by a set of parametric equations.

7. $P(3, 0, 5)$, $Q(2, -2, 3)$
8. $P(-2, -3, 8)$, $Q(5, 1, -2)$

Sketching a Curve In Exercises 9–12, sketch the curve represented by the vector-valued function and give the orientation of the curve.

9. $\mathbf{r}(t) = \langle \pi\cos t, \pi\sin t\rangle$
10. $\mathbf{r}(t) = \langle t+2, t^2-1\rangle$
11. $\mathbf{r}(t) = (t+1)\mathbf{i} + (3t-1)\mathbf{j} + 2t\,\mathbf{k}$
12. $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + t\,\mathbf{j} + 2\sin t\,\mathbf{k}$

Representing a Graph by a Vector-Valued Function In Exercises 13 and 14, represent the plane curve by a vector-valued function. (There are many correct answers.)

13. $3x + 4y - 12 = 0$
14. $y = 9 - x^2$

Representing a Graph by a Vector-Valued Function In Exercises 15 and 16, sketch the space curve represented by the intersection of the surfaces. Use the parameter $x = t$ to find a vector-valued function for the space curve.

15. $z = x^2 + y^2$, $x + y = 0$
16. $x^2 + z^2 = 4$, $x - y = 0$

Finding a Limit In Exercises 17 and 18, find the limit.

17. $\displaystyle\lim_{t\to 4^-}\left(t\,\mathbf{i} + \sqrt{4-t}\,\mathbf{j} + \mathbf{k}\right)$
18. $\displaystyle\lim_{t\to 0}\left(\frac{\sin 2t}{t}\mathbf{i} + e^{-t}\mathbf{j} + e^{t}\mathbf{k}\right)$

Higher-Order Differentiation In Exercises 19 and 20, find (a) $\mathbf{r}'(t)$, (b) $\mathbf{r}''(t)$, and (c) $\mathbf{r}'(t)\cdot\mathbf{r}''(t)$.

19. $\mathbf{r}(t) = (t^2+4t)\mathbf{i} - 3t^2\mathbf{j}$
20. $\mathbf{r}(t) = 5\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j}$

Higher-Order Differentiation In Exercises 21 and 22, find (a) $\mathbf{r}'(t)$, (b) $\mathbf{r}''(t)$, (c) $\mathbf{r}'(t)\cdot\mathbf{r}''(t)$, and (d) $\mathbf{r}'(t)\times\mathbf{r}''(t)$.

21. $\mathbf{r}(t) = 2t^3\mathbf{i} + 4t\mathbf{j} - t^2\mathbf{k}$
22. $\mathbf{r}(t) = (4t+3)\mathbf{i} + t^2\mathbf{j} + (2t^2+4)\mathbf{k}$

Using Properties of the Derivative In Exercises 23 and 24, use the properties of the derivative to find the following.

(a) $\mathbf{r}'(t)$ (b) $\dfrac{d}{dt}[\mathbf{u}(t) - 2\mathbf{r}(t)]$ (c) $\dfrac{d}{dt}(3t)\mathbf{r}(t)$

(d) $\dfrac{d}{dt}[\mathbf{r}(t)\cdot\mathbf{u}(t)]$ (e) $\dfrac{d}{dt}[\mathbf{r}(t)\times\mathbf{u}(t)]$ (f) $\dfrac{d}{dt}\mathbf{u}(2t)$

23. $\mathbf{r}(t) = 3t\,\mathbf{i} + (t-1)\mathbf{j}$, $\mathbf{u}(t) = t\,\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k}$
24. $\mathbf{r}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + t\,\mathbf{k}$, $\mathbf{u}(t) = \sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \dfrac{1}{t}\mathbf{k}$

Finding an Indefinite Integral In Exercises 25–28, find the indefinite integral.

25. $\displaystyle\int (\mathbf{i} + 3\mathbf{j} + 4t\mathbf{k})\,dt$
26. $\displaystyle\int (t^2\mathbf{i} + 5t\mathbf{j} + 8t^3\mathbf{k})\,dt$
27. $\displaystyle\int \left(3\sqrt{t}\,\mathbf{i} + \frac{2}{t}\mathbf{j} + \mathbf{k}\right)dt$
28. $\displaystyle\int (\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + e^{2t}\mathbf{k})\,dt$

Evaluating a Definite Integral In Exercises 29–32, evaluate the definite integral.

29. $\displaystyle\int_{-2}^{2} (3t\,\mathbf{i} + 2t^2\mathbf{j} - t^3\mathbf{k})\,dt$
30. $\displaystyle\int_{0}^{1} (t\,\mathbf{i} + \sqrt{t}\,\mathbf{j} + 4t\,\mathbf{k})\,dt$
31. $\displaystyle\int_{0}^{2} (e^{t/2}\mathbf{i} - 3t^2\mathbf{j} - \mathbf{k})\,dt$
32. $\displaystyle\int_{0}^{\pi/3} (2\cos t\,\mathbf{i} + \sin t\,\mathbf{j} + 3\mathbf{k})\,dt$

Finding an Antiderivative In Exercises 33 and 34, find $\mathbf{r}(t)$ that satisfies the initial condition(s).

33. $\mathbf{r}'(t) = 2t\,\mathbf{i} + e^t\mathbf{j} + e^{-t}\mathbf{k}$, $\mathbf{r}(0) = \mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$
34. $\mathbf{r}'(t) = \sec t\,\mathbf{i} + \tan t\,\mathbf{j} + t^2\mathbf{k}$, $\mathbf{r}(0) = 3\mathbf{k}$

Finding Velocity and Acceleration Vectors **In Exercises 35–38, the position vector r describes the path of an object moving in space.**

(a) Find the velocity vector, speed, and acceleration vector of the object.

(b) Evaluate the velocity vector and acceleration vector of the object at the given value of *t*.

	Position Vector	Time
35.	$\mathbf{r}(t) = 4t\mathbf{i} + t^3\mathbf{j} - t\mathbf{k}$	$t = 1$
36.	$\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + 5t\mathbf{j} + 2t^2\mathbf{k}$	$t = 4$
37.	$\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t, 3t \rangle$	$t = \pi$
38.	$\mathbf{r}(t) = \langle t, -\tan t, e^t \rangle$	$t = 0$

Projectile Motion **In Exercises 39–42, use the model for projectile motion, assuming there is no air resistance. [$a(t) = -32$ feet per second per second or $a(t) = -9.8$ meters per second per second]**

39. A projectile is fired from ground level with an initial velocity of 84 feet per second at an angle of 30° with the horizontal. Find the range of the projectile.

40. A baseball is hit from a height of 3.5 feet above the ground with an initial velocity of 120 feet per second and at an angle of 30° above the horizontal. Find the maximum height reached by the baseball. Determine whether it will clear an 8-foot-high fence located 375 feet from home plate.

41. A projectile is fired from ground level at an angle of 20° with the horizontal. The projectile has a range of 95 meters. Find the minimum initial velocity.

42. Use a graphing utility to graph the paths of a projectile for $v_0 = 20$ meters per second, $h = 0$ and (a) $\theta = 30°$, (b) $\theta = 45°$, and (c) $\theta = 60°$. Use the graphs to approximate the maximum height and range of the projectile for each case.

Finding the Unit Tangent Vector **In Exercises 43 and 44, find the unit tangent vector to the curve at the specified value of the parameter.**

43. $\mathbf{r}(t) = 3t\mathbf{i} + 3t^3\mathbf{j}, \quad t = 1$

44. $\mathbf{r}(t) = 2\sin t\mathbf{i} + 4\cos t\mathbf{j}, \quad t = \dfrac{\pi}{6}$

Finding a Tangent Line **In Exercises 45 and 46, find the unit tangent vector T(*t*) and find a set of parametric equations for the line tangent to the space curve at point *P*.**

45. $\mathbf{r}(t) = 2\cos t\mathbf{i} + 2\sin t\mathbf{j} + t\mathbf{k}, \quad P\left(1, \sqrt{3}, \dfrac{\pi}{3}\right)$

46. $\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + \frac{2}{3}t^3\mathbf{k}, \quad P\left(2, 4, \frac{16}{3}\right)$

Finding the Principal Unit Normal Vector **In Exercises 47–50, find the principal unit normal vector to the curve at the specified value of the parameter.**

47. $\mathbf{r}(t) = 2t\mathbf{i} + 3t^2\mathbf{j}, \quad t = 1$

48. $\mathbf{r}(t) = t\mathbf{i} + \ln t\mathbf{j}, \quad t = 2$

49. $\mathbf{r}(t) = 3\cos 2t\mathbf{i} + 3\sin 2t\mathbf{j} + 3\mathbf{k}, \quad t = \dfrac{\pi}{4}$

50. $\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + \mathbf{k}, \quad t = \dfrac{2\pi}{3}$

Finding Tangential and Normal Components of Acceleration **In Exercises 51 and 52, find T(*t*), N(*t*), $a_{\mathbf{T}}$, and $a_{\mathbf{N}}$ at the given time *t* for the plane curve *r*(*t*).**

51. $\mathbf{r}(t) = \dfrac{3}{t}\mathbf{i} - 6t\mathbf{j}, \quad t = 3$

52. $\mathbf{r}(t) = 3\cos 2t\mathbf{i} + 3\sin 2t\mathbf{j}, \quad t = \dfrac{\pi}{6}$

Finding the Arc Length of a Plane Curve **In Exercises 53–56, sketch the plane curve and find its length over the given interval.**

	Vector-Valued Function	Interval
53.	$\mathbf{r}(t) = 2t\mathbf{i} - 3t\mathbf{j}$	$[0, 5]$
54.	$\mathbf{r}(t) = t^2\mathbf{i} + 2t\mathbf{k}$	$[0, 3]$
55.	$\mathbf{r}(t) = 10\cos^3 t\mathbf{i} + 10\sin^3 t\mathbf{j}$	$[0, 2\pi]$
56.	$\mathbf{r}(t) = 10\cos t\mathbf{i} + 10\sin t\mathbf{j}$	$[0, 2\pi]$

Finding the Arc Length of a Curve in Space **In Exercises 57–60, sketch the space curve and find its length over the given interval.**

	Vector-Valued Function	Interval
57.	$\mathbf{r}(t) = -3t\mathbf{i} + 2t\mathbf{j} + 4t\mathbf{k}$	$[0, 3]$
58.	$\mathbf{r}(t) = t\mathbf{i} + t^2\mathbf{j} + 2t\mathbf{k}$	$[0, 2]$
59.	$\mathbf{r}(t) = \langle 8\cos t, 8\sin t, t \rangle$	$\left[0, \dfrac{\pi}{2}\right]$
60.	$\mathbf{r}(t) = \langle 2(\sin t - t\cos t), 2(\cos t + t\sin t), t \rangle$	$\left[0, \dfrac{\pi}{2}\right]$

Finding Curvature **In Exercises 61–64, find the curvature *K* of the curve.**

61. $\mathbf{r}(t) = 3t\mathbf{i} + 2t\mathbf{j}$

62. $\mathbf{r}(t) = 2\sqrt{t}\,\mathbf{i} + 3t\mathbf{j}$

63. $\mathbf{r}(t) = 2t\mathbf{i} + \frac{1}{2}t^2\mathbf{j} + t^2\mathbf{k}$

64. $\mathbf{r}(t) = 2t\mathbf{i} + 5\cos t\mathbf{j} + 5\sin t\mathbf{k}$

Finding Curvature **In Exercises 65 and 66, find the curvature *K* of the curve at the point *P*.**

65. $\mathbf{r}(t) = \frac{1}{2}t^2\mathbf{i} + t\mathbf{j} + \frac{1}{3}t^3\mathbf{k}, \quad P\left(\frac{1}{2}, 1, \frac{1}{3}\right)$

66. $\mathbf{r}(t) = 4\cos t\mathbf{i} + 3\sin t\mathbf{j} + t\mathbf{k}, \quad P(-4, 0, \pi)$

Finding Curvature in Rectangular Coordinates **In Exercises 67–70, find the curvature and radius of curvature of the plane curve at the given value of *x*.**

67. $y = \frac{1}{2}x^2 + 2, \quad x = 4$

68. $y = e^{-x/2}, \quad x = 0$

69. $y = \ln x, \quad x = 1$

70. $y = \tan x, \quad x = \dfrac{\pi}{4}$

71. Frictional Force A 7200-pound vehicle is driven at a speed of 25 miles per hour on a circular interchange of radius 150 feet. To keep the vehicle from skidding off course, what frictional force must the road surface exert on the tires?

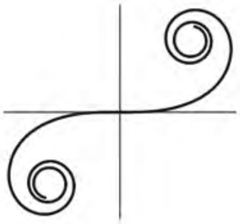
P.S. Problem Solving

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

1. Cornu Spiral The **cornu spiral** is given by

$$x(t) = \int_0^t \cos\left(\frac{\pi u^2}{2}\right) du \quad \text{and} \quad y(t) = \int_0^t \sin\left(\frac{\pi u^2}{2}\right) du.$$

The spiral shown in the figure was plotted over the interval $-\pi \le t \le \pi$.



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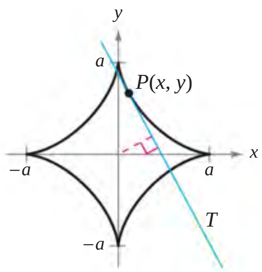
(a) Find the arc length of this curve from $t = 0$ to $t = a$.

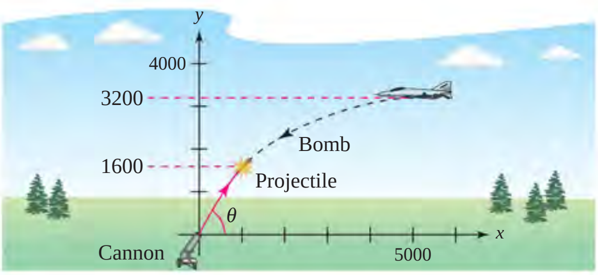
(b) Find the curvature of the graph when $t = a$.

(c) The cornu spiral was discovered by James Bernoulli. He found that the spiral has an amazing relationship between curvature and arc length. What is this relationship?

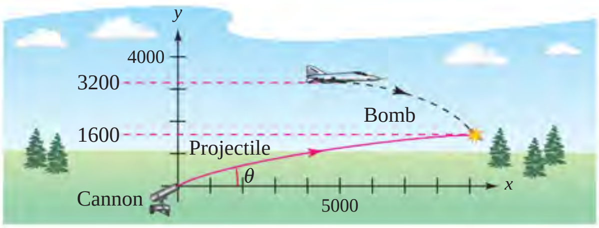
2. Radius of Curvature Let T be the tangent line at the point $P(x, y)$ to the graph of the curve $x^{2/3} + y^{2/3} = a^{2/3}$, $a > 0$, as shown in the figure. Show that the radius of curvature at P is three times the distance from the origin to the tangent line T.



3. Projectile Motion A bomber is flying horizontally at an altitude of 3200 feet with a velocity of 400 feet per second when it releases a bomb. A projectile is launched 5 seconds later from a cannon at a site facing the bomber and 5000 feet from the point that was directly beneath the bomber when the bomb was released, as shown in the figure. The projectile is to intercept the bomb at an altitude of 1600 feet. Determine the required initial speed and angle of inclination of the projectile. (Ignore air resistance.)



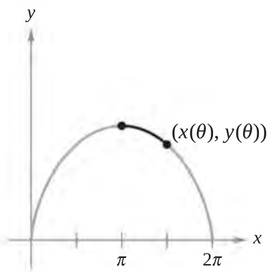
4. Projectile Motion Repeat Exercise 3 for the case in which the bomber is facing *away* from the launch site, as shown in the figure.



5. Cycloid Consider one arch of the cycloid

$$\mathbf{r}(\theta) = (\theta - \sin\theta)\mathbf{i} + (1 - \cos\theta)\mathbf{j}, \quad 0 \le \theta \le 2\pi$$

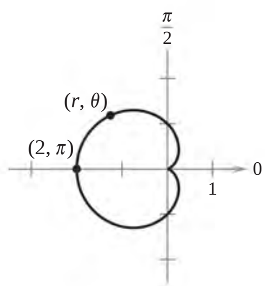
as shown in the figure. Let $s(\theta)$ be the arc length from the highest point on the arch to the point $(x(\theta), y(\theta))$, and let $\rho(\theta) = 1/K$ be the radius of curvature at the point $(x(\theta), y(\theta))$. Show that s and ρ are related by the equation $s^2 + \rho^2 = 16$. (This equation is called a *natural equation* for the curve.)



6. Cardioid Consider the cardioid

$$r = 1 - \cos\theta, \quad 0 \le \theta \le 2\pi$$

as shown in the figure. Let $s(\theta)$ be the arc length from the point $(2, \pi)$ on the cardioid to the point (r, θ), and let $p(\theta) = 1/K$ be the radius of curvature at the point (r, θ). Show that s and ρ are related by the equation $s^2 + 9\rho^2 = 16$. (This equation is called a *natural equation* for the curve.)



7. Proof If $\mathbf{r}(t)$ is a nonzero differentiable function of t, prove that

$$\frac{d}{dt}(\|\mathbf{r}(t)\|) = \frac{1}{\|\mathbf{r}(t)\|}\mathbf{r}(t) \cdot \mathbf{r}'(t).$$

8. Satellite A communications satellite moves in a circular orbit around Earth at a distance of 42,000 kilometers from the center of Earth. The angular velocity

$$\frac{d\theta}{dt} = \omega = \frac{\pi}{12} \text{ radian per hour}$$

is constant.

(a) Use polar coordinates to show that the acceleration vector is given by

$$\mathbf{a} = \frac{d^2\mathbf{r}}{dt^2} = \left[\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2\right]\mathbf{u}_\mathbf{r} + \left[r\frac{d^2\theta}{dt^2} + 2\frac{dr}{dt}\frac{d\theta}{dt}\right]\mathbf{u}_\theta$$

where $\mathbf{u}_\mathbf{r} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$ is the unit vector in the radial direction and $\mathbf{u}_\theta = -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}$.

(b) Find the radial and angular components of acceleration for the satellite.

Binormal Vector **In Exercises 9–11, use the binormal vector defined by the equation $\mathbf{B} = \mathbf{T} \times \mathbf{N}$.**

9. Find the unit tangent, unit normal, and binormal vectors for the helix

$$\mathbf{r}(t) = 4\cos t\mathbf{i} + 4\sin t\mathbf{j} + 3t\mathbf{k}$$

at $t = \pi/2$. Sketch the helix together with these three mutually orthogonal unit vectors.

10. Find the unit tangent, unit normal, and binormal vectors for the curve

$$\mathbf{r}(t) = \cos t\mathbf{i} + \sin t\mathbf{j} - \mathbf{k}$$

at $t = \pi/4$. Sketch the curve together with these three mutually orthogonal unit vectors.

11. (a) Prove that there exists a scalar τ, called the **torsion,** such that $d\mathbf{B}/ds = -\tau\mathbf{N}$.

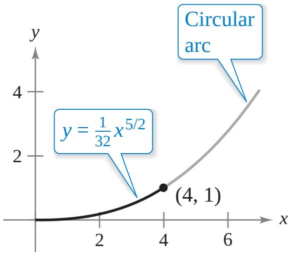
(b) Prove that $\dfrac{d\mathbf{N}}{ds} = -K\mathbf{T} + \tau\mathbf{B}$.

(The three equations $d\mathbf{T}/ds = K\mathbf{N}$, $d\mathbf{N}/ds = -K\mathbf{T} + \tau\mathbf{B}$, and $d\mathbf{B}/ds = -\tau\mathbf{N}$ are called the *Frenet-Serret formulas.*)

12. Exit Ramp A highway has an exit ramp that begins at the origin of a coordinate system and follows the curve

$$y = \frac{1}{32}x^{5/2}$$

to the point (4, 1) (see figure). Then it follows a circular path whose curvature is that given by the curve at (4, 1). What is the radius of the circular arc? Explain why the curve and the circular arc should have the same curvature at (4, 1).



13. Arc Length and Curvature Consider the vector-valued function

$$\mathbf{r}(t) = \langle t\cos\pi t, t\sin\pi t\rangle, \quad 0 \le t \le 2.$$

(a) Use a graphing utility to graph the function.

(b) Find the length of the arc in part (a).

(c) Find the curvature K as a function of t. Find the curvatures for t-values of 0, 1, and 2.

(d) Use a graphing utility to graph the function K.

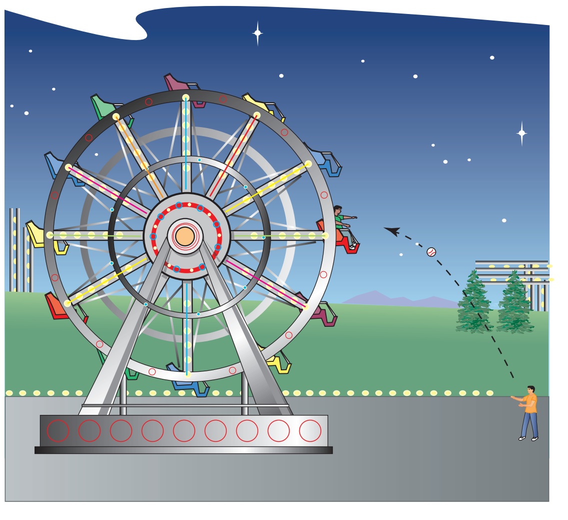
(e) Find (if possible) $\lim_{t\to\infty} K$.

(f) Using the result of part (e), make a conjecture about the graph of $\mathbf{r}$ as $t \to \infty$.

14. Ferris Wheel You want to toss an object to a friend who is riding a Ferris wheel (see figure). The following parametric equations give the path of the friend $\mathbf{r}_1(t)$ and the path of the object $\mathbf{r}_2(t)$. Distance is measured in meters and time is measured in seconds.

$$\mathbf{r}_1(t) = 15\left(\sin\frac{\pi t}{10}\right)\mathbf{i} + \left(16 - 15\cos\frac{\pi t}{10}\right)\mathbf{j}$$

$$\mathbf{r}_2(t) = [22 - 8.03(t - t_0)]\mathbf{i} + [1 + 11.47(t - t_0) - 4.9(t - t_0)^2]\mathbf{j}$$



(a) Locate your friend's position on the Ferris wheel at time $t = 0$.

(b) Determine the number of revolutions per minute of the Ferris wheel.

(c) What are the speed and angle of inclination (in degrees) at which the object is thrown at time $t = t_0$?

(d) Use a graphing utility to graph the vector-valued functions using a value of t_0 that allows your friend to be within reach of the object. (Do this by trial and error.) Explain the significance of t_0.

(e) Find the approximate time your friend should be able to catch the object. Approximate the speeds of your friend and the object at that time.