6 Differential Equations

Intravenous Feeding (Exercise 30, p. 421)

Sailing (Exercise 85, p. 408)

Wildlife Population (Example 4, p. 399)

Forestry (Exercise 62, p. 396)

Radioactive Decay (Example 3, p. 391)

Clockwise from top left, Web Picture Blog/Shutterstock.com; Auremar/Shutterstock.com;
Stephen Aaron Rees/Shutterstock.com; KIMIMASA MAYAMA/EPA/Newscom; franzfoto.com/Alamy

6.1 Slope Fields and Euler's Method

- **Use initial conditions to find particular solutions of differential equations.**
- **Use slope fields to approximate solutions of differential equations.**
- **Use Euler's Method to approximate solutions of differential equations.**

General and Particular Solutions

In this text, you will learn that physical phenomena can be described by differential equations. Recall that a **differential equation** in x and y is an equation that involves x, y, and derivatives of y. For example,

$$2xy' - 3y = 0 \qquad \text{Differential equation}$$

is a differential equation. In Section 6.2, you will see that problems involving radioactive decay, population growth, and Newton's Law of Cooling can be formulated in terms of differential equations.

A function $y = f(x)$ is called a **solution** of a differential equation if the equation is satisfied when y and its derivatives are replaced by $f(x)$ and its derivatives. For example, differentiation and substitution would show that $y = e^{-2x}$ is a solution of the differential equation $y' + 2y = 0$. It can be shown that every solution of this differential equation is of the form

$$y = Ce^{-2x} \qquad \text{General solution of } y' + 2y = 0$$

where C is any real number. This solution is called the **general solution.** Some differential equations have **singular solutions** that cannot be written as special cases of the general solution. Such solutions, however, are not considered in this text. The **order** of a differential equation is determined by the highest-order derivative in the equation. For instance, $y' = 4y$ is a first-order differential equation. First-order linear differential equations are discussed in Section 6.5.

In Section 5.1, Example 9, you saw that the second-order differential equation $s''(t) = -32$ has the general solution

$$s(t) = -16t^2 + C_1t + C_2 \qquad \text{General solution of } s''(t) = -32$$

which contains two arbitrary constants. It can be shown that a differential equation of order n has a general solution with n arbitrary constants.

EXAMPLE 1 Verifying Solutions

Determine whether the function is a solution of the differential equation $y'' - y = 0$.

a. $y = \sin x$ **b.** $y = 4e^{-x}$ **c.** $y = Ce^x$

Solution

a. Because $y = \sin x$, $y' = \cos x$, and $y'' = -\sin x$, it follows that

$$y'' - y = -\sin x - \sin x = -2 \sin x \neq 0.$$

So, $y = \sin x$ is *not* a solution.

b. Because $y = 4e^{-x}$, $y' = -4e^{-x}$, and $y'' = 4e^{-x}$, it follows that

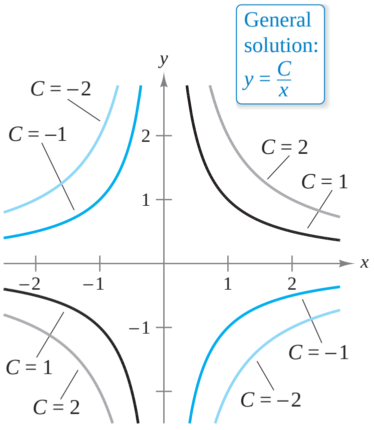
$$y'' - y = 4e^{-x} - 4e^{-x} = 0.$$

So, $y = 4e^{-x}$ is a solution.

c. Because $y = Ce^x$, $y' = Ce^x$, and $y'' = Ce^x$, it follows that

$$y'' - y = Ce^x - Ce^x = 0.$$

So, $y = Ce^x$ is a solution for any value of C.



Solution curves for $xy' + y = 0$
Figure 6.1

Geometrically, the general solution of a first-order differential equation represents a family of curves known as **solution curves,** one for each value assigned to the arbitrary constant. For instance, you can verify that every function of the form

$$y = \frac{C}{x} \qquad \text{General solution of } xy' + y = 0$$

is a solution of the differential equation

$$xy' + y = 0.$$

Figure 6.1 shows four of the solution curves corresponding to different values of C.

As discussed in Section 5.1, **particular solutions** of a differential equation are obtained from **initial conditions** that give the values of the dependent variable or one of its derivatives for particular values of the independent variable. The term "initial condition" stems from the fact that, often in problems involving time, the value of the dependent variable or one of its derivatives is known at the *initial* time $t = 0$. For instance, the second-order differential equation

$$s''(t) = -32$$

having the general solution

$$s(t) = -16t^2 + C_1t + C_2 \qquad \text{General solution of } s''(t) = -32$$

might have the following initial conditions.

$$s(0) = 80, \quad s'(0) = 64 \qquad \text{Initial conditions}$$

In this case, the initial conditions yield the particular solution

$$s(t) = -16t^2 + 64t + 80. \qquad \text{Particular solution}$$

EXAMPLE 2 Finding a Particular Solution

See LarsonCalculus.com for an interactive version of this type of example.

For the differential equation

$$xy' - 3y = 0$$

verify that $y = Cx^3$ is a solution. Then find the particular solution determined by the initial condition $y = 2$ when $x = -3$.

Solution You know that $y = Cx^3$ is a solution because $y' = 3Cx^2$ and

$$xy' - 3y = x(3Cx^2) - 3(Cx^3) = 0.$$

Furthermore, the initial condition $y = 2$ when $x = -3$ yields

$$y = Cx^3 \qquad \text{General solution}$$

$$2 = C(-3)^3 \qquad \text{Substitute initial condition.}$$

$$-\frac{2}{27} = C \qquad \text{Solve for } C.$$

and you can conclude that the particular solution is

$$y = -\frac{2x^3}{27}. \qquad \text{Particular solution}$$

Try checking this solution by substituting for y and y' in the original differential equation.

Note that to determine a particular solution, the number of initial conditions must match the number of constants in the general solution.

Slope Fields

Solving a differential equation analytically can be difficult or even impossible. However, there is a graphical approach you can use to learn a lot about the solution of a differential equation. Consider a differential equation of the form

$$y' = F(x, y) \qquad \text{Differential equation}$$

where $F(x, y)$ is some expression in x and y. At each point (x, y) in the xy-plane where F is defined, the differential equation determines the slope $y' = F(x, y)$ of the solution at that point. If you draw short line segments with slope $F(x, y)$ at selected points (x, y) in the domain of F, then these line segments form a **slope field,** or a *direction field*, for the differential equation $y' = F(x, y)$. Each line segment has the same slope as the solution curve through that point. A slope field shows the general shape of all the solutions and can be helpful in getting a visual perspective of the directions of the solutions of a differential equation.

EXAMPLE 3 Sketching a Slope Field

Sketch a slope field for the differential equation $y' = x - y$ for the points $(-1, 1)$, $(0, 1)$, and $(1, 1)$.

Solution The slope of the solution curve at any point (x, y) is

$$F(x, y) = x - y. \qquad \text{Slope at } (x, y)$$

So, the slope at each point can be found as shown.

Slope at $(-1, 1)$: $y' = -1 - 1 = -2$

Slope at $(0, 1)$: $y' = 0 - 1 = -1$

Slope at $(1, 1)$: $y' = 1 - 1 = 0$

Draw short line segments at the three points with their respective slopes, as shown in Figure 6.2.

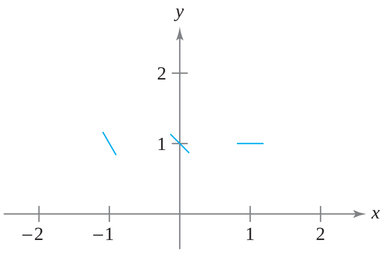
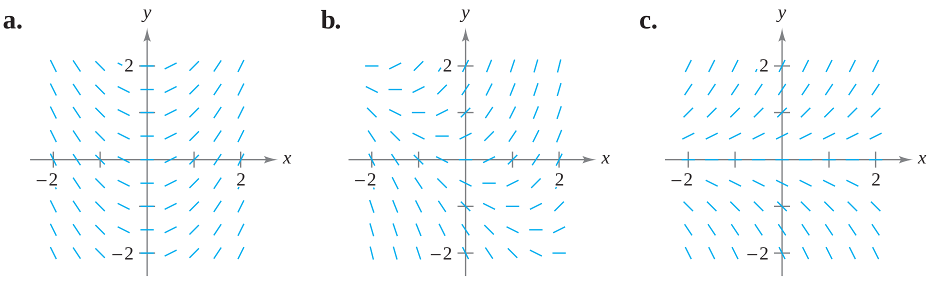


Figure 6.2

EXAMPLE 4 Identifying Slope Fields for Differential Equations

Match each slope field with its differential equation.



i. $y' = x + y$ **ii.** $y' = x$ **iii.** $y' = y$

Solution

a. You can see that the slope at any point along the y-axis is 0. The only equation that satisfies this condition is $y' = x$. So, the graph matches equation (ii).

b. You can see that the slope at the point $(1, -1)$ is 0. The only equation that satisfies this condition is $y' = x + y$. So, the graph matches equation (i).

c. You can see that the slope at any point along the x-axis is 0. The only equation that satisfies this condition is $y' = y$. So, the graph matches equation (iii).

A solution curve of a differential equation $y' = F(x, y)$ is simply a curve in the xy-plane whose tangent line at each point (x, y) has slope equal to $F(x, y)$. This is illustrated in Example 5.

EXAMPLE 5 Sketching a Solution Using a Slope Field

Sketch a slope field for the differential equation

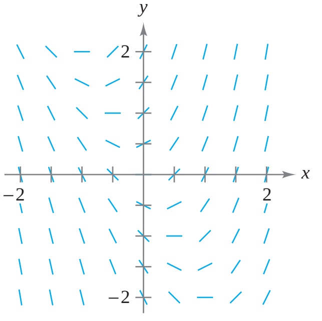
$$y' = 2x + y.$$

Use the slope field to sketch the solution that passes through the point $(1, 1)$.

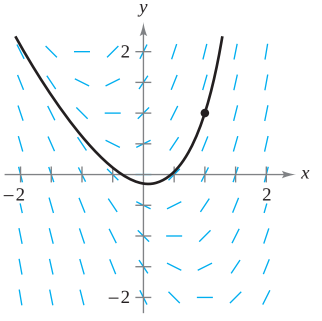
Solution Make a table showing the slopes at several points. The table shown is a small sample. The slopes at many other points should be calculated to get a representative slope field.

x	-2	-2	-1	-1	0	0	1	1	2	2
y	-1	1	-1	1	-1	1	-1	1	-1	1
$y' = 2x + y$	-5	-3	-3	-1	-1	1	1	3	3	5

Next, draw line segments at the points with their respective slopes, as shown in Figure 6.3.



Slope field for $y' = 2x + y$
Figure 6.3

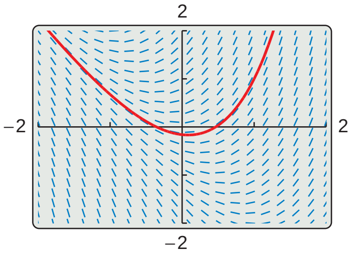


Particular solution for $y' = 2x + y$ passing through $(1, 1)$
Figure 6.4

After the slope field is drawn, start at the initial point $(1, 1)$ and move to the right in the direction of the line segment. Continue to draw the solution curve so that it moves parallel to the nearby line segments. Do the same to the left of $(1, 1)$. The resulting solution is shown in Figure 6.4.

In Example 5, note that the slope field shows that y' increases to infinity as x increases.

TECHNOLOGY Drawing a slope field by hand is tedious. In practice, slope fields are usually drawn using a graphing utility. If you have access to a graphing utility that can graph slope fields, try graphing the slope field for the differential equation in Example 5. One example of a slope field drawn by a graphing utility is shown at the right.



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Euler's Method

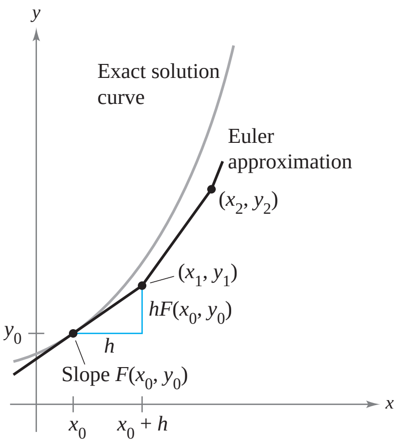


Figure 6.5

Euler's Method is a numerical approach to approximating the particular solution of the differential equation

$$y' = F(x, y)$$

that passes through the point (x_0, y_0). From the given information, you know that the graph of the solution passes through the point (x_0, y_0) and has a slope of $F(x_0, y_0)$ at this point. This gives you a "starting point" for approximating the solution.

From this starting point, you can proceed in the direction indicated by the slope. Using a small step h, move along the tangent line until you arrive at the point (x_1, y_1), where

$$x_1 = x_0 + h \quad \text{and} \quad y_1 = y_0 + hF(x_0, y_0)$$

as shown in Figure 6.5. Then, using (x_1, y_1) as a new starting point, you can repeat the process to obtain a second point (x_2, y_2). The values of x_i and y_i are shown below.

$$\begin{aligned} x_1 &= x_0 + h & y_1 &= y_0 + hF(x_0, y_0) \\ x_2 &= x_1 + h & y_2 &= y_1 + hF(x_1, y_1) \\ &\vdots & &\vdots \\ x_n &= x_{n-1} + h & y_n &= y_{n-1} + hF(x_{n-1}, y_{n-1}) \end{aligned}$$

When using this method, note that you can obtain better approximations of the exact solution by choosing smaller and smaller step sizes.

EXAMPLE 6 Approximating a Solution Using Euler's Method

Use Euler's Method to approximate the particular solution of the differential equation

$$y' = x - y$$

passing through the point $(0, 1)$. Use a step of $h = 0.1$.

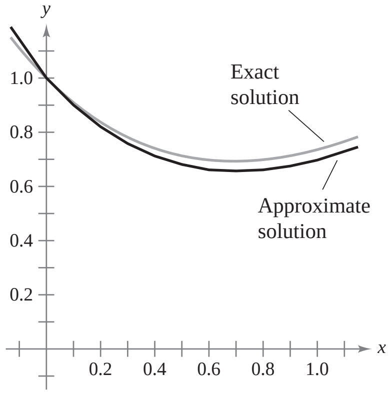


Figure 6.6

Solution Using $h = 0.1$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = x - y$, you have

$$x_0 = 0, \quad x_1 = 0.1, \quad x_2 = 0.2, \quad x_3 = 0.3,$$

and the first three approximations are

$$\begin{aligned} y_1 &= y_0 + hF(x_0, y_0) = 1 + (0.1)(0 - 1) = 0.9 \\ y_2 &= y_1 + hF(x_1, y_1) = 0.9 + (0.1)(0.1 - 0.9) = 0.82 \\ y_3 &= y_2 + hF(x_2, y_2) = 0.82 + (0.1)(0.2 - 0.82) = 0.758. \end{aligned}$$

The first ten approximations are shown in the table. You can plot these values to see a graph of the approximate solution, as shown in Figure 6.6.

n	0	1	2	3	4	5	6	7	8	9	10
x_n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
y_n	1	0.900	0.820	0.758	0.712	0.681	0.663	0.657	0.661	0.675	0.697

For the differential equation in Example 6, you can verify the exact solution to be the equation

$$y = x - 1 + 2e^{-x}.$$

Figure 6.6 compares this exact solution with the approximate solution obtained in Example 6.

6.1 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Verifying a Solution **In Exercises 1–8, verify the solution of the differential equation.**

Solution	Differential Equation
1. $y = Ce^{4x}$	$y' = 4y$
2. $y = e^{-2x}$	$3y' + 5y = -e^{-2x}$
3. $x^2 + y^2 = Cy$	$y' = \dfrac{2xy}{x^2 - y^2}$
4. $y^2 - 2\ln y = x^2$	$\dfrac{dy}{dx} = \dfrac{xy}{y^2 - 1}$
5. $y = C_1 \sin x - C_2 \cos x$	$y'' + y = 0$
6. $y = C_1 e^{-x}\cos x + C_2 e^{-x} \sin x$	$y'' + 2y' + 2y = 0$
7. $y = -\cos x \ln\lvert\sec x + \tan x\rvert$	$y'' + y = \tan x$
8. $y = \frac{2}{5}(e^{-4x} + e^x)$	$y'' + 4y' = 2e^x$

Verifying a Particular Solution **In Exercises 9–12, verify the particular solution of the differential equation.**

Solution	Differential Equation and Initial Condition
9. $y = \sin x \cos x - \cos^2 x$	$2y + y' = 2\sin(2x) - 1$ $y\left(\dfrac{\pi}{4}\right) = 0$
10. $y = 6x - 4\sin x + 1$	$y' = 6 - 4\cos x$ $y(0) = 1$
11. $y = 4e^{-6x^2}$	$y' = -12xy$ $y(0) = 4$
12. $y = e^{-\cos x}$	$y' = y \sin x$ $y\left(\dfrac{\pi}{2}\right) = 1$

Determining a Solution **In Exercises 13–20, determine whether the function is a solution of the differential equation $y^{(4)} - 16y = 0$.**

13. $y = 3\cos x$

14. $y = 2\sin x$

15. $y = 3\cos 2x$

16. $y = 3\sin 2x$

17. $y = e^{-2x}$

18. $y = 5\ln x$

19. $y = C_1 e^{2x} + C_2 e^{-2x} + C_3 \sin 2x + C_4 \cos 2x$

20. $y = 3e^{2x} - 4\sin 2x$

Determining a Solution **In Exercises 21–28, determine whether the function is a solution of the differential equation $xy' - 2y = x^3 e^x$.**

21. $y = x^2$

22. $y = x^3$

23. $y = x^2 e^x$

24. $y = x^2(2 + e^x)$

25. $y = \sin x$

26. $y = \cos x$

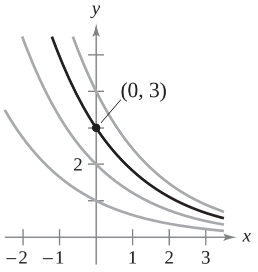
27. $y = \ln x$

28. $y = x^2 e^x - 5x^2$

Finding a Particular Solution **In Exercises 29–32, some of the curves corresponding to different values of C in the general solution of the differential equation are shown in the graph. Find the particular solution that passes through the point shown on the graph.**

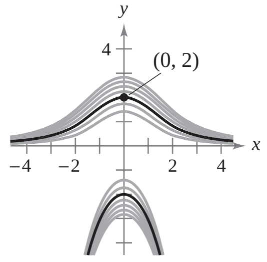
29. $y^2 = Ce^{-x/2}$

$2y' + y = 0$



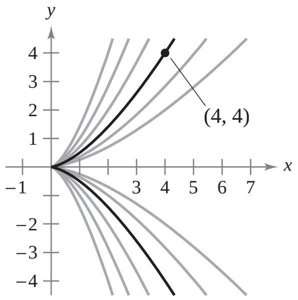
30. $y(x^2 + y) = C$

$2xy + (x^2 + 2y)y' = 0$



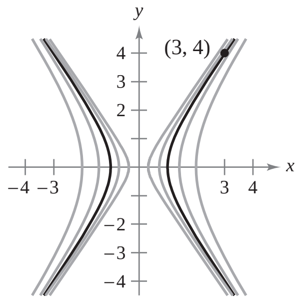
31. $y^2 = Cx^3$

$2xy' - 3y = 0$



32. $2x^2 - y^2 = C$

$yy' - 2x = 0$



Graphs of Particular Solutions **In Exercises 33 and 34, the general solution of the differential equation is given. Use a graphing utility to graph the particular solutions for the given values of C.**

33. $4yy' - x = 0$

$4y^2 - x^2 = C$

$C = 0, C = \pm 1, C = \pm 4$

34. $yy' + x = 0$

$x^2 + y^2 = C$

$C = 0, C = 1, C = 4$

Finding a Particular Solution **In Exercises 35–40, verify that the general solution satisfies the differential equation. Then find the particular solution that satisfies the initial condition(s).**

35. $y = Ce^{-2x}$

$y' + 2y = 0$

$y = 3$ when $x = 0$

36. $3x^2 + 2y^2 = C$

$3x + 2yy' = 0$

$y = 3$ when $x = 1$

37. $y = C_1 \sin 3x + C_2 \cos 3x$

$y'' + 9y = 0$

$y = 2$ when $x = \dfrac{\pi}{6}$

$y' = 1$ when $x = \dfrac{\pi}{6}$

38. $y = C_1 + C_2 \ln x$

$xy'' + y' = 0$

$y = 0$ when $x = 2$

$y' = \dfrac{1}{2}$ when $x = 2$

39. $y = C_1x + C_2x^3$

$x^2y'' - 3xy' + 3y = 0$

$y = 0$ when $x = 2$

$y' = 4$ when $x = 2$

40. $y = e^{2x/3}(C_1 + C_2x)$

$9y'' - 12y' + 4y = 0$

$y = 4$ when $x = 0$

$y = 0$ when $x = 3$

Finding a General Solution **In Exercises 41–52, use integration to find a general solution of the differential equation.**

41. $\dfrac{dy}{dx} = 6x^2$ **42.** $\dfrac{dy}{dx} = 10x^4 - 2x^3$

43. $\dfrac{dy}{dx} = \dfrac{x}{1 + x^2}$ **44.** $\dfrac{dy}{dx} = \dfrac{e^x}{4 + e^x}$

45. $\dfrac{dy}{dx} = \dfrac{x - 2}{x}$ **46.** $\dfrac{dy}{dx} = x \cos x^2$

47. $\dfrac{dy}{dx} = \sin 2x$ **48.** $\dfrac{dy}{dx} = \tan^2 x$

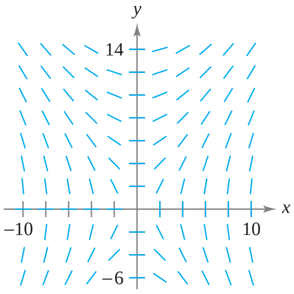
49. $\dfrac{dy}{dx} = x\sqrt{x - 6}$ **50.** $\dfrac{dy}{dx} = 2x\sqrt{4x^2 + 1}$

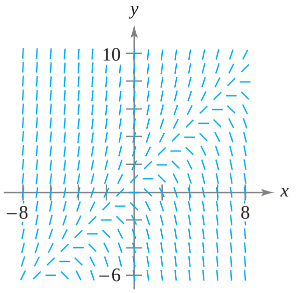
51. $\dfrac{dy}{dx} = xe^{x^2}$ **52.** $\dfrac{dy}{dx} = 5e^{-x/2}$

Slope Field **In Exercises 53–56, a differential equation and its slope field are given. Complete the table by determining the slopes (if possible) in the slope field at the given points.**

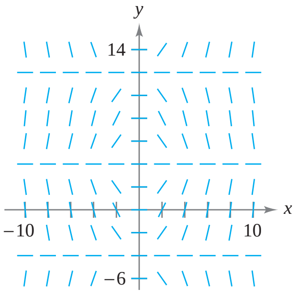
x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx						

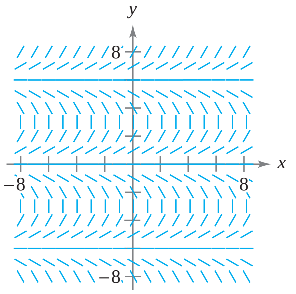
53. $\dfrac{dy}{dx} = \dfrac{2x}{y}$ **54.** $\dfrac{dy}{dx} = y - x$



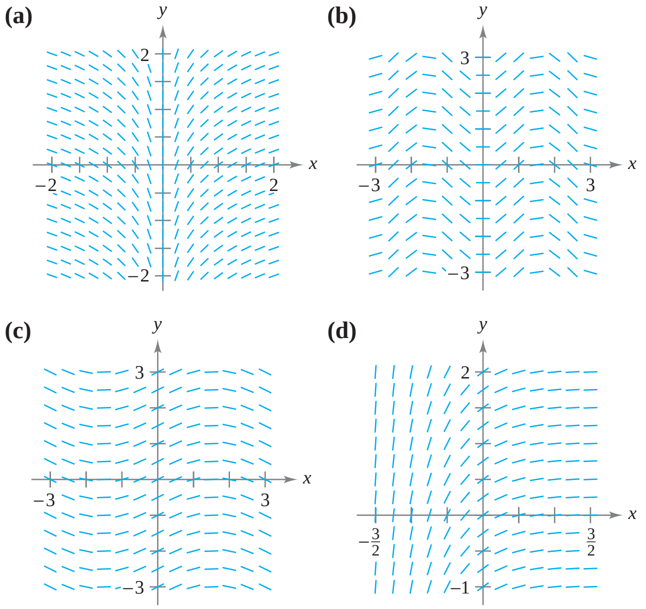


55. $\dfrac{dy}{dx} = x \cos \dfrac{\pi y}{8}$ **56.** $\dfrac{dy}{dx} = \tan\left(\dfrac{\pi y}{6}\right)$





Matching **In Exercises 57–60, match the differential equation with its slope field. [The slope fields are labeled (a), (b), (c), and (d).]**



57. $\dfrac{dy}{dx} = \sin(2x)$ **58.** $\dfrac{dy}{dx} = \dfrac{1}{2}\cos x$

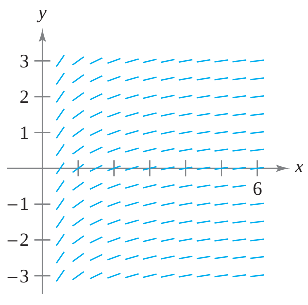
59. $\dfrac{dy}{dx} = e^{-2x}$ **60.** $\dfrac{dy}{dx} = \dfrac{1}{x}$

Slope Field **In Exercises 61–64, (a) sketch the slope field for the differential equation, (b) use the slope field to sketch the solution that passes through the given point, and (c) discuss the graph of the solution as $x \to \infty$ and $x \to -\infty$. Use a graphing utility to verify your results. To print a blank graph, go to *MathGraphs.com*.**

61. $y' = 3 - x$, $(4, 2)$ **62.** $y' = \frac{1}{3}x^2 - \frac{1}{2}x$, $(1, 1)$

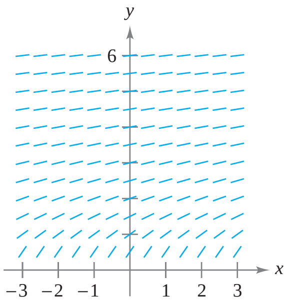
63. $y' = y - 4x$, $(2, 2)$ **64.** $y' = y + xy$, $(0, -4)$

65. Slope Field Use the slope field for the differential equation $y' = 1/x$, where $x > 0$, to sketch the graph of the solution that satisfies each given initial condition. Then make a conjecture about the behavior of a particular solution of $y' = 1/x$ as $x \to \infty$. To print an enlarged copy of the graph, go to *MathGraphs.com.*



(a) $(1, 0)$ (b) $(2, -1)$

66. Slope Field Use the slope field for the differential equation $y' = 1/y$, where $y > 0$, to sketch the graph of the solution that satisfies each given initial condition. Then make a conjecture about the behavior of a particular solution of $y' = 1/y$ as $x \to \infty$. To print an enlarged copy of the graph, go to *MathGraphs.com*.



(a) $(0, 1)$ (b) $(1, 1)$

Slope Field In Exercises 67–72, use a computer algebra system to (a) graph the slope field for the differential equation and (b) graph the solution satisfying the specified initial condition.

67. $\dfrac{dy}{dx} = 0.25y, \quad y(0) = 4$

68. $\dfrac{dy}{dx} = 4 - y, \quad y(0) = 6$

69. $\dfrac{dy}{dx} = 0.02y(10 - y), \quad y(0) = 2$

70. $\dfrac{dy}{dx} = 0.2x(2 - y), \quad y(0) = 9$

71. $\dfrac{dy}{dx} = 0.4y(3 - x), \quad y(0) = 1$

72. $\dfrac{dy}{dx} = \dfrac{1}{2}e^{-x/8} \sin \dfrac{\pi y}{4}, \quad y(0) = 2$

Euler's Method In Exercises 73–78, use Euler's Method to make a table of values for the approximate solution of the differential equation with the specified initial value. Use *n* steps of size *h*.

73. $y' = x + y, \quad y(0) = 2, \quad n = 10, \quad h = 0.1$

74. $y' = x + y, \quad y(0) = 2, \quad n = 20, \quad h = 0.05$

75. $y' = 3x - 2y, \quad y(0) = 3, \quad n = 10, \quad h = 0.05$

76. $y' = 0.5x(3 - y), \quad y(0) = 1, \quad n = 5, \quad h = 0.4$

77. $y' = e^{xy}, \quad y(0) = 1, \quad n = 10, \quad h = 0.1$

78. $y' = \cos x + \sin y, \quad y(0) = 5, \quad n = 10, \quad h = 0.1$

Euler's Method In Exercises 79–81, complete the table using the exact solution of the differential equation and two approximations obtained using Euler's Method to approximate the particular solution of the differential equation. Use $h = 0.2$ and $h = 0.1$, and compute each approximation to four decimal places.

x	0	0.2	0.4	0.6	0.8	1
$y(x)$ (exact)						
$y(x)$ $(h = 0.2)$						
$y(x)$ $(h = 0.1)$						

Table for 79–81

	Differential Equation	Initial Condition	Exact Solution
79.	$\dfrac{dy}{dx} = y$	$(0, 3)$	$y = 3e^x$
80.	$\dfrac{dy}{dx} = \dfrac{2x}{y}$	$(0, 2)$	$y = \sqrt{2x^2 + 4}$
81.	$\dfrac{dy}{dx} = y + \cos(x)$	$(0, 0)$	$y = \dfrac{1}{2}(\sin x - \cos x + e^x)$

82. Euler's Method Compare the values of the approximations in Exercises 79–81 with the values given by the exact solution. How does the error change as h increases?

83. Temperature At time $t = 0$ minutes, the temperature of an object is 140°F. The temperature of the object is changing at the rate given by the differential equation

$$\frac{dy}{dt} = -\frac{1}{2}(y - 72).$$

(a) Use a graphing utility and Euler's Method to approximate the particular solutions of this differential equation at $t = 1$, 2, and 3. Use a step size of $h = 0.1$. (A graphing utility program for Euler's Method is available at the website *college.hmco.com*.)

(b) Compare your results with the exact solution

$$y = 72 + 68e^{-t/2}.$$

(c) Repeat parts (a) and (b) using a step size of $h = 0.05$. Compare the results.

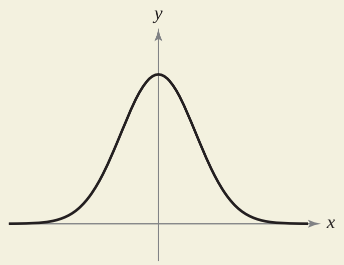
84. HOW DO YOU SEE IT? The graph shows a solution of one of the following differential equations. Determine the correct equation. Explain your reasoning.

(a) $y' = xy$

(b) $y' = \dfrac{4x}{y}$

(c) $y' = -4xy$

(d) $y' = 4 - xy$



WRITING ABOUT CONCEPTS

85. General and Particular Solutions In your own words, describe the difference between a general solution of a differential equation and a particular solution.

86. Slope Field Explain how to interpret a slope field.

87. Euler's Method Describe how to use Euler's Method to approximate a particular solution of a differential equation.

88. Finding Values It is known that $y = Ce^{kx}$ is a solution of the differential equation $y' = 0.07y$. Is it possible to determine C or k from the information given? If so, find its value.

True or False? **In Exercises 89–92, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

89. If $y = f(x)$ is a solution of a first-order differential equation, then $y = f(x) + C$ is also a solution.

90. The general solution of a differential equation is $y = -4.9x^2 + C_1x + C_2$. To find a particular solution, you must be given two initial conditions.

91. Slope fields represent the general solutions of differential equations.

92. A slope field shows that the slope at the point $(1, 1)$ is 6. This slope field represents the family of solutions for the differential equation $y' = 4x + 2y$.

93. Errors and Euler's Method The exact solution of the differential equation

$$\frac{dy}{dx} = -2y$$

where $y(0) = 4$, is $y = 4e^{-2x}$.

(a) Use a graphing utility to complete the table, where y is the exact value of the solution, y_1 is the approximate solution using Euler's Method with $h = 0.1$, y_2 is the approximate solution using Euler's Method with $h = 0.2$, e_1 is the absolute error $|y - y_1|$, e_2 is the absolute error $|y - y_2|$, and r is the ratio e_1/e_2.

x	0	0.2	0.4	0.6	0.8	1
y						
y_1						
y_2						
e_1						
e_2						
r						

(b) What can you conclude about the ratio r as h changes?

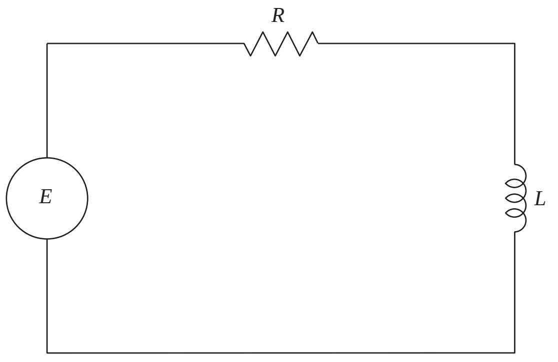
(c) Predict the absolute error when $h = 0.05$.

94. Errors and Euler's Method Repeat Exercise 93 for which the exact solution of the differential equation

$$\frac{dy}{dx} = x - y$$

where $y(0) = 1$, is $y = x - 1 + 2e^{-x}$.

95. Electric Circuit The diagram shows a simple electric circuit consisting of a power source, a resistor, and an inductor.



A model of the current I, in amperes (A), at time t is given by the first-order differential equation

$$L\frac{dI}{dt} + RI = E(t)$$

where $E(t)$ is the voltage (V) produced by the power source, R is the resistance, in ohms (Ω), and L is the inductance, in henrys (H). Suppose the electric circuit consists of a 24-V power source, a 12-Ω resistor, and a 4-H inductor.

(a) Sketch a slope field for the differential equation.

(b) What is the limiting value of the current? Explain.

96. Think About It It is known that $y = e^{kt}$ is a solution of the differential equation $y'' - 16y = 0$. Find the values of k.

97. Think About It It is known that $y = A \sin \omega t$ is a solution of the differential equation $y'' + 16y = 0$. Find the values of ω.

PUTNAM EXAM CHALLENGE

98. Let f be a twice-differentiable real-valued function satisfying

$$f(x) + f''(x) = -xg(x)f'(x)$$

where $g(x) \geq 0$ for all real x. Prove that $|f(x)|$ is bounded.

99. Prove that if the family of integral curves of the differential equation

$$\frac{dy}{dx} + p(x)y = q(x), \quad p(x) \cdot q(x) \neq 0$$

is cut by the line $x = k$, the tangents at the points of intersection are concurrent.

These problems were composed by the Committee on the Putnam Prize Competition. © The Mathematical Association of America. All rights reserved.

6.2 Differential Equations: Growth and Decay

- **Use separation of variables to solve a simple differential equation.**
- **Use exponential functions to model growth and decay in applied problems.**

Differential Equations

In Section 6.1, you learned to analyze the solutions visually of differential equations using slope fields and to approximate solutions numerically using Euler's Method. Analytically, you have learned to solve only two types of differential equations—those of the forms $y' = f(x)$ and $y'' = f(x)$. In this section, you will learn how to solve a more general type of differential equation. The strategy is to rewrite the equation so that each variable occurs on only one side of the equation. This strategy is called *separation of variables*. (You will study this strategy in detail in Section 6.3.)

EXAMPLE 1 Solving a Differential Equation

$$y' = \frac{2x}{y} \qquad \text{Original equation}$$

$$yy' = 2x \qquad \text{Multiply both sides by } y.$$

$$\int yy'\,dx = \int 2x\,dx \qquad \text{Integrate with respect to } x.$$

$$\int y\,dy = \int 2x\,dx \qquad dy = y'\,dx$$

$$\frac{1}{2}y^2 = x^2 + C_1 \qquad \text{Apply Power Rule.}$$

$$y^2 - 2x^2 = C \qquad \text{Rewrite, letting } C = 2C_1.$$

So, the general solution is $y^2 - 2x^2 = C$.

REMARK You can use implicit differentiation to check the solution in Example 1.

When you integrate both sides of the equation in Example 1, you do not need to add a constant of integration to both sides. When you do, you still obtain the same result.

$$\int y\,dy = \int 2x\,dx$$

$$\frac{1}{2}y^2 + C_2 = x^2 + C_3$$

$$\frac{1}{2}y^2 = x^2 + (C_3 - C_2)$$

$$\frac{1}{2}y^2 = x^2 + C_1$$

Some people prefer to use Leibniz notation and differentials when applying separation of variables. The solution to Example 1 is shown below using this notation.

$$\frac{dy}{dx} = \frac{2x}{y}$$

$$y\,dy = 2x\,dx$$

$$\int y\,dy = \int 2x\,dx$$

$$\frac{1}{2}y^2 = x^2 + C_1$$

$$y^2 - 2x^2 = C$$

Exploration

In Example 1, the general solution of the differential equation is

$$y^2 - 2x^2 = C.$$

Use a graphing utility to sketch the particular solutions for $C = \pm 2$, $C = \pm 1$, and $C = 0$. Describe the solutions graphically. Is the following statement true of each solution?

The slope of the graph at the point (x, y) is equal to twice the ratio of x and y.

Explain your reasoning. Are all curves for which this statement is true represented by the general solution?

Growth and Decay Models

In many applications, the rate of change of a variable y is proportional to the value of y. When y is a function of time t, the proportion can be written as shown.

Rate of change of y — is — proportional to y.

$$\frac{dy}{dt} = ky$$

The general solution of this differential equation is given in the next theorem.

> **THEOREM 6.1 Exponential Growth and Decay Model**
>
> If y is a differentiable function of t such that $y > 0$ and $y' = ky$ for some constant k, then
>
> $$y = Ce^{kt}$$
>
> where C is the **initial value** of y, and k is the **proportionality constant. Exponential growth** occurs when $k > 0$, and **exponential decay** occurs when $k < 0$.

Proof

$y' = ky$	Write original equation.
$\dfrac{y'}{y} = k$	Separate variables.
$\displaystyle\int \frac{y'}{y}\,dt = \int k\,dt$	Integrate with respect to t.
$\displaystyle\int \frac{1}{y}\,dy = \int k\,dt$	$dy = y'\,dt$
$\ln y = kt + C_1$	Find antiderivative of each side.
$y = e^{kt}e^{C_1}$	Solve for y.
$y = Ce^{kt}$	Let $C = e^{C_1}$.

So, all solutions of $y' = ky$ are of the form $y = Ce^{kt}$. Remember that you can differentiate the function $y = Ce^{kt}$ with respect to t to verify that $y' = ky$.

See LarsonCalculus.com for Bruce Edwards's video of this proof.

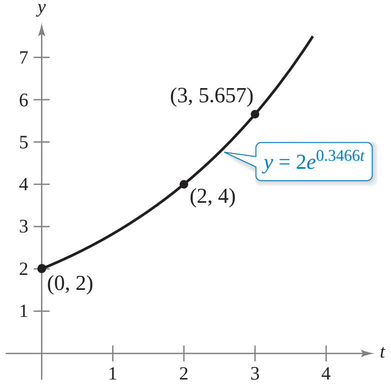
EXAMPLE 2 Using an Exponential Growth Model

The rate of change of y is proportional to y. When $t = 0$, $y = 2$, and when $t = 2$, $y = 4$. What is the value of y when $t = 3$?

Solution Because $y' = ky$, you know that y and t are related by the equation $y = Ce^{kt}$. You can find the values of the constants C and k by applying the initial conditions.

$2 = Ce^0$ ⇨	$C = 2$	When $t = 0$, $y = 2$.
$4 = 2e^{2k}$ ⇨	$k = \frac{1}{2}\ln 2 \approx 0.3466$	When $t = 2$, $y = 4$.

So, the model is $y = 2e^{0.3466t}$. When $t = 3$, the value of y is $2e^{0.3466(3)} \approx 5.657$ (see Figure 6.7).



If the rate of change of y is proportional to y, then y follows an exponential model.
Figure 6.7

Using logarithmic properties, the value of k in Example 2 can also be written as $\ln\sqrt{2}$. So, the model becomes $y = 2e^{(\ln\sqrt{2})t}$, which can be rewritten as $y = 2(\sqrt{2})^t$.

TECHNOLOGY Most graphing utilities have curve-fitting capabilities that can be used to find models that represent data. Use the *exponential regression* feature of a graphing utility and the information in Example 2 to find a model for the data. How does your model compare with the given model?

Radioactive decay is measured in terms of *half-life*—the number of years required for half of the atoms in a sample of radioactive material to decay. The rate of decay is proportional to the amount present. The half-lives of some common radioactive isotopes are listed below.

Uranium (^{238}U)	4,470,000,000 years
Plutonium (^{239}Pu)	24,100 years
Carbon (^{14}C)	5715 years
Radium (^{226}Ra)	1599 years
Einsteinium (^{254}Es)	276 days
Radon (^{222}Rn)	3.82 days
Nobelium (^{257}No)	25 seconds

EXAMPLE 3 Radioactive Decay

The Fukushima Daiichi nuclear disaster occurred after an earthquake and tsunami. Several of the reactors at the plant experienced full meltdowns.

Ten grams of the plutonium isotope ^{239}Pu were released in a nuclear accident. How long will it take for the 10 grams to decay to 1 gram?

Solution Let y represent the mass (in grams) of the plutonium. Because the rate of decay is proportional to y, you know that

$$y = Ce^{kt}$$

where t is the time in years. To find the values of the constants C and k, apply the initial conditions. Using the fact that $y = 10$ when $t = 0$, you can write

$$10 = Ce^{k(0)} \quad \Rightarrow \quad 10 = Ce^0$$

which implies that $C = 10$. Next, using the fact that the half-life of ^{239}Pu is 24,100 years, you have $y = 10/2 = 5$ when $t = 24,100$, so you can write

$$\begin{aligned} 5 &= 10e^{k(24,100)} \\ \frac{1}{2} &= e^{24,100k} \\ \frac{1}{24,100} \ln \frac{1}{2} &= k \\ -0.000028761 &\approx k. \end{aligned}$$

So, the model is

$$y = 10e^{-0.000028761t}. \qquad \text{Half-life model}$$

To find the time it would take for 10 grams to decay to 1 gram, you can solve for t in the equation

$$1 = 10e^{-0.000028761t}.$$

The solution is approximately 80,059 years.

REMARK The exponential decay model in Example 3 could also be written as $y = 10\left(\frac{1}{2}\right)^{t/24,100}$. This model is much easier to derive, but for some applications it is not as convenient to use.

From Example 3, notice that in an exponential growth or decay problem, it is easy to solve for C when you are given the value of y at $t = 0$. The next example demonstrates a procedure for solving for C and k when you do not know the value of y at $t = 0$.

KIMIMASA MAYAMA/EPA/Newscom

EXAMPLE 4 Population Growth

See LarsonCalculus.com for an interactive version of this type of example.

An experimental population of fruit flies increases according to the law of exponential growth. There were 100 flies after the second day of the experiment and 300 flies after the fourth day. Approximately how many flies were in the original population?

Solution Let $y = Ce^{kt}$ be the number of flies at time t, where t is measured in days. Note that y is continuous, whereas the number of flies is discrete. Because $y = 100$ when $t = 2$ and $y = 300$ when $t = 4$, you can write

$$100 = Ce^{2k} \quad \text{and} \quad 300 = Ce^{4k}.$$

From the first equation, you know that

$$C = 100e^{-2k}.$$

Substituting this value into the second equation produces the following.

$$\begin{aligned} 300 &= 100e^{-2k}e^{4k} \\ 300 &= 100e^{2k} \\ 3 &= e^{2k} \\ \ln 3 &= 2k \\ \frac{1}{2}\ln 3 &= k \\ 0.5493 &\approx k \end{aligned}$$

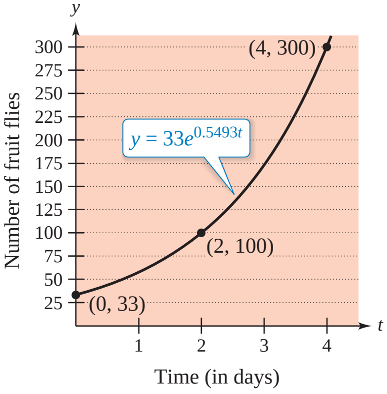


Figure 6.8

So, the exponential growth model is

$$y = Ce^{0.5493t}.$$

To solve for C, reapply the condition $y = 100$ when $t = 2$ and obtain

$$\begin{aligned} 100 &= Ce^{0.5493(2)} \\ C &= 100e^{-1.0986} \\ C &\approx 33. \end{aligned}$$

So, the original population (when $t = 0$) consisted of approximately $y = C = 33$ flies, as shown in Figure 6.8.

EXAMPLE 5 Declining Sales

Four months after it stops advertising, a manufacturing company notices that its sales have dropped from 100,000 units per month to 80,000 units per month. The sales follow an exponential pattern of decline. What will the sales be after another 2 months?

Solution Use the exponential decay model $y = Ce^{kt}$, where t is measured in months. From the initial condition $(t = 0)$, you know that $C = 100{,}000$. Moreover, because $y = 80{,}000$ when $t = 4$, you have

$$\begin{aligned} 80{,}000 &= 100{,}000e^{4k} \\ 0.8 &= e^{4k} \\ \ln(0.8) &= 4k \\ -0.0558 &\approx k. \end{aligned}$$

So, after 2 more months $(t = 6)$, you can expect the monthly sales rate to be

$$\begin{aligned} y &= 100{,}000e^{-0.0558(6)} \\ &\approx 71{,}500 \text{ units.} \end{aligned}$$

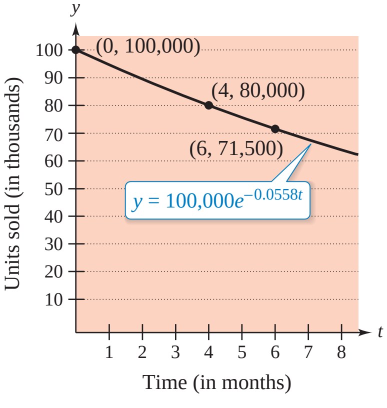


Figure 6.9

See Figure 6.9.

In Examples 2 through 5, you did not actually have to solve the differential equation $y' = ky$. (This was done once in the proof of Theorem 6.1.) The next example demonstrates a problem whose solution involves the separation of variables technique. The example concerns **Newton's Law of Cooling,** which states that the rate of change in the temperature of an object is proportional to the difference between the object's temperature and the temperature of the surrounding medium.

EXAMPLE 6 Newton's Law of Cooling

Let y represent the temperature (in °F) of an object in a room whose temperature is kept at a constant 60°. The object cools from 100° to 90° in 10 minutes. How much longer will it take for the temperature of the object to decrease to 80°?

Solution From Newton's Law of Cooling, you know that the rate of change in y is proportional to the difference between y and 60. This can be written as

$$y' = k(y - 60), \quad 80 \le y \le 100.$$

To solve this differential equation, use separation of variables, as shown.

$$\frac{dy}{dt} = k(y - 60) \qquad \text{Differential equation}$$

$$\left(\frac{1}{y - 60}\right) dy = k\,dt \qquad \text{Separate variables.}$$

$$\int \frac{1}{y - 60}\,dy = \int k\,dt \qquad \text{Integrate each side.}$$

$$\ln|y - 60| = kt + C_1 \qquad \text{Find antiderivative of each side.}$$

Because $y > 60$, $|y - 60| = y - 60$, and you can omit the absolute value signs. Using exponential notation, you have

$$y - 60 = e^{kt + C_1}$$

$$y = 60 + Ce^{kt}. \qquad C = e^{C_1}$$

Using $y = 100$ when $t = 0$, you obtain

$$100 = 60 + Ce^{k(0)} = 60 + C$$

which implies that $C = 40$. Because $y = 90$ when $t = 10$,

$$90 = 60 + 40e^{k(10)}$$

$$30 = 40e^{10k}$$

$$k = \frac{1}{10} \ln \frac{3}{4}.$$

So, $k \approx -0.02877$ and the model is

$$y = 60 + 40e^{-0.02877t}. \qquad \text{Cooling model}$$

When $y = 80$, you obtain

$$80 = 60 + 40e^{-0.02877t}$$

$$20 = 40e^{-0.02877t}$$

$$\frac{1}{2} = e^{-0.02877t}$$

$$\ln \frac{1}{2} = -0.02877t$$

$$t \approx 24.09 \text{ minutes.}$$

So, it will require about 14.09 *more* minutes for the object to cool to a temperature of 80° (see Figure 6.10).

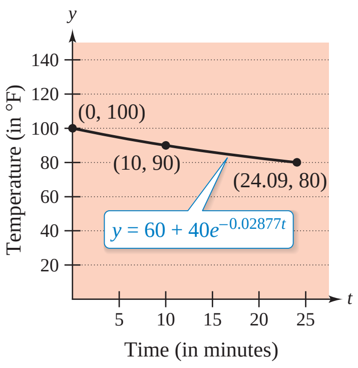


Figure 6.10

6.2 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Solving a Differential Equation **In Exercises 1–10, solve the differential equation.**

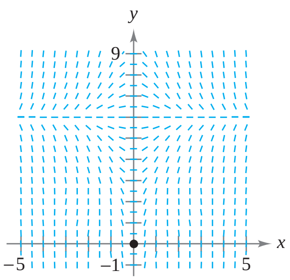
1. $\dfrac{dy}{dx} = x + 3$
2. $\dfrac{dy}{dx} = 5 - 8x$
3. $\dfrac{dy}{dx} = y + 3$
4. $\dfrac{dy}{dx} = 6 - y$
5. $y' = \dfrac{5x}{y}$
6. $y' = -\dfrac{\sqrt{x}}{4y}$
7. $y' = \sqrt{x}\,y$
8. $y' = x(1 + y)$
9. $(1 + x^2)y' - 2xy = 0$
10. $xy + y' = 100x$

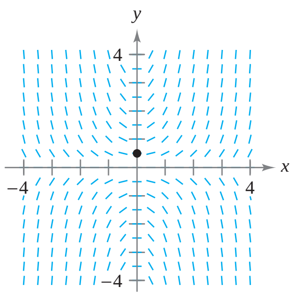
Writing and Solving a Differential Equation **In Exercises 11 and 12, write and solve the differential equation that models the verbal statement.**

11. The rate of change of Q with respect to t is inversely proportional to the square of t.
12. The rate of change of P with respect to t is proportional to $25 - t$.

Slope Field **In Exercises 13 and 14, a differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Use integration to find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketch in part (a). To print an enlarged copy of the graph, go to *MathGraphs.com*.**

13. $\dfrac{dy}{dx} = x(6 - y),\quad (0, 0)$
14. $\dfrac{dy}{dx} = xy,\quad \left(0, \dfrac{1}{2}\right)$





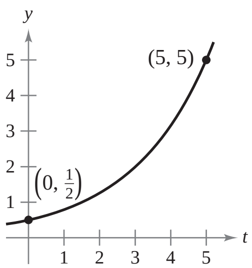
Finding a Particular Solution **In Exercises 15–18, find the function $y = f(t)$ passing through the point $(0, 10)$ with the given first derivative. Use a graphing utility to graph the solution.**

15. $\dfrac{dy}{dt} = \dfrac{1}{2}t$
16. $\dfrac{dy}{dt} = -9\sqrt{t}$
17. $\dfrac{dy}{dt} = -\dfrac{1}{2}y$
18. $\dfrac{dy}{dt} = \dfrac{3}{4}y$

Writing and Solving a Differential Equation **In Exercises 19 and 20, write and solve the differential equation that models the verbal statement. Evaluate the solution at the specified value of the independent variable.**

19. The rate of change of N is proportional to N. When $t = 0$, $N = 250$, and when $t = 1$, $N = 400$. What is the value of N when $t = 4$?
20. The rate of change of P is proportional to P. When $t = 0$, $P = 5000$, and when $t = 1$, $P = 4750$. What is the value of P when $t = 5$?

Finding an Exponential Function **In Exercises 21–24, find the exponential function $y = Ce^{kt}$ that passes through the two given points.**

21.


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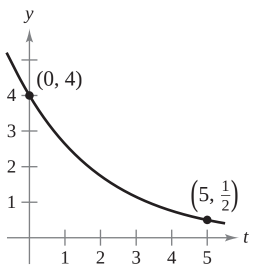

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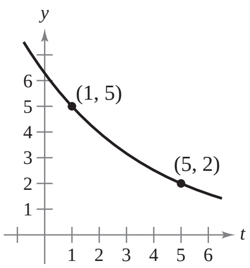

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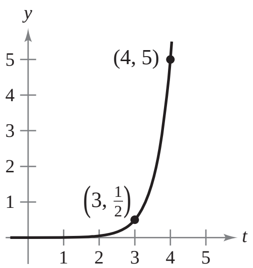


WRITING ABOUT CONCEPTS

25. **Describing Values** Describe what the values of C and k represent in the exponential growth and decay model, $y = Ce^{kt}$.
26. **Exponential Growth and Decay** Give the differential equation that models exponential growth and decay.

Increasing Function **In Exercises 27 and 28, determine the quadrants in which the solution of the differential equation is an increasing function. Explain. (Do not solve the differential equation.)**

27. $\dfrac{dy}{dx} = \dfrac{1}{2}xy$
28. $\dfrac{dy}{dx} = \dfrac{1}{2}x^2y$

Radioactive Decay **In Exercises 29–36, complete the table for the radioactive isotope.**

	Isotope	Half-life (in years)	Initial Quantity	Amount After 1000 Years	Amount After 10,000 Years
29.	^{226}Ra	1599	20 g		
30.	^{226}Ra	1599		1.5 g	
31.	^{226}Ra	1599			0.1 g
32.	^{14}C	5715			3 g
33.	^{14}C	5715	5 g		
34.	^{14}C	5715		1.6 g	
35.	^{239}Pu	24,100		2.1 g	
36.	^{239}Pu	24,100			0.4 g

37. Radioactive Decay Radioactive radium has a half-life of approximately 1599 years. What percent of a given amount remains after 100 years?

38. Carbon Dating Carbon-14 dating assumes that the carbon dioxide on Earth today has the same radioactive content as it did centuries ago. If this is true, the amount of ^{14}C absorbed by a tree that grew several centuries ago should be the same as the amount of ^{14}C absorbed by a tree growing today. A piece of ancient charcoal contains only 15% as much of the radioactive carbon as a piece of modern charcoal. How long ago was the tree burned to make the ancient charcoal? (The half-life of ^{14}C is 5715 years.)

Compound Interest **In Exercises 39–44, complete the table for a savings account in which interest is compounded continuously.**

	Initial Investment	Annual Rate	Time to Double	Amount After 10 Years
39.	\$4000	6%		
40.	\$18,000	$5\frac{1}{2}\%$		
41.	\$750		$7\frac{3}{4}$ yr	
42.	\$12,500		20 yr	
43.	\$500			\$1292.85
44.	\$6000			\$8950.95

Compound Interest **In Exercises 45–48, find the principal *P* that must be invested at rate *r*, compounded monthly, so that \$1,000,000 will be available for retirement in *t* years.**

45. $r = 7\frac{1}{2}\%, \quad t = 20$

46. $r = 6\%, \quad t = 40$

47. $r = 8\%, \quad t = 35$

48. $r = 9\%, \quad t = 25$

Compound Interest **In Exercises 49 and 50, find the time necessary for \$1000 to double when it is invested at a rate of *r* compounded (a) annually, (b) monthly, (c) daily, and (d) continuously.**

49. $r = 7\%$ **50.** $r = 5.5\%$

Population **In Exercises 51–54, the population (in millions) of a country in 2011 and the expected continuous annual rate of change *k* of the population are given. (*Source: U.S. Census Bureau, International Data Base*)**

(a) Find the exponential growth model

$$P = Ce^{kt}$$

for the population by letting $t = 0$ correspond to 2010.

(b) Use the model to predict the population of the country in 2020.

(c) Discuss the relationship between the sign of *k* and the change in population for the country.

	Country	2011 Population	k
51.	Latvia	2.2	-0.006
52.	Egypt	82.1	0.020
53.	Uganda	34.6	0.036
54.	Hungary	10.0	-0.002

55. Modeling Data One hundred bacteria are started in a culture and the number *N* of bacteria is counted each hour for 5 hours. The results are shown in the table, where *t* is the time in hours.

t	0	1	2	3	4	5
N	100	126	151	198	243	297

(a) Use the regression capabilities of a graphing utility to find an exponential model for the data.

(b) Use the model to estimate the time required for the population to quadruple in size.

56. Bacteria Growth The number of bacteria in a culture is increasing according to the law of exponential growth. There are 125 bacteria in the culture after 2 hours and 350 bacteria after 4 hours.

(a) Find the initial population.

(b) Write an exponential growth model for the bacteria population. Let *t* represent time in hours.

(c) Use the model to determine the number of bacteria after 8 hours.

(d) After how many hours will the bacteria count be 25,000?

57. Learning Curve The management at a certain factory has found that a worker can produce at most 30 units in a day. The learning curve for the number of units *N* produced per day after a new employee has worked *t* days is

$$N = 30(1 - e^{kt}).$$

After 20 days on the job, a particular worker produces 19 units.

(a) Find the learning curve for this worker.

(b) How many days should pass before this worker is producing 25 units per day?

58. Learning Curve Suppose the management in Exercise 57 requires a new employee to produce at least 20 units per day after 30 days on the job.

(a) Find the learning curve that describes this minimum requirement.

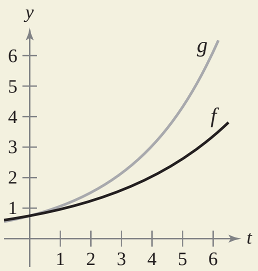
(b) Find the number of days before a minimal achiever is producing 25 units per day.

59. Insect Population

(a) Suppose an insect population increases by a constant number each month. Explain why the number of insects can be represented by a linear function.

(b) Suppose an insect population increases by a constant percentage each month. Explain why the number of insects can be represented by an exponential function.

60. HOW DO YOU SEE IT? The functions f and g are both of the form $y = Ce^{kt}$.



(a) Do the functions f and g represent exponential growth or exponential decay? Explain.

(b) Assume both functions have the same value of C. Which function has a greater value of k? Explain.

61. Modeling Data The table shows the resident populations P (in millions) of the United States from 1920 to 2010. (*Source: U.S. Census Bureau*)

Year	1920	1930	1940	1950	1960
Population, P	106	123	132	151	179

Year	1970	1980	1990	2000	2010
Population, P	203	227	249	281	309

(a) Use the 1920 and 1930 data to find an exponential model P_1 for the data. Let $t = 0$ represent 1920.

(b) Use a graphing utility to find an exponential model P_2 for all the data. Let $t = 0$ represent 1920.

(c) Use a graphing utility to plot the data and graph models P_1 and P_2 in the same viewing window. Compare the actual data with the predictions. Which model better fits the data?

(d) Use the model chosen in part (c) to estimate when the resident population will be 400 million.

Stephen Aaron Rees/Shutterstock.com

62. Forestry

The value of a tract of timber is

$$V(t) = 100{,}000e^{0.8\sqrt{t}}$$

where t is the time in years, with $t = 0$ corresponding to 2010. If money earns interest continuously at 10%, then the present value of the timber at any time t is

$$A(t) = V(t)e^{-0.10t}.$$

Find the year in which the timber should be harvested to maximize the present value function.

63. Sound Intensity The level of sound β (in decibels) with an intensity of I is

$$\beta(I) = 10 \log_{10}\left(\frac{I}{I_0}\right)$$

where I_0 is an intensity of 10^{-16} watt per square centimeter, corresponding roughly to the faintest sound that can be heard. Determine $\beta(I)$ for the following.

(a) $I = 10^{-14}$ watt per square centimeter (whisper)

(b) $I = 10^{-9}$ watt per square centimeter (busy street corner)

(c) $I = 10^{-6.5}$ watt per square centimeter (air hammer)

(d) $I = 10^{-4}$ watt per square centimeter (threshold of pain)

64. Noise Level With the installation of noise suppression materials, the noise level in an auditorium was reduced from 93 to 80 decibels. Use the function in Exercise 63 to find the percent decrease in the intensity level of the noise as a result of the installation of these materials.

65. Newton's Law of Cooling When an object is removed from a furnace and placed in an environment with a constant temperature of 80°F, its core temperature is 1500°F. One hour after it is removed, the core temperature is 1120°F. Find the core temperature 5 hours after the object is removed from the furnace.

66. Newton's Law of Cooling A container of hot liquid is placed in a freezer that is kept at a constant temperature of 20°F. The initial temperature of the liquid is 160°F. After 5 minutes, the liquid's temperature is 60°F. How much longer will it take for its temperature to decrease to 30°F?

True or False? **In Exercises 67–70, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

67. In exponential growth, the rate of growth is constant.

68. In linear growth, the rate of growth is constant.

69. If prices are rising at a rate of 0.5% per month, then they are rising at a rate of 6% per year.

70. The differential equation modeling exponential growth is $dy/dx = ky$, where k is a constant.

6.3 Differential Equations: Separation of Variables

- **Recognize and solve differential equations that can be solved by separation of variables.**
- **Use differential equations to model and solve applied problems.**

Separation of Variables

Consider a differential equation that can be written in the form

$$M(x) + N(y)\frac{dy}{dx} = 0$$

where M is a continuous function of x alone and N is a continuous function of y alone. As you saw in Section 6.2, for this type of equation, all x terms can be collected with dx and all y terms with dy, and a solution can be obtained by integration. Such equations are said to be **separable,** and the solution procedure is called **separation of variables.** Below are some examples of differential equations that are separable.

Original Differential Equation	Rewritten with Variables Separated
$x^2 + 3y\dfrac{dy}{dx} = 0$	$3y\,dy = -x^2\,dx$
$(\sin x)y' = \cos x$	$dy = \cot x\,dx$
$\dfrac{xy'}{e^y + 1} = 2$	$\dfrac{1}{e^y + 1}\,dy = \dfrac{2}{x}\,dx$

EXAMPLE 1 Separation of Variables

See LarsonCalculus.com for an interactive version of this type of example.

Find the general solution of

$$(x^2 + 4)\frac{dy}{dx} = xy.$$

Solution To begin, note that $y = 0$ is a solution. To find other solutions, assume that $y \neq 0$ and separate variables as shown.

$$(x^2 + 4)\,dy = xy\,dx \qquad \text{Differential form}$$

$$\frac{dy}{y} = \frac{x}{x^2 + 4}\,dx \qquad \text{Separate variables.}$$

Now, integrate to obtain

$$\int \frac{dy}{y} = \int \frac{x}{x^2 + 4}\,dx \qquad \text{Integrate.}$$

$$\ln|y| = \frac{1}{2}\ln(x^2 + 4) + C_1$$

$$\ln|y| = \ln\sqrt{x^2 + 4} + C_1$$

$$|y| = e^{C_1}\sqrt{x^2 + 4}$$

$$y = \pm e^{C_1}\sqrt{x^2 + 4}.$$

Because $y = 0$ is also a solution, you can write the general solution as

$$y = C\sqrt{x^2 + 4}. \qquad \text{General solution}$$

REMARK Be sure to check your solutions throughout this chapter. In Example 1, you can check the solution

$$y = C\sqrt{x^2 + 4}$$

by differentiating and substituting into the original equation.

$$(x^2 + 4)\frac{dy}{dx} = xy$$

$$(x^2 + 4)\frac{Cx}{\sqrt{x^2 + 4}} \stackrel{?}{=} x\left(C\sqrt{x^2 + 4}\right)$$

$$Cx\sqrt{x^2 + 4} = Cx\sqrt{x^2 + 4}$$

So, the solution checks.

In some cases, it is not feasible to write the general solution in the explicit form $y = f(x)$. The next example illustrates such a solution. Implicit differentiation can be used to verify this solution.

FOR FURTHER INFORMATION For an example (from engineering) of a differential equation that is separable, see the article "Designing a Rose Cutter" by J. S. Hartzler in *The College Mathematics Journal*. To view this article, go to *MathArticles.com*.

EXAMPLE 2 Finding a Particular Solution

Given the initial condition $y(0) = 1$, find the particular solution of the equation

$$xy\,dx + e^{-x^2}(y^2 - 1)\,dy = 0.$$

Solution Note that $y = 0$ is a solution of the differential equation—but this solution does not satisfy the initial condition. So, you can assume that $y \neq 0$. To separate variables, you must rid the first term of y and the second term of e^{-x^2}. So, you should multiply by e^{x^2}/y and obtain the following.

$$\begin{aligned} xy\,dx + e^{-x^2}(y^2 - 1)\,dy &= 0 \\ e^{-x^2}(y^2 - 1)\,dy &= -xy\,dx \\ \int\left(y - \frac{1}{y}\right)dy &= \int -xe^{x^2}\,dx \\ \frac{y^2}{2} - \ln|y| &= -\frac{1}{2}e^{x^2} + C \end{aligned}$$

From the initial condition $y(0) = 1$, you have

$$\frac{1}{2} - 0 = -\frac{1}{2} + C$$

which implies that $C = 1$. So, the particular solution has the implicit form

$$\frac{y^2}{2} - \ln|y| = -\frac{1}{2}e^{x^2} + 1$$

$$y^2 - \ln y^2 + e^{x^2} = 2.$$

You can check this by differentiating and rewriting to get the original equation.

EXAMPLE 3 Finding a Particular Solution Curve

Find the equation of the curve that passes through the point $(1, 3)$ and has a slope of y/x^2 at any point (x, y).

Solution Because the slope of the curve is y/x^2, you have

$$\frac{dy}{dx} = \frac{y}{x^2}$$

with the initial condition $y(1) = 3$. Separating variables and integrating produces

$$\int \frac{dy}{y} = \int \frac{dx}{x^2}, \quad y \neq 0$$

$$\ln|y| = -\frac{1}{x} + C_1$$

$$y = e^{-(1/x) + C_1}$$

$$y = Ce^{-1/x}.$$

Because $y = 3$ when $x = 1$, it follows that $3 = Ce^{-1}$ and $C = 3e$. So, the equation of the specified curve is

$$y = (3e)e^{-1/x} \quad \Rightarrow \quad y = 3e^{(x-1)/x}, \quad x > 0.$$

Because the solution is not defined at $x = 0$ and the initial condition is given at $x = 1$, x is restricted to positive values. See Figure 6.11.

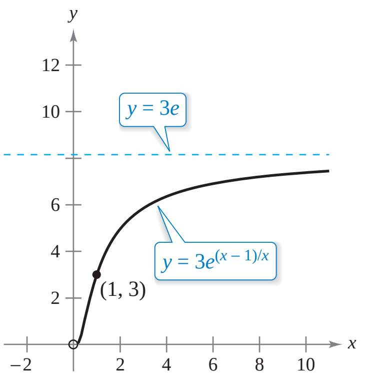


Figure 6.11

Applications

EXAMPLE 4 **Wildlife Population**

The rate of change of the number of coyotes $N(t)$ in a population is directly proportional to $650 - N(t)$, where t is the time in years. When $t = 0$, the population is 300, and when $t = 2$, the population has increased to 500. Find the population when $t = 3$.

Solution Because the rate of change of the population is proportional to $650 - N(t)$, or $650 - N$, you can write the differential equation

$$\frac{dN}{dt} = k(650 - N).$$

You can solve this equation using separation of variables.

$$dN = k(650 - N)\,dt \qquad \text{Differential form}$$

$$\frac{dN}{650 - N} = k\,dt \qquad \text{Separate variables.}$$

$$-\ln|650 - N| = kt + C_1 \qquad \text{Integrate.}$$

$$\ln|650 - N| = -kt - C_1$$

$$650 - N = e^{-kt - C_1} \qquad \text{Assume } N < 650.$$

$$N = 650 - Ce^{-kt} \qquad \text{General solution}$$

Using $N = 300$ when $t = 0$, you can conclude that $C = 350$, which produces

$$N = 650 - 350e^{-kt}.$$

Then, using $N = 500$ when $t = 2$, it follows that

$$500 = 650 - 350e^{-2k} \quad \Longrightarrow \quad e^{-2k} = \frac{3}{7} \quad \Longrightarrow \quad k \approx 0.4236.$$

So, the model for the coyote population is

$$N = 650 - 350e^{-0.4236t}. \qquad \text{Model for population}$$

When $t = 3$, you can approximate the population to be

$$N = 650 - 350e^{-0.4236(3)}$$
$$\approx 552 \text{ coyotes.}$$

The model for the population is shown in Figure 6.12. Note that $N = 650$ is the horizontal asymptote of the graph and is the *carrying capacity* of the model. You will learn more about carrying capacity in Section 6.4.

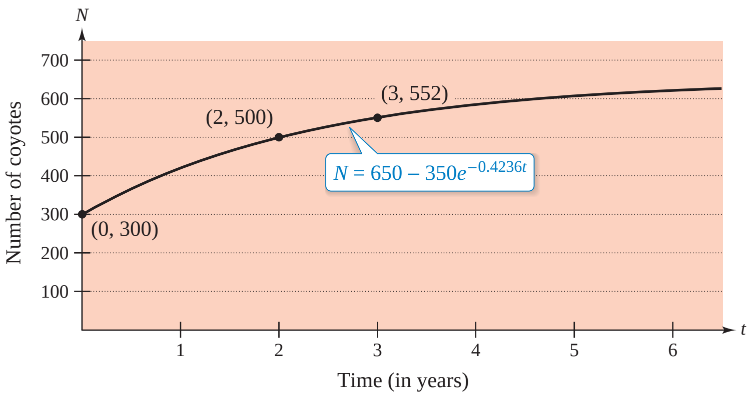


Figure 6.12

franzfoto.com/Alamy

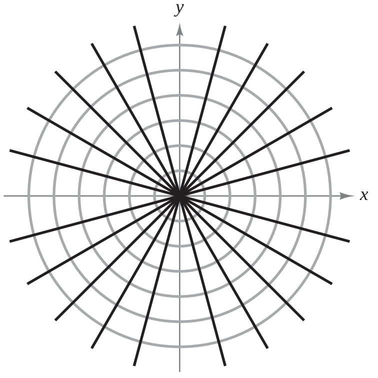
A common problem in electrostatics, thermodynamics, and hydrodynamics involves finding a family of curves, each of which is orthogonal to all members of a given family of curves. For example, Figure 6.13 shows a family of circles

$$x^2 + y^2 = C \qquad \text{Family of circles}$$

each of which intersects the lines in the family

$$y = Kx \qquad \text{Family of lines}$$

at right angles. Two such families of curves are said to be **mutually orthogonal,** and each curve in one of the families is called an **orthogonal trajectory** of the other family. In electrostatics, lines of force are orthogonal to the *equipotential curves*. In thermodynamics, the flow of heat across a plane surface is orthogonal to the *isothermal curves*. In hydrodynamics, the flow (stream) lines are orthogonal trajectories of the *velocity potential curves*.



Each line $y = Kx$ is an orthogonal trajectory of the family of circles.
Figure 6.13

EXAMPLE 5 Finding Orthogonal Trajectories

Describe the orthogonal trajectories for the family of curves given by

$$y = \frac{C}{x}$$

for $C \neq 0$. Sketch several members of each family.

Solution First, solve the given equation for C and write $xy = C$. Then, by differentiating implicitly with respect to x, you obtain the differential equation

$$x\frac{dy}{dx} + y = 0 \qquad \text{Differential equation}$$

$$x\frac{dy}{dx} = -y$$

$$\frac{dy}{dx} = -\frac{y}{x}. \qquad \text{Slope of given family}$$

Because dy/dx represents the slope of the given family of curves at (x, y), it follows that the orthogonal family has the negative reciprocal slope x/y. So,

$$\frac{dy}{dx} = \frac{x}{y}. \qquad \text{Slope of orthogonal family}$$

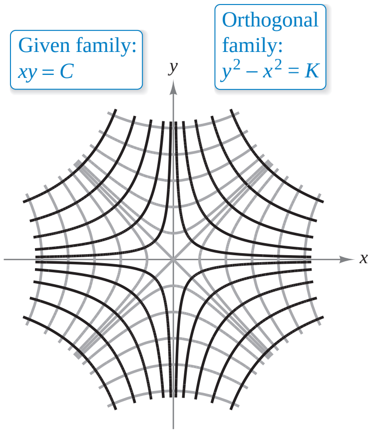
Now you can find the orthogonal family by separating variables and integrating.

$$\int y\,dy = \int x\,dx$$

$$\frac{y^2}{2} = \frac{x^2}{2} + C_1$$

$$y^2 - x^2 = K$$

The centers are at the origin, and the transverse axes are vertical for $K > 0$ and horizontal for $K < 0$. When $K = 0$, the orthogonal trajectories are the lines $y = \pm x$. When $K \neq 0$, the orthogonal trajectories are hyperbolas. Several trajectories are shown in Figure 6.14.



Orthogonal trajectories
Figure 6.14

EXAMPLE 6 Modeling Advertising Awareness

A new cereal product is introduced through an advertising campaign to a population of 1 million potential customers. The rate at which the population hears about the product is assumed to be proportional to the number of people who are not yet aware of the product. By the end of 1 year, half of the population has heard of the product. How many will have heard of it by the end of 2 years?

Solution Let y be the number (in millions) of people at time t who have heard of the product. This means that $(1 - y)$ is the number of people who have not heard of it, and dy/dt is the rate at which the population hears about the product. From the given assumption, you can write the differential equation as shown.

$$\underbrace{\frac{dy}{dt}}_{\text{Rate of change of } y} \underbrace{=\ k}_{\text{is proportional to}} \underbrace{(1 - y)}_{\text{the difference between 1 and } y.}$$

You can solve this equation using separation of variables.

$$dy = k(1 - y)\,dt \qquad \text{Differential form}$$

$$\frac{dy}{1 - y} = k\,dt \qquad \text{Separate variables.}$$

$$-\ln|1 - y| = kt + C_1 \qquad \text{Integrate.}$$

$$\ln|1 - y| = -kt - C_1 \qquad \text{Multiply each side by } -1.$$

$$1 - y = e^{-kt - C_1} \qquad \text{Assume } y < 1.$$

$$y = 1 - Ce^{-kt} \qquad \text{General solution}$$

To solve for the constants C and k, use the initial conditions. That is, because $y = 0$ when $t = 0$, you can determine that $C = 1$. Similarly, because $y = 0.5$ when $t = 1$, it follows that $0.5 = 1 - e^{-k}$, which implies that

$$k = \ln 2 \approx 0.693.$$

So, the particular solution is

$$y = 1 - e^{-0.693t}. \qquad \text{Particular solution}$$

This model is shown in Figure 6.15. Using the model, you can determine that the number of people who have heard of the product after 2 years is

$$\begin{aligned} y &= 1 - e^{-0.693(2)} \\ &\approx 0.75 \text{ or } 750{,}000 \text{ people.} \end{aligned}$$

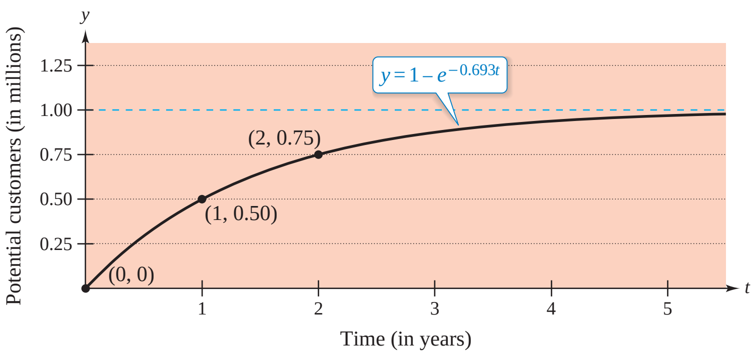


Figure 6.15

EXAMPLE 7 Modeling a Chemical Reaction

During a chemical reaction, substance A is converted into substance B at a rate that is proportional to the square of the amount of A. When $t = 0$, 60 grams of A is present, and after 1 hour ($t = 1$), only 10 grams of A remains unconverted. How much of A is present after 2 hours?

Solution Let y be the amount of unconverted substance A at any time t. From the given assumption about the conversion rate, you can write the differential equation as shown.

$$\frac{dy}{dt} = ky^2$$

Rate of change of y — is proportional to — the square of y.

You can solve this equation using separation of variables.

$$dy = ky^2\,dt \qquad \text{Differential form}$$

$$\frac{dy}{y^2} = k\,dt \qquad \text{Separate variables.}$$

$$-\frac{1}{y} = kt + C \qquad \text{Integrate.}$$

$$y = \frac{-1}{kt + C} \qquad \text{General solution}$$

To solve for the constants C and k, use the initial conditions. That is, because $y = 60$ when $t = 0$, you can determine that $C = -\frac{1}{60}$. Similarly, because $y = 10$ when $t = 1$, it follows that

$$10 = \frac{-1}{k - (1/60)}$$

which implies that $k = -\frac{1}{12}$. So, the particular solution is

$$y = \frac{-1}{(-1/12)t - (1/60)} \qquad \text{Substitute for } k \text{ and } C.$$

$$= \frac{60}{5t + 1}. \qquad \text{Particular solution}$$

Using the model, you can determine that the unconverted amount of substance A after 2 hours is

$$y = \frac{60}{5(2) + 1}$$

$$\approx 5.45 \text{ grams.}$$

In Figure 6.16, note that the chemical conversion is occurring rapidly during the first hour. Then, as more and more of substance A is converted, the conversion rate slows down.

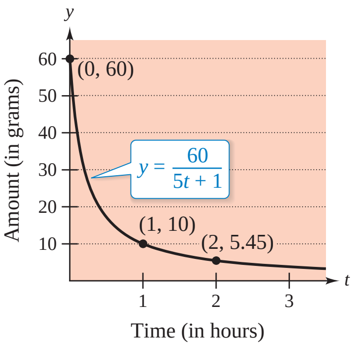


Figure 6.16

Exploration

In Example 7, the rate of conversion was assumed to be proportional to the *square* of the unconverted amount. How would the result change if the rate of conversion were assumed to be proportional to the unconverted amount?

The next example describes a growth model called a **Gompertz growth model.** This model assumes that the rate of change of y is proportional to the product of y and the natural log of L/y, where L is the population limit.

EXAMPLE 8 Modeling Population Growth

A population of 20 wolves has been introduced into a national park. The forest service estimates that the maximum population the park can sustain is 200 wolves. After 3 years, the population is estimated to be 40 wolves. According to a Gompertz growth model, how many wolves will there be 10 years after their introduction?

Solution Let y be the number of wolves at any time t. From the given assumption about the rate of growth of the population, you can write the differential equation as shown.

$$\frac{dy}{dt} = ky \ln \frac{200}{y}$$

Rate of change of y | is proportional to | the product of y and | the natural log of the ratio of 200 and y.

> **TECHNOLOGY** If you have access to a computer algebra system, try using it to find the general solution and the particular solution to Example 8.

Using separation of variables *or* a computer algebra system, you can find the general solution to be

$$y = 200e^{-Ce^{-kt}}. \qquad \text{General solution}$$

To solve for the constants C and k, use the initial conditions. That is, because $y = 20$ when $t = 0$, you can determine that

$$\begin{aligned} C &= \ln 10 \\ &\approx 2.3026. \end{aligned}$$

Similarly, because $y = 40$ when $t = 3$, it follows that

$$40 = 200e^{-2.3026e^{-3k}}$$

which implies that $k \approx 0.1194$. So, the particular solution is

$$y = 200e^{-2.3026e^{-0.1194t}}. \qquad \text{Particular solution}$$

Using the model, you can estimate the wolf population after 10 years to be

$$\begin{aligned} y &= 200e^{-2.3026e^{-0.1194(10)}} \\ &\approx 100 \text{ wolves.} \end{aligned}$$

In Figure 6.17, note that after 10 years the population has reached about half of the estimated maximum population. Try checking the growth model to see that it yields $y = 20$ when $t = 0$ and $y = 40$ when $t = 3$.

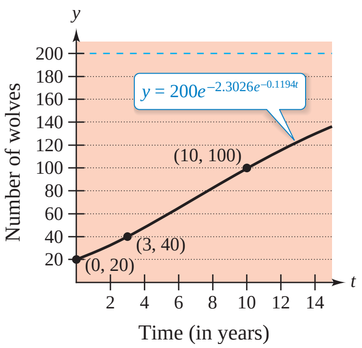


Figure 6.17

In genetics, a commonly used hybrid selection model is based on the differential equation

$$\frac{dy}{dt} = ky(1 - y)(a - by).$$

In this model, y represents the portion of the population that has a certain characteristic and t represents the time (measured in generations). The numbers a, b, and k are constants that depend on the genetic characteristic that is being studied.

EXAMPLE 9 Modeling Hybrid Selection

You are studying a population of beetles to determine how quickly characteristic D will pass from one generation to the next. At the beginning of your study ($t = 0$), you find that half the population has characteristic D. After four generations ($t = 4$), you find that 80% of the population has characteristic D. Use the hybrid selection model above with $a = 2$ and $b = 1$ to find the percent of the population that will have characteristic D after 10 generations.

Solution Using $a = 2$ and $b = 1$, the differential equation for the hybrid selection model is

$$\frac{dy}{dt} = ky(1 - y)(2 - y).$$

Using separation of variables or a computer algebra system, you can find the general solution to be

$$\frac{y(2 - y)}{(1 - y)^2} = Ce^{2kt}. \qquad \text{General solution}$$

To solve for the constants C and k, use the initial conditions. That is, because $y = 0.5$ when $t = 0$, you can determine that $C = 3$. Similarly, because $y = 0.8$ when $t = 4$, it follows that

$$\frac{0.8(1.2)}{(0.2)^2} = 3e^{8k}$$

which implies that

$$k = \frac{1}{8}\ln 8 \approx 0.2599.$$

So, the particular solution is

$$\frac{y(2 - y)}{(1 - y)^2} = 3e^{0.5199t}. \qquad \text{Particular solution}$$

Using the model, you can estimate the percent of the population that will have characteristic D after 10 generations to be given by

$$\frac{y(2 - y)}{(1 - y)^2} = 3e^{0.5199(10)}.$$

Using a computer algebra system, you can solve this equation for y to obtain

$$y \approx 0.96$$

or 96% of the population. The graph of the model is shown in Figure 6.18.

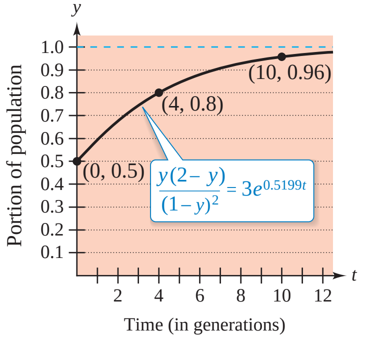


Figure 6.18

6.3 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Finding a General Solution Using Separation of Variables In Exercises 1–14, find the general solution of the differential equation.

1. $\dfrac{dy}{dx} = \dfrac{x}{y}$

2. $\dfrac{dy}{dx} = \dfrac{3x^2}{y^2}$

3. $x^2 + 5y\dfrac{dy}{dx} = 0$

4. $\dfrac{dy}{dx} = \dfrac{6 - x^2}{2y^3}$

5. $\dfrac{dr}{ds} = 0.75r$

6. $\dfrac{dr}{ds} = 0.75s$

7. $(2 + x)y' = 3y$

8. $xy' = y$

9. $yy' = 4 \sin x$

10. $yy' = -8 \cos(\pi x)$

11. $\sqrt{1 - 4x^2}\,y' = x$

12. $\sqrt{x^2 - 16}\,y' = 11x$

13. $y \ln x - xy' = 0$

14. $12yy' - 7e^x = 0$

Finding a Particular Solution Using Separation of Variables In Exercises 15–24, find the particular solution that satisfies the initial condition.

	Differential Equation	Initial Condition
15.	$yy' - 2e^x = 0$	$y(0) = 3$
16.	$\sqrt{x} + \sqrt{y}\,y' = 0$	$y(1) = 9$
17.	$y(x + 1) + y' = 0$	$y(-2) = 1$
18.	$2xy' - \ln x^2 = 0$	$y(1) = 2$
19.	$y(1 + x^2)y' - x(1 + y^2) = 0$	$y(0) = \sqrt{3}$
20.	$y\sqrt{1 - x^2}\,y' - x\sqrt{1 - y^2} = 0$	$y(0) = 1$
21.	$\dfrac{du}{dv} = uv \sin v^2$	$u(0) = 1$
22.	$\dfrac{dr}{ds} = e^{r-2s}$	$r(0) = 0$
23.	$dP - kP\,dt = 0$	$P(0) = P_0$
24.	$dT + k(T - 70)\,dt = 0$	$T(0) = 140$

Finding a Particular Solution In Exercises 25–28, find an equation of the graph that passes through the point and has the given slope.

25. $(0, 2), \quad y' = \dfrac{x}{4y}$

26. $(1, 1), \quad y' = -\dfrac{9x}{16y}$

27. $(9, 1), \quad y' = \dfrac{y}{2x}$

28. $(8, 2), \quad y' = \dfrac{2y}{3x}$

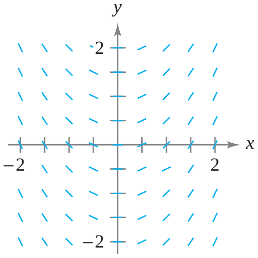
Using Slope In Exercises 29 and 30, find all functions f having the indicated property.

29. The tangent to the graph of f at the point (x, y) intersects the x-axis at $(x + 2, 0)$.

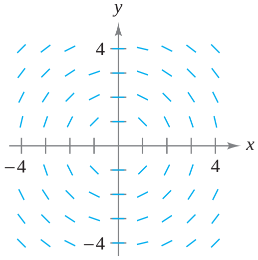
30. All tangents to the graph of f pass through the origin.

Slope Field In Exercises 31–34, sketch a few solutions of the differential equation on the slope field and then find the general solution analytically. To print an enlarged copy of the graph, go to *MathGraphs.com*.

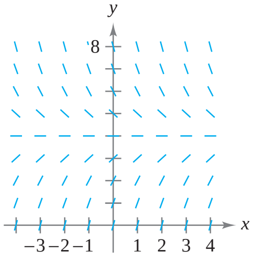
31. $\dfrac{dy}{dx} = x$



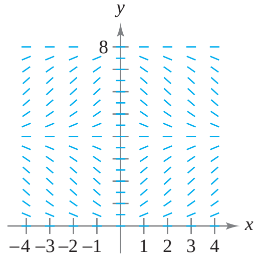
32. $\dfrac{dy}{dx} = -\dfrac{x}{y}$



33. $\dfrac{dy}{dx} = 4 - y$



34. $\dfrac{dy}{dx} = 0.25x(4 - y)$



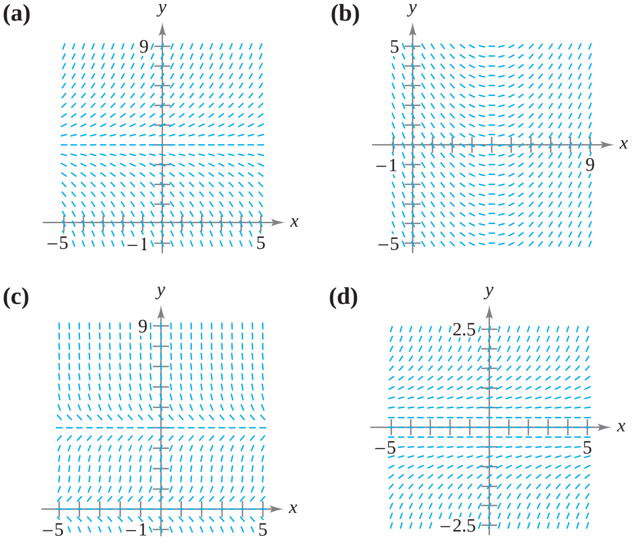
Euler's Method In Exercises 35–38, (a) use Euler's Method with a step size of $h = 0.1$ to approximate the particular solution of the initial value problem at the given x-value, (b) find the exact solution of the differential equation analytically, and (c) compare the solutions at the given x-value.

	Differential Equation	Initial Condition	x-value
35.	$\dfrac{dy}{dx} = -6xy$	$(0, 5)$	$x = 1$
36.	$\dfrac{dy}{dx} + 6xy^2 = 0$	$(0, 3)$	$x = 1$
37.	$\dfrac{dy}{dx} = \dfrac{2x + 12}{3y^2 - 4}$	$(1, 2)$	$x = 2$
38.	$\dfrac{dy}{dx} = 2x(1 + y^2)$	$(1, 0)$	$x = 1.5$

39. **Radioactive Decay** The rate of decomposition of radioactive radium is proportional to the amount present at any time. The half-life of radioactive radium is 1599 years. What percent of a present amount will remain after 50 years?

40. **Chemical Reaction** In a chemical reaction, a certain compound changes into another compound at a rate proportional to the unchanged amount. There is 40 grams of the original compound initially and 35 grams after 1 hour. When will 75 percent of the compound be changed?

Slope Field **In Exercises 41–44, (a) write a differential equation for the statement, (b) match the differential equation with a possible slope field, and (c) verify your result by using a graphing utility to graph a slope field for the differential equation. [The slope fields are labeled (a), (b), (c), and (d).] To print an enlarged copy of the graph, go to *MathGraphs.com*.**



41. The rate of change of y with respect to x is proportional to the difference between y and 4.

42. The rate of change of y with respect to x is proportional to the difference between x and 4.

43. The rate of change of y with respect to x is proportional to the product of y and the difference between y and 4.

44. The rate of change of y with respect to x is proportional to y^2.

45. Weight Gain A calf that weighs 60 pounds at birth gains weight at the rate

$$\frac{dw}{dt} = k(1200 - w)$$

where w is weight in pounds and t is time in years.

(a) Solve the differential equation.

(b) Use a graphing utility to graph the particular solutions for $k = 0.8$, 0.9, and 1.

(c) The animal is sold when its weight reaches 800 pounds. Find the time of sale for each of the models in part (b).

(d) What is the maximum weight of the animal for each of the models in part (b)?

46. Weight Gain A calf that weighs w_0 pounds at birth gains weight at the rate $dw/dt = 1200 - w$, where w is weight in pounds and t is time in years. Solve the differential equation.

Finding Orthogonal Trajectories **In Exercises 47–52, find the orthogonal trajectories of the family. Use a graphing utility to graph several members of each family.**

47. $x^2 + y^2 = C$

48. $x^2 - 2y^2 = C$

49. $x^2 = Cy$

50. $y^2 = 2Cx$

51. $y^2 = Cx^3$

52. $y = Ce^x$

53. Biology At any time t, the rate of growth of the population N of deer in a state park is proportional to the product of N and $L - N$, where $L = 500$ is the maximum number of deer the park can sustain. When $t = 0$, $N = 100$, and when $t = 4$, $N = 200$. Write N as a function of t.

54. Sales Growth The rate of change in sales S (in thousands of units) of a new product is proportional to the product of S and $L - S$, where L (in thousands of units) is the estimated maximum level of sales. When $t = 0$, $S = 10$. Write and solve the differential equation for this sales model.

Advertising Awareness **In Exercises 55 and 56, use the advertising awareness model described in Example 6 to find the number of people y (in millions) aware of the product as a function of time t (in years).**

55. $y = 0$ when $t = 0$; $y = 0.75$ when $t = 1$

56. $y = 0$ when $t = 0$; $y = 0.9$ when $t = 2$

Chemical Reaction **In Exercises 57 and 58, use the chemical reaction model given in Example 7 to find the amount y as a function of t, and use a graphing utility to graph the function.**

57. $y = 45$ grams when $t = 0$; $y = 4$ grams when $t = 2$

58. $y = 75$ grams when $t = 0$; $y = 12$ grams when $t = 1$

Using a Gompertz Growth Model **In Exercises 59 and 60, use the Gompertz growth model described on page 403 to find the growth function, and sketch its graph.**

59. $L = 500$; $y = 100$ when $t = 0$; $y = 150$ when $t = 2$

60. $L = 5000$; $y = 500$ when $t = 0$; $y = 625$ when $t = 1$

61. Biology A population of eight beavers has been introduced into a new wetlands area. Biologists estimate that the maximum population the wetlands can sustain is 60 beavers. After 3 years, the population is 15 beavers. According to a Gompertz growth model, how many beavers will be present in the wetlands after 10 years?

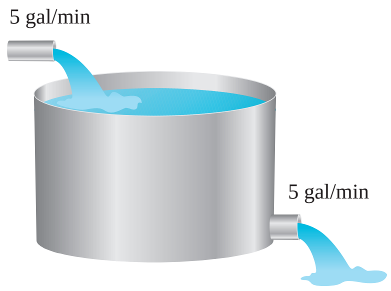
62. Biology A population of 30 rabbits has been introduced into a new region. It is estimated that the maximum population the region can sustain is 400 rabbits. After 1 year, the population is estimated to be 90 rabbits. According to a Gompertz growth model, how many rabbits will be present after 3 years?

Biology **In Exercises 63 and 64, use the hybrid selection model described on page 404 to find the percent of the population that has the given characteristic. (Assume $a = 2$ and $b = 1$.)**

63. You are studying a population of mayflies to determine how quickly characteristic A will pass from one generation to the next. At the start of the study, half the population has characteristic A. After four generations, 75% of the population has characteristic A. Find the percent of the population that will have characteristic A after 10 generations.

64. A research team is studying a population of snails to determine how quickly characteristic B will pass from one generation to the next. At the start of the study, 40% of the snails have characteristic B. After five generations, 80% of the population has characteristic B. Find the percent of the population that will have characteristic B after eight generations.

65. Chemical Mixture A 100-gallon tank is full of a solution containing 25 pounds of a concentrate. Starting at time $t = 0$, distilled water is admitted to the tank at the rate of 5 gallons per minute, and the well-stirred solution is withdrawn at the same rate, as shown in the figure.



(a) Find the amount Q of the concentrate in the solution as a function of t. (*Hint*: $Q' + Q/20 = 0$)

(b) Find the time when the amount of concentrate in the tank reaches 15 pounds.

66. Chemical Mixture A 200-gallon tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 pound of concentrate per gallon enters the tank at the rate of 5 gallons per minute, and the well-stirred mixture is withdrawn at the same rate. Find the amount Q of concentrate in the tank after 30 minutes. (*Hint*: $Q' + Q/20 = 5/2$)

67. Chemical Reaction In a chemical reaction, a compound changes into another compound at a rate proportional to the unchanged amount, according to the model

$$\frac{dy}{dt} = ky.$$

(a) Solve the differential equation.

(b) The initial amount of the original compound is 20 grams, and the amount remaining after 1 hour is 16 grams. When will 75% of the compound have been changed?

68. Snow Removal The rate of change in the number of miles s of road cleared per hour by a snowplow is inversely proportional to the depth h of snow. That is,

$$\frac{ds}{dh} = \frac{k}{h}.$$

Find s as a function of h given that $s = 25$ miles when $h = 2$ inches and $s = 12$ miles when $h = 6$ inches $(2 \le h \le 15)$.

69. Chemistry A wet towel hung from a clothesline to dry loses moisture through evaporation at a rate proportional to its moisture content. After 1 hour, the towel has lost 40% of its original moisture content. After how long will it have lost 80%?

70. Biology Let x and y be the sizes of two internal organs of a particular mammal at time t. Empirical data indicate that the relative growth rates of these two organs are equal, and can be modeled by

$$\frac{1}{x}\frac{dx}{dt} = \frac{1}{y}\frac{dy}{dt}.$$

Use this differential equation to write y as a function of x.

71. Population Growth When predicting population growth, demographers must consider birth and death rates as well as the net change caused by the difference between the rates of immigration and emigration. Let P be the population at time t and let N be the net increase per unit time due to the difference between immigration and emigration. The rate of growth of the population is given by

$$\frac{dP}{dt} = kP + N, \quad N \text{ is constant.}$$

Solve this differential equation to find P as a function of time.

72. Meteorology The barometric pressure y (in inches of mercury) at an altitude of x miles above sea level decreases at a rate proportional to the current pressure according to the model

$$\frac{dy}{dx} = -0.2y$$

where $y = 29.92$ inches when $x = 0$. Find the barometric pressure (a) at the top of Mt. St. Helens (8364 feet) and (b) at the top of Mt. McKinley (20,320 feet).

73. Investment A large corporation starts at time $t = 0$ to invest part of its receipts at a rate of P dollars per year in a fund for future corporate expansion. The fund earns r percent interest per year compounded continuously. The rate of growth of the amount A in the fund is given by

$$\frac{dA}{dt} = rA + P$$

where $A = 0$ when $t = 0$. Solve this differential equation for A as a function of t.

Investment **In Exercises 74–76, use the result of Exercise 73.**

74. Find A for $P = \$275{,}000$, $r = 8\%$, and $t = 10$ years.

75. The corporation needs \$260,000,000 in 8 years and the fund earns $7\frac{1}{4}\%$ interest compounded continuously. Find P.

76. The corporation needs \$1,000,000 and it can invest \$125,000 per year in a fund earning 8% interest compounded continuously. Find t.

Using a Gompertz Growth Model **In Exercises 77 and 78, use the Gompertz growth model described in Example 8.**

77. (a) Use a graphing utility to graph the slope field for the growth model when $k = 0.02$ and $L = 5000$.

(b) Describe the behavior of the graph as $t \to \infty$.

(c) Solve the growth model for $L = 5000$, $y_0 = 500$, and $k = 0.02$.

(d) Graph the equation you found in part (c). Determine the concavity of the graph.

78. (a) Use a graphing utility to graph the slope field for the growth model when $k = 0.05$ and $L = 1000$.

(b) Describe the behavior of the graph as $t \to \infty$.

(c) Solve the growth model for $L = 1000$, $y_0 = 100$, and $k = 0.05$.

(d) Graph the equation you found in part (c). Determine the concavity of the graph.

WRITING ABOUT CONCEPTS

79. Separation of Variables In your own words, describe how to recognize and solve differential equations that can be solved by separation of variables.

80. Mutually Orthogonal In your own words, describe the relationship between two families of curves that are mutually orthogonal.

Separation of Variables **In Exercises 81–84, determine whether the differential equation is separable. If the equation is separable, rewrite it in the form $N(y)\,dy = M(x)\,dx$. (Do not solve the differential equation.)**

81. $y(1 + x)\,dx + x\,dy = 0$

82. $y' = y^{1/2}$

83. $y' + xy = 5$

84. $y' = x - xy - y + 1$

85. Sailing

Ignoring resistance, a sailboat starting from rest accelerates (dv/dt) at a rate proportional to the difference between the velocities of the wind and the boat.

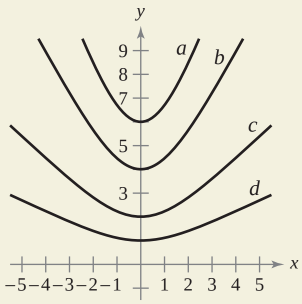
(a) The wind is blowing at 20 knots, and after 1 half-hour, the boat is moving at 10 knots. Write the velocity v as a function of time t.

(b) Use the result of part (a) to write the distance traveled by the boat as a function of time.

86. HOW DO YOU SEE IT? Recall from Example 1 that the general solution of

$$(x^2 + 4)\frac{dy}{dx} = xy$$

is $y = C\sqrt{x^2 + 4}$. The graphs below show the particular solutions for $C = 0.5$, 1, 2, and 3. Match the value of C with each graph. Explain your reasoning.



Determining if a Function Is Homogeneous **In Exercises 87–94, determine whether the function is homogeneous, and if it is, determine its degree. A function $f(x, y)$ is *homogeneous of degree n* if $f(tx, ty) = t^n f(x, y)$.**

87. $f(x, y) = x^3 - 4xy^2 + y^3$

88. $f(x, y) = x^3 + 3x^2y^2 - 2y^2$

89. $f(x, y) = \dfrac{x^2y^2}{\sqrt{x^2 + y^2}}$

90. $f(x, y) = \dfrac{xy}{\sqrt{x^2 + y^2}}$

91. $f(x, y) = 2 \ln xy$

92. $f(x, y) = \tan(x + y)$

93. $f(x, y) = 2 \ln \dfrac{x}{y}$

94. $f(x, y) = \tan \dfrac{y}{x}$

Solving a Homogeneous Differential Equation **In Exercises 95–100, solve the homogeneous differential equation in terms of x and y. A *homogeneous differential equation* is an equation of the form $M(x, y)\,dx + N(x, y)\,dy = 0$, where M and N are homogeneous functions of the same degree. To solve an equation of this form by the method of separation of variables, use the substitutions $y = vx$ and $dy = x\,dv + v\,dx$.**

95. $(x + y)\,dx - 2x\,dy = 0$

96. $(x^3 + y^3)\,dx - xy^2\,dy = 0$

97. $(x - y)\,dx - (x + y)\,dy = 0$

98. $(x^2 + y^2)\,dx - 2xy\,dy = 0$

99. $xy\,dx + (y^2 - x^2)\,dy = 0$

100. $(2x + 3y)\,dx - x\,dy = 0$

True or False? **In Exercises 101–103, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

101. The function $y = 0$ is always a solution of a differential equation that can be solved by separation of variables.

102. The differential equation $y' = xy - 2y + x - 2$ can be written in separated variables form.

103. The families $x^2 + y^2 = 2Cy$ and $x^2 + y^2 = 2Kx$ are mutually orthogonal.

PUTNAM EXAM CHALLENGE

104. A not uncommon calculus mistake is to believe that the product rule for derivatives says that $(fg)' = f'g'$. If $f(x) = e^{x^2}$, determine, with proof, whether there exists an open interval (a, b) and a nonzero function g defined on (a, b) such that this wrong product rule is true for x in (a, b).

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6.4 The Logistic Equation

- **Solve and analyze logistic differential equations.**
- **Use logistic differential equations to model and solve applied problems.**

Logistic Differential Equation

In Section 6.2, the exponential growth model was derived from the fact that the rate of change of a variable y is proportional to the value of y. You observed that the differential equation $dy/dt = ky$ has the general solution $y = Ce^{kt}$. Exponential growth is unlimited, but when describing a population, there often exists some upper limit L past which growth cannot occur. This upper limit L is called the **carrying capacity,** which is the maximum population $y(t)$ that can be sustained or supported as time t increases. A model that is often used to describe this type of growth is the **logistic differential equation**

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \qquad \text{Logistic differential equation}$$

where k and L are positive constants. A population that satisfies this equation does not grow without bound, but approaches the carrying capacity L as t increases.

From the equation, you can see that if y is between 0 and the carrying capacity L, then $dy/dt > 0$, and the population increases. If y is greater than L, then $dy/dt < 0$, and the population decreases. The general solution of the logistic differential equation is derived in the next example.

EXAMPLE 1 Deriving the General Solution

Solve the logistic differential equation $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{L}\right)$.

Solution Begin by separating variables.

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right) \qquad \text{Write differential equation.}$$

$$\frac{1}{y(1 - y/L)}\,dy = k\,dt \qquad \text{Separate variables.}$$

$$\int \frac{1}{y(1 - y/L)}\,dy = \int k\,dt \qquad \text{Integrate each side.}$$

$$\int \left(\frac{1}{y} + \frac{1}{L - y}\right) dy = \int k\,dt \qquad \text{Rewrite left side using partial fractions.}$$

$$\ln|y| - \ln|L - y| = kt + C \qquad \text{Find antiderivative of each side.}$$

$$\ln\left|\frac{L - y}{y}\right| = -kt - C \qquad \text{Multiply each side by } -1 \text{ and simplify.}$$

$$\left|\frac{L - y}{y}\right| = e^{-kt - C} \qquad \text{Exponentiate each side.}$$

$$\left|\frac{L - y}{y}\right| = e^{-C}e^{-kt} \qquad \text{Property of exponents}$$

$$\frac{L - y}{y} = be^{-kt} \qquad \text{Let } \pm e^{-C} = b.$$

Solving this equation for y produces the general solution $y = \dfrac{L}{1 + be^{-kt}}$.

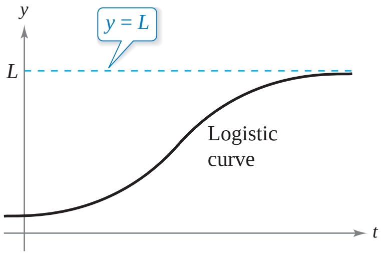
REMARK A review of the method of partial fractions is given in Section 8.5.

> **Exploration**
>
> Use a graphing utility to investigate the effects of the values of L, b, and k on the graph of
>
> $$y = \frac{L}{1 + be^{-kt}}.$$
>
> Include some examples to support your results.

From Example 1, you can conclude that all solutions of the logistic differential equation are of the general form

$$y = \frac{L}{1 + be^{-kt}}.$$

The graph of the function y is called the *logistic curve*, as shown in Figure 6.19. In the next example, you will verify a particular solution of a logistic differential equation and find the initial condition.



Note that as $t \to \infty$, $y \to L$.
Figure 6.19

EXAMPLE 2 Verifying a Particular Solution

Verify that the equation

$$y = \frac{4}{1 + 2e^{-3t}}$$

satisfies the logistic differential equation, and find the initial condition.

Solution Comparing the given equation with the general form derived in Example 1, you know that $L = 4$, $b = 2$, and $k = 3$. You can verify that y satisfies the logistic differential equation as follows.

$$y = 4(1 + 2e^{-3t})^{-1} \qquad \text{Rewrite using negative exponent.}$$

$$y' = 4(-1)(1 + 2e^{-3t})^{-2}(-6e^{-3t}) \qquad \text{Apply Power Rule.}$$

$$= 3\left(\frac{4}{1 + 2e^{-3t}}\right)\left(\frac{2e^{-3t}}{1 + 2e^{-3t}}\right) \qquad \text{Rewrite.}$$

$$= 3y\left(\frac{2e^{-3t}}{1 + 2e^{-3t}}\right) \qquad \text{Rewrite using } y = \frac{4}{1 + 2e^{-3t}}.$$

$$= 3y\left(1 - \frac{1}{1 + 2e^{-3t}}\right) \qquad \text{Rewrite fraction using long division.}$$

$$= 3y\left(1 - \frac{4}{4(1 + 2e^{-3t})}\right) \qquad \text{Multiply fraction by } \frac{4}{4}.$$

$$= 3y\left(1 - \frac{y}{4}\right) \qquad \text{Rewrite using } y = \frac{4}{1 + 2e^{-3t}}.$$

So, y satisfies the logistic differential equation $y' = 3y\left(1 - \frac{y}{4}\right)$. The initial condition can be found by letting $t = 0$ in the given equation.

$$y = \frac{4}{1 + 2e^{-3(0)}} \qquad \text{Let } t = 0.$$

$$= \frac{4}{3} \qquad \text{Simplify.}$$

So, the initial condition is $y(0) = \frac{4}{3}$.

EXAMPLE 3 Verifying the Upper Limit

Verify that the upper limit of

$$y = \frac{4}{1 + 2e^{-3t}}$$

is 4.

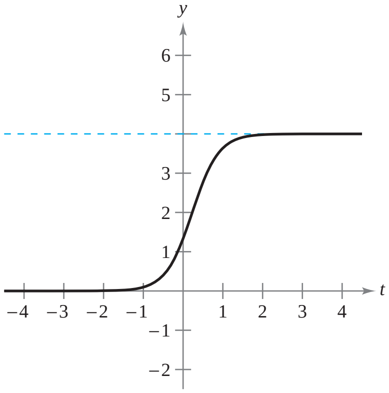


Figure 6.20

Solution In Figure 6.20, you can see that the values of y appear to approach 4 as t increases without bound. You can come to this conclusion numerically, as shown in the table.

t	0	1	2	5	10	100
y	1.3333	3.6378	3.9803	4.0000	4.0000	4.0000

You can obtain the same results analytically, as follows.

$$\lim_{t\to\infty} y = \lim_{t\to\infty} \frac{4}{1 + 2e^{-3t}} = \frac{\displaystyle\lim_{t\to\infty} 4}{\displaystyle\lim_{t\to\infty}(1 + 2e^{-3t})} = \frac{4}{1 + 0} = 4$$

The upper limit of y is 4, which is also the carrying capacity $L = 4$.

EXAMPLE 4 Determining the Point of Inflection

Sketch a graph of

$$y = \frac{4}{1 + 2^{-3t}}.$$

Calculate y'' in terms of y and y'. Then determine the point of inflection.

Solution From Example 2, you know that

$$y' = 3y\left(1 - \frac{y}{4}\right).$$

Now calculate y'' in terms of y and y'.

$$y'' = 3y\left(-\frac{y'}{4}\right) + \left(1 - \frac{y}{4}\right)3y' \qquad \text{Differentiate using Product Rule.}$$

$$y'' = 3y'\left(1 - \frac{y}{2}\right) \qquad \text{Factor and simplify.}$$

When $2 < y < 4$, $y'' < 0$ and the graph of y is concave downward. When $0 < y < 2$, $y'' > 0$ and the graph of y is concave upward. So, a point of inflection must occur at $y = 2$. The corresponding t-value is

$$2 = \frac{4}{1 + 2e^{-3t}} \;\Rightarrow\; 1 + 2e^{-3t} = 2 \;\Rightarrow\; e^{-3t} = \frac{1}{2} \;\Rightarrow\; t = \frac{1}{3}\ln 2.$$

The point of inflection is $\left(\frac{1}{3}\ln 2, 2\right)$, as shown in Figure 6.21.

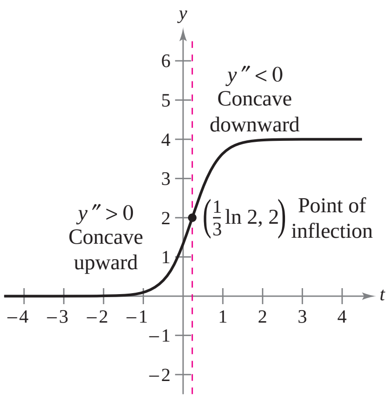


Figure 6.21

In Example 4, the point of inflection occurs at $y = \frac{L}{2}$. This is true for any logistic growth curve for which the solution starts below the carrying capacity L (see Exercise 37).

EXAMPLE 5 Graphing a Slope Field and Solution Curves

Graph a slope field for the logistic differential equation

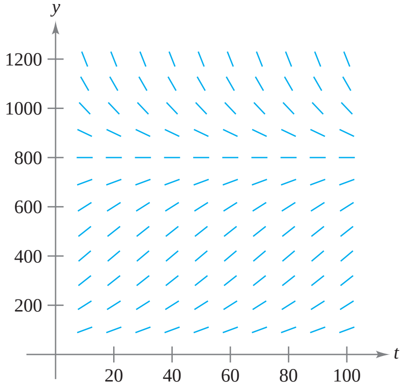
$$y' = 0.05y\left(1 - \frac{y}{800}\right).$$

Then graph solution curves for the initial conditions $y(0) = 200$, $y(0) = 1200$, and $y(0) = 800$.

Solution You can use a graphing utility to graph the slope field shown in Figure 6.22. The solution curves for the initial conditions

$$y(0) = 200, \quad y(0) = 1200, \quad \text{and} \quad y(0) = 800$$

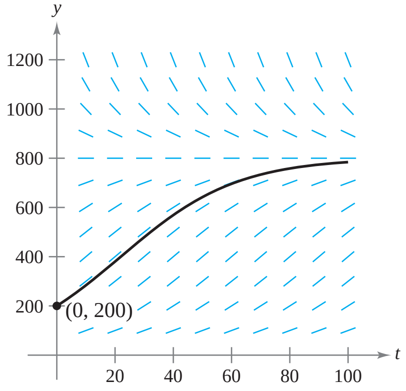
are shown in Figures 6.23–6.25.



Slope field for

$$y' = 0.05y\left(1 - \frac{y}{800}\right)$$

Figure 6.22

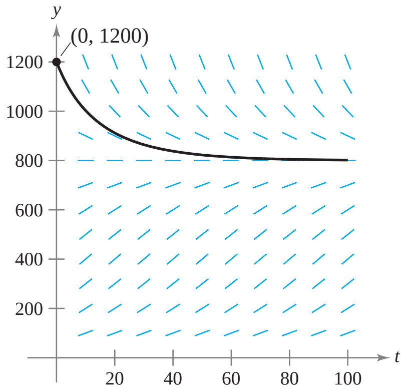


Particular solution for

$$y' = 0.05y\left(1 - \frac{y}{800}\right)$$

and initial condition $y(0) = 200$

Figure 6.23

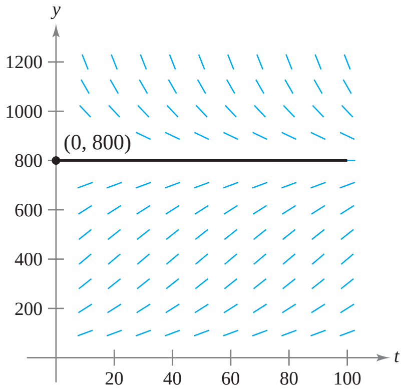


Particular solution for

$$y' = 0.05y\left(1 - \frac{y}{800}\right)$$

and initial condition $y(0) = 1200$

Figure 6.24



Particular solution for

$$y' = 0.05y\left(1 - \frac{y}{800}\right)$$

and initial condition $y(0) = 800$

Figure 6.25

Note that as t increases without bound, the solution curves in Figures 6.23–6.25 all tend to the same limit, which is the carrying capacity of 800.

Application

EXAMPLE 6 Solving a Logistic Differential Equation

A state game commission releases 40 elk into a game refuge. After 5 years, the elk population is 104. The commission believes that the environment can support no more than 4000 elk. The growth rate of the elk population p is

$$\frac{dp}{dt} = kp\left(1 - \frac{p}{4000}\right), \quad 40 \le p \le 4000$$

where t is the number of years.

a. Write a model for the elk population in terms of t.

b. Graph the slope field for the differential equation and the solution that passes through the point $(0, 40)$.

c. Use the model to estimate the elk population after 15 years.

d. Find the limit of the model as $t \to \infty$.

Solution

a. You know that $L = 4000$. So, the solution of the equation is of the form

$$p = \frac{4000}{1 + be^{-kt}}.$$

Because $p(0) = 40$, you can solve for b as follows.

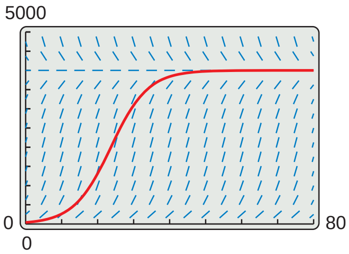
$$40 = \frac{4000}{1 + be^{-k(0)}} \quad \Rightarrow \quad 40 = \frac{4000}{1 + b} \quad \Rightarrow \quad b = 99$$

Then, because $p = 104$ when $t = 5$, you can solve for k.

$$104 = \frac{4000}{1 + 99e^{-k(5)}} \quad \Rightarrow \quad k \approx 0.194$$

So, a model for the elk population is

$$p = \frac{4000}{1 + 99e^{-0.194t}}.$$



Slope field for
$\frac{dp}{dt} = 0.194p\left(1 - \frac{p}{4000}\right)$
and the solution passing through $(0, 40)$
Figure 6.26

b. Using a graphing utility, you can graph the slope field of

$$\frac{dp}{dt} = 0.194p\left(1 - \frac{p}{4000}\right)$$

and the solution that passes through $(0, 40)$, as shown in Figure 6.26.

c. To estimate the elk population after 15 years, substitute 15 for t in the model.

$$p = \frac{4000}{1 + 99e^{-0.194(15)}} \qquad \text{Substitute 15 for } t.$$

$$= \frac{4000}{1 + 99e^{-2.91}} \qquad \text{Simplify.}$$

$$\approx 626 \qquad \text{Simplify.}$$

d. As t increases without bound, the denominator of

$$\frac{4000}{1 + 99e^{-0.194t}}$$

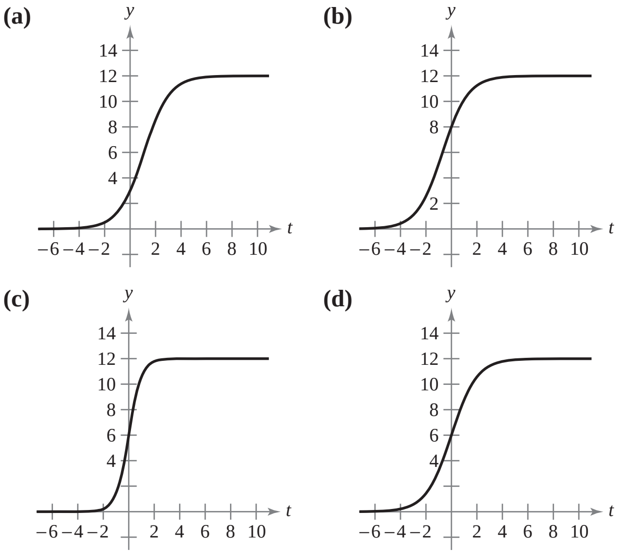
gets closer and closer to 1. So,

$$\lim_{t \to \infty} \frac{4000}{1 + 99e^{-0.194t}} = 4000.$$

6.4 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Matching In Exercises 1–4, match the logistic equation with its graph. [The graphs are labeled (a), (b), (c), and (d).]



1. $y = \dfrac{12}{1 + e^{-t}}$

2. $y = \dfrac{12}{1 + 3e^{-t}}$

3. $y = \dfrac{12}{1 + \frac{1}{2}e^{-t}}$

4. $y = \dfrac{12}{1 + e^{-2t}}$

Verifying a Particular Solution In Exercises 5–8, verify that the equation satisfies the logistic differential equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right).$$

Then find the initial condition.

5. $y = \dfrac{8}{1 + e^{-2t}}$

6. $y = \dfrac{10}{1 + 3e^{-4t}}$

7. $y = \dfrac{12}{1 + 6e^{-t}}$

8. $y = \dfrac{14}{1 + 5e^{-3t}}$

Using a Logistic Equation In Exercises 9–12, the logistic equation models the growth of a population. Use the equation to (a) find the value of k, (b) find the carrying capacity, (c) find the initial population, (d) determine when the population will reach 50% of its carrying capacity, and (e) write a logistic differential equation that has the solution $P(t)$.

9. $P(t) = \dfrac{2100}{1 + 29e^{-0.75t}}$

10. $P(t) = \dfrac{5000}{1 + 39e^{-0.2t}}$

11. $P(t) = \dfrac{6000}{1 + 4999e^{-0.8t}}$

12. $P(t) = \dfrac{1000}{1 + 8e^{-0.2t}}$

Using a Logistic Differential Equation In Exercises 13–16, the logistic differential equation models the growth rate of a population. Use the equation to (a) find the value of k, (b) find the carrying capacity, (c) use a computer algebra system to graph a slope field, and (d) determine the value of P at which the population growth rate is the greatest.

13. $\dfrac{dP}{dt} = 3P\left(1 - \dfrac{P}{100}\right)$

14. $\dfrac{dP}{dt} = 0.5P\left(1 - \dfrac{P}{250}\right)$

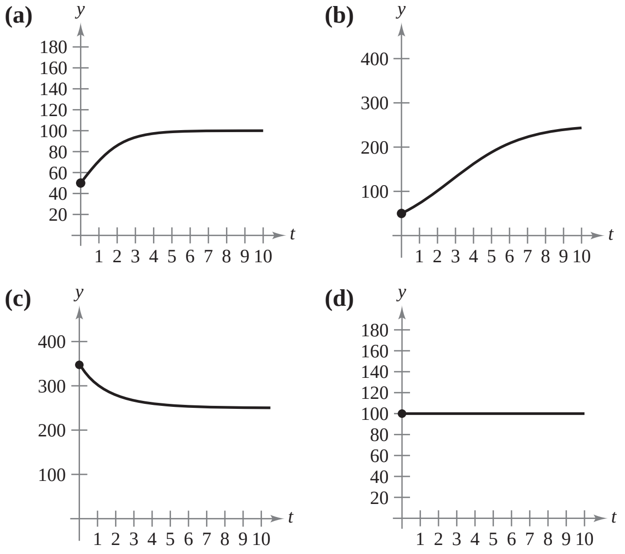
15. $\dfrac{dP}{dt} = 0.1P - 0.0004P^2$

16. $\dfrac{dP}{dt} = 0.4P - 0.00025P^2$

Solving a Logistic Differential Equation In Exercises 17–20, find the logistic equation that satisfies the initial condition. Then use the logistic equation to find y when $t = 5$ and $t = 100$.

Logistic Differential Equation	Initial Condition
17. $\dfrac{dy}{dt} = y\left(1 - \dfrac{y}{36}\right)$	(0, 4)
18. $\dfrac{dy}{dt} = 2.8y\left(1 - \dfrac{y}{10}\right)$	(0, 7)
19. $\dfrac{dy}{dt} = \dfrac{4y}{5} - \dfrac{y^2}{150}$	(0, 8)
20. $\dfrac{dy}{dt} = \dfrac{3y}{20} - \dfrac{y^2}{1600}$	(0, 15)

Matching In Exercises 21–24, match the logistic differential equation and initial condition with the graph of its solution. [The graphs are labeled (a), (b), (c), and (d).]



21. $\dfrac{dy}{dt} = 0.5y\left(1 - \dfrac{y}{250}\right)$, (0, 350)

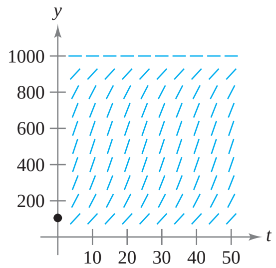
22. $\dfrac{dy}{dt} = 0.9y\left(1 - \dfrac{y}{100}\right)$, (0, 100)

23. $\dfrac{dy}{dt} = 0.5y\left(1 - \dfrac{y}{250}\right)$, (0, 50)

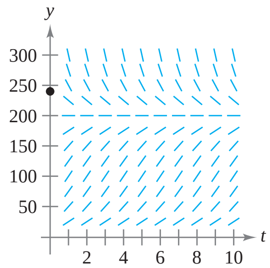
24. $\dfrac{dy}{dt} = 0.9y\left(1 - \dfrac{y}{100}\right)$, (0, 50)

Slope Field **In Exercises 25 and 26, a logistic differential equation, a point, and a slope field are given. (a) Sketch two approximate solutions of the differential equation on the slope field, one of which passes through the given point. (b) Find the particular solution of the differential equation and use a graphing utility to graph the solution. Compare the result with the sketch in part (a). To print an enlarged copy of the graph, go to *MathGraphs.com*.**

25. $\dfrac{dy}{dt} = 0.2y\left(1 - \dfrac{y}{1000}\right)$, $(0, 105)$



26. $\dfrac{dy}{dt} = 0.9y\left(1 - \dfrac{y}{200}\right)$, $(0, 240)$



WRITING ABOUT CONCEPTS

27. Describing a Value Describe what the value of L represents in the logistic differential equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right).$$

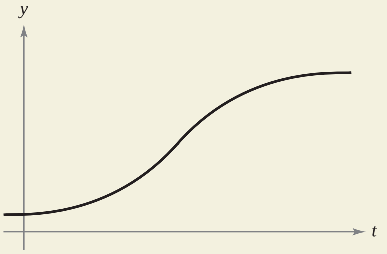
28. Determining Values It is known that $y = \dfrac{L}{1 + be^{-kt}}$ is a solution of the logistic differential equation

$$\frac{dy}{dt} = 0.75y\left(1 - \frac{y}{2500}\right).$$

Is it possible to determine L, k, and b from the information given? If so, find their values. If not, which value(s) cannot be determined and what information do you need to determine the value(s)?

29. Separation of Variables Is the logistic differential equation separable? Explain.

30. HOW DO YOU SEE IT? The growth of a population is modeled by a logistic equation as shown in the graph below. What happens to the rate of growth as the population increases? What do you think causes this to occur in real-life situations, such as animal or human populations?



31. Endangered Species A conservation organization releases 25 Florida panthers into a game preserve. After 2 years, there are 39 panthers in the preserve. The Florida preserve has a carrying capacity of 200 panthers.

(a) Write a logistic equation that models the population of panthers in the preserve.

(b) Find the population after 5 years.

(c) When will the population reach 100?

(d) Write a logistic differential equation that models the growth rate of the panther population. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answer.

(e) After how many years is the panther population growing most rapidly? Explain.

32. Bacteria Growth At time $t = 0$, a bacterial culture weighs 1 gram. Two hours later, the culture weighs 4 grams. The maximum weight of the culture is 20 grams.

(a) Write a logistic equation that models the weight of the bacterial culture.

(b) Find the culture's weight after 5 hours.

(c) When will the culture's weight reach 18 grams?

(d) Write a logistic differential equation that models the growth rate of the culture's weight. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answer.

(e) After how many hours is the culture's weight increasing most rapidly? Explain.

True or False? **In Exercises 33 and 34, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

33. For the logistic differential equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$$

if $y > L$, then $dy/dt > 0$ and the population increases.

34. For the logistic differential equation

$$\frac{dy}{dt} = ky\left(1 - \frac{y}{L}\right)$$

if $0 < y < L$, then $dy/dt > 0$ and the population increases.

35. Think About It The growth of a population is modeled by a logistic equation. As the population increases, its rate of growth decreases. What do you think causes this to occur in real-life situations such as in animal or human populations?

36. Finding a Derivative Show that if $y = \dfrac{1}{1 + be^{-kt}}$, then

$$\frac{dy}{dt} = ky(1 - y).$$

37. Point of Inflection For any logistic growth curve, show that the point of inflection occurs at $y = \dfrac{L}{2}$ when the solution starts below the carrying capacity L.

6.5 First-Order Linear Differential Equations

- **Solve a first-order linear differential equation, and use linear differential equations to solve applied problems.**

First-Order Linear Differential Equations

In this section, you will see how to solve a very important class of first-order differential equations—first-order linear differential equations.

ANNA JOHNSON PELL WHEELER (1883–1966)

Anna Johnson Pell Wheeler was awarded a master's degree in 1904 from the University of Iowa for her thesis *The Extension of Galois Theory to Linear Differential Equations*. Influenced by David Hilbert, she worked on integral equations while studying infinite linear spaces.

Definition of First-Order Linear Differential Equation

A **first-order linear differential equation** is an equation of the form

$$\frac{dy}{dx} + P(x)y = Q(x)$$

where P and Q are continuous functions of x. This first-order linear differential equation is said to be in **standard form.**

To solve a linear differential equation, write it in standard form to identify the functions $P(x)$ and $Q(x)$. Then integrate $P(x)$ and form the expression

$$u(x) = e^{\int P(x)\,dx} \qquad \text{Integrating factor}$$

which is called an **integrating factor.** The general solution of the equation is

$$y = \frac{1}{u(x)}\int Q(x)u(x)\,dx. \qquad \text{General solution}$$

It is instructive to see why the integrating factor helps solve a linear differential equation of the form $y' + P(x)y = Q(x)$. When both sides of the equation are multiplied by the integrating factor $u(x) = e^{\int P(x)\,dx}$, the left-hand side becomes the derivative of a product.

$$y'e^{\int P(x)\,dx} + P(x)ye^{\int P(x)\,dx} = Q(x)e^{\int P(x)\,dx}$$

$$\left[ye^{\int P(x)\,dx}\right]' = Q(x)e^{\int P(x)\,dx}$$

Integrating both sides of this second equation and dividing by $u(x)$ produce the general solution.

EXAMPLE 1 Solving a Linear Differential Equation

Find the general solution of

$$y' + y = e^x.$$

Solution For this equation, $P(x) = 1$ and $Q(x) = e^x$. So, the integrating factor is

$$u(x) = e^{\int P(x)\,dx} = e^{\int dx} = e^x.$$

This implies that the general solution is

$$\begin{aligned} y &= \frac{1}{u(x)}\int Q(x)u(x)\,dx \\ &= \frac{1}{e^x}\int e^x(e^x)\,dx \\ &= e^{-x}\left(\frac{1}{2}e^{2x} + C\right) \\ &= \frac{1}{2}e^x + Ce^{-x}. \end{aligned}$$

Courtesy of the Visual Collections, Canaday Library, Bryn Mawr College.

REMARK Rather than memorizing the formula in Theorem 6.2, just remember that multiplication by the integrating factor $e^{\int P(x)\,dx}$ converts the left side of the differential equation into the derivative of the product $ye^{\int P(x)\,dx}$.

THEOREM 6.2 Solution of a First-Order Linear Differential Equation

An integrating factor for the first-order linear differential equation

$$y' + P(x)y = Q(x)$$

is $u(x) = e^{\int P(x)\,dx}$. The solution of the differential equation is

$$ye^{\int P(x)\,dx} = \int Q(x)e^{\int P(x)\,dx}\,dx + C.$$

EXAMPLE 2 Solving a First-Order Linear Differential Equation

See LarsonCalculus.com for an interactive version of this type of example.

Find the general solution of $xy' - 2y = x^2$.

Solution The standard form of the equation is

$$y' + \left(-\frac{2}{x}\right)y = x. \qquad \text{Standard form}$$

So, $P(x) = -2/x$, and you have

$$\int P(x)\,dx = -\int \frac{2}{x}\,dx = -\ln x^2$$

which implies that the integrating factor is

$$e^{\int P(x)\,dx} = e^{-\ln x^2} = \frac{1}{e^{\ln x^2}} = \frac{1}{x^2}. \qquad \text{Integrating factor}$$

So, multiplying each side of the standard form by $1/x^2$ yields

$$\frac{y'}{x^2} - \frac{2y}{x^3} = \frac{1}{x}$$

$$\frac{d}{dx}\left[\frac{y}{x^2}\right] = \frac{1}{x}$$

$$\frac{y}{x^2} = \int \frac{1}{x}\,dx$$

$$\frac{y}{x^2} = \ln|x| + C$$

$$y = x^2(\ln|x| + C). \qquad \text{General solution}$$

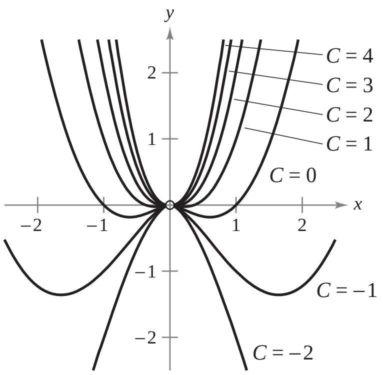


Figure 6.27

Several solution curves (for $C = -2, -1, 0, 1, 2, 3,$ and 4) are shown in Figure 6.27.

In most falling-body problems discussed so far in the text, air resistance has been neglected. The next example includes this factor. In the example, the air resistance on the falling object is assumed to be proportional to its velocity v. If g is the gravitational constant, the downward force F on a falling object of mass m is given by the difference $mg - kv$. If a is the acceleration of the object, then by Newton's Second Law of Motion,

$$F = ma = m\frac{dv}{dt}$$

which yields the following differential equation.

$$m\frac{dv}{dt} = mg - kv \quad \Longrightarrow \quad \frac{dv}{dt} + \frac{kv}{m} = g$$

EXAMPLE 3 A Falling Object with Air Resistance

An object of mass m is dropped from a hovering helicopter. The air resistance is proportional to the velocity of the object. Find the velocity of the object as a function of time t.

Solution The velocity v satisfies the equation

$$\frac{dv}{dt} + \frac{kv}{m} = g. \qquad g = \text{gravitational constant, } k = \text{constant of proportionality}$$

Letting $b = k/m$, you can *separate variables* to obtain

$$\begin{aligned} dv &= (g - bv)\,dt \\ \int \frac{dv}{g - bv} &= \int dt \\ -\frac{1}{b}\ln|g - bv| &= t + C_1 \\ \ln|g - bv| &= -bt - bC_1 \\ g - bv &= Ce^{-bt}. \qquad C = e^{-bC_1} \end{aligned}$$

Because the object was dropped, $v = 0$ when $t = 0$; so $g = C$, and it follows that

$$-bv = -g + ge^{-bt} \quad \Longrightarrow \quad v = \frac{g - ge^{-bt}}{b} = \frac{mg}{k}\left(1 - e^{-kt/m}\right).$$

REMARK Notice in Example 3 that the velocity approaches a limit of mg/k as a result of the air resistance. For falling-body problems in which air resistance is neglected, the velocity increases without bound.

A simple electric circuit consists of an electric current I (in amperes), a resistance R (in ohms), an inductance L (in henrys), and a constant electromotive force E (in volts), as shown in Figure 6.28. According to Kirchhoff's Second Law, if the switch S is closed when $t = 0$, then the applied electromotive force (voltage) is equal to the sum of the voltage drops in the rest of the circuit. This, in turn, means that the current I satisfies the differential equation

$$L\frac{dI}{dt} + RI = E.$$

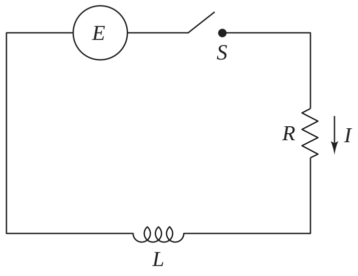


Figure 6.28

EXAMPLE 4 An Electric Circuit Problem

Find the current I as a function of time t (in seconds), given that I satisfies the differential equation $L(dI/dt) + RI = \sin 2t$, where R and L are nonzero constants.

Solution In standard form, the given linear equation is

$$\frac{dI}{dt} + \frac{R}{L}I = \frac{1}{L}\sin 2t.$$

Let $P(t) = R/L$, so that $e^{\int P(t)\,dt} = e^{(R/L)t}$, and, by Theorem 6.2,

$$\begin{aligned} Ie^{(R/L)t} &= \frac{1}{L}\int e^{(R/L)t}\sin 2t\,dt \\ &= \frac{1}{4L^2 + R^2}e^{(R/L)t}(R\sin 2t - 2L\cos 2t) + C. \end{aligned}$$

So, the general solution is

$$\begin{aligned} I &= e^{-(R/L)t}\left[\frac{1}{4L^2 + R^2}e^{(R/L)t}(R\sin 2t - 2L\cos 2t) + C\right] \\ &= \frac{1}{4L^2 + R^2}(R\sin 2t - 2L\cos 2t) + Ce^{-(R/L)t}. \end{aligned}$$

TECHNOLOGY The integral in Example 4 was found using a computer algebra system. If you have access to *Maple*, *Mathematica*, or the *TI-Nspire*, try using it to integrate

$$\frac{1}{L}\int e^{(R/L)t}\sin 2t\,dt.$$

In Chapter 8, you will learn how to integrate functions of this type using integration by parts.

One type of problem that can be described in terms of a differential equation involves chemical mixtures, as illustrated in the next example.

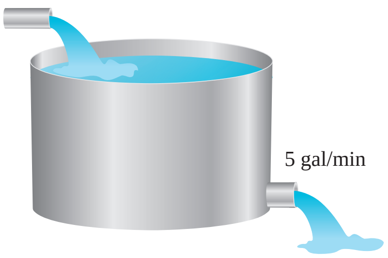


Figure 6.29

EXAMPLE 5 A Mixture Problem

A tank contains 50 gallons of a solution composed of 90% water and 10% alcohol. A second solution containing 50% water and 50% alcohol is added to the tank at the rate of 4 gallons per minute. As the second solution is being added, the tank is being drained at a rate of 5 gallons per minute, as shown in Figure 6.29. The solution in the tank is stirred constantly. How much alcohol is in the tank after 10 minutes?

Solution Let y be the number of gallons of alcohol in the tank at any time t. You know that $y = 5$ when $t = 0$. Because the number of gallons of solution in the tank at any time is $50 - t$, and the tank loses 5 gallons of solution per minute, it must lose

$$\left(\frac{5}{50 - t}\right)y$$

gallons of alcohol per minute. Furthermore, because the tank is gaining 2 gallons of alcohol per minute, the rate of change of alcohol in the tank is

$$\frac{dy}{dt} = 2 - \left(\frac{5}{50 - t}\right)y \quad \Longrightarrow \quad \frac{dy}{dt} + \left(\frac{5}{50 - t}\right)y = 2.$$

To solve this linear differential equation, let

$$P(t) = \frac{5}{50 - t}$$

and obtain

$$\int P(t)\,dt = \int \frac{5}{50 - t}\,dt = -5\ln|50 - t|.$$

Because $t < 50$, you can drop the absolute value signs and conclude that

$$e^{\int P(t)\,dt} = e^{-5\ln(50 - t)} = \frac{1}{(50 - t)^5}.$$

So, the general solution is

$$\frac{y}{(50 - t)^5} = \int \frac{2}{(50 - t)^5}\,dt$$

$$\frac{y}{(50 - t)^5} = \frac{1}{2(50 - t)^4} + C$$

$$y = \frac{50 - t}{2} + C(50 - t)^5.$$

Because $y = 5$ when $t = 0$, you have

$$5 = \frac{50}{2} + C(50)^5 \quad \Longrightarrow \quad -\frac{20}{50^5} = C$$

which means that the particular solution is

$$y = \frac{50 - t}{2} - 20\left(\frac{50 - t}{50}\right)^5.$$

Finally, when $t = 10$, the amount of alcohol in the tank is

$$y = \frac{50 - 10}{2} - 20\left(\frac{50 - 10}{50}\right)^5 \approx 13.45 \text{ gal}$$

which represents a solution containing 33.6% alcohol.

6.5 Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

Determining Whether a Differential Equation Is Linear **In Exercises 1–4, determine whether the differential equation is linear. Explain your reasoning.**

1. $x^3y' + xy = e^x + 1$

2. $2xy - y' \ln x = y$

3. $y' - y \sin x = xy^2$

4. $\dfrac{2 - y'}{y} = 5x$

Solving a First-Order Linear Differential Equation **In Exercises 5–14, solve the first-order linear differential equation.**

5. $\dfrac{dy}{dx} + \left(\dfrac{1}{x}\right)y = 6x + 2$

6. $\dfrac{dy}{dx} + \left(\dfrac{2}{x}\right)y = 3x - 5$

7. $y' - y = 16$

8. $y' + 2xy = 10x$

9. $(y + 1) \cos x\, dx - dy = 0$

10. $(y - 1) \sin x\, dx - dy = 0$

11. $(x - 1)y' + y = x^2 - 1$

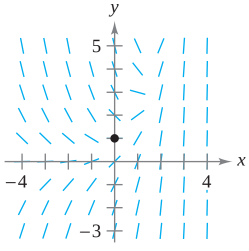
12. $y' + 3y = e^{3x}$

13. $y' - 3x^2y = e^{x^3}$

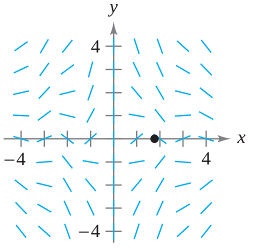
14. $y' + y \tan x = \sec x$

Slope Field **In Exercises 15 and 16, (a) sketch an approximate solution of the differential equation satisfying the given initial condition by hand on the slope field, (b) find the particular solution that satisfies the given initial condition, and (c) use a graphing utility to graph the particular solution. Compare the graph with the hand-drawn graph in part (a). To print an enlarged copy of the graph, go to *MathGraphs.com*.**

15. $\dfrac{dy}{dx} = e^x - y$, $(0, 1)$



16. $y' + \left(\dfrac{1}{x}\right)y = \sin x^2$, $(\sqrt{\pi}, 0)$



Finding a Particular Solution **In Exercises 17–24, find the particular solution of the differential equation that satisfies the initial condition.**

	Differential Equation	Initial Condition
17.	$y' \cos^2 x + y - 1 = 0$	$y(0) = 5$
18.	$x^3y' + 2y = e^{1/x^2}$	$y(1) = e$
19.	$y' + y \tan x = \sec x + \cos x$	$y(0) = 1$
20.	$y' + y \sec x = \sec x$	$y(0) = 4$
21.	$y' + \left(\dfrac{1}{x}\right)y = 0$	$y(2) = 2$
22.	$y' + (2x - 1)y = 0$	$y(1) = 2$
23.	$x\, dy = (x + y + 2)\, dx$	$y(1) = 10$
24.	$2xy' - y = x^3 - x$	$y(4) = 2$

25. Population Growth When predicting population growth, demographers must consider birth and death rates as well as the net change caused by the difference between the rates of immigration and emigration. Let P be the population at time t and let N be the net increase per unit time resulting from the difference between immigration and emigration. So, the rate of growth of the population is given by

$$\frac{dP}{dt} = kP + N$$

where N is constant. Solve this differential equation to find P as a function of time, when at time $t = 0$ the size of the population is P_0.

26. Investment Growth A large corporation starts at time $t = 0$ to invest part of its receipts continuously at a rate of P dollars per year in a fund for future corporate expansion. Assume that the fund earns r percent interest per year compounded continuously. So, the rate of growth of the amount A in the fund is given by

$$\frac{dA}{dt} = rA + P$$

where $A = 0$ when $t = 0$. Solve this differential equation for A as a function of t.

Investment Growth **In Exercises 27 and 28, use the result of Exercise 26.**

27. Find A for the following.

(a) $P = \$275{,}000$, $r = 8\%$, $t = 10$ years

(b) $P = \$550{,}000$, $r = 5.9\%$, $t = 25$ years

28. Find t if the corporation needs \$1,000,000 and it can invest \$125,000 per year in a fund earning 8% interest compounded continuously.

29. Learning Curve The management at a certain factory has found that the maximum number of units a worker can produce in a day is 75. The rate of increase in the number of units N produced with respect to time t in days by a new employee is proportional to $75 - N$.

(a) Determine the differential equation describing the rate of change of performance with respect to time.

(b) Solve the differential equation from part (a).

(c) Find the particular solution for a new employee who produced 20 units on the first day at the factory and 35 units on the twentieth day.

30. Intravenous Feeding

Glucose is added intravenously to the bloodstream at the rate of q units per minute, and the body removes glucose from the bloodstream at a rate proportional to the amount present. Assume that $Q(t)$ is the amount of glucose in the bloodstream at time t.

(a) Determine the differential equation describing the rate of change of glucose in the bloodstream with respect to time.

(b) Solve the differential equation from part (a), letting $Q = Q_0$ when $t = 0$.

(c) Find the limit of $Q(t)$ as $t \to \infty$.

Falling Object In Exercises 31 and 32, consider an eight-pound object dropped from a height of 5000 feet, where the air resistance is proportional to the velocity.

31. Write the velocity of the object as a function of time when the velocity after 5 seconds is approximately -101 feet per second. What is the limiting value of the velocity function?

32. Use the result of Exercise 31 to write the position of the object as a function of time. Approximate the velocity of the object when it reaches ground level.

Electric Circuits In Exercises 33 and 34, use the differential equation for electric circuits given by

$$L\frac{dI}{dt} + RI = E.$$

In this equation, I is the current, R is the resistance, L is the inductance, and E is the electromotive force (voltage).

33. Solve the differential equation for the current given a constant voltage E_0.

34. Use the result of Exercise 33 to find the equation for the current when $I(0) = 0$, $E_0 = 120$ volts, $R = 600$ ohms, and $L = 4$ henrys. When does the current reach 90% of its limiting value?

Mixture In Exercises 35–38, consider a tank that at time $t = 0$ contains v_0 gallons of a solution of which, by weight, q_0 pounds is soluble concentrate. Another solution containing q_1 pounds of the concentrate per gallon is running into the tank at the rate of r_1 gallons per minute. The solution in the tank is kept well stirred and is withdrawn at the rate of r_2 gallons per minute.

35. Let Q be the amount of concentrate in the solution at any time t. Show that

$$\frac{dQ}{dt} + \frac{r_2 Q}{v_0 + (r_1 - r_2)t} = q_1 r_1.$$

36. Let Q be the amount of concentrate in the solution at any time t. Write the differential equation for the rate of change of Q with respect to t when $r_1 = r_2 = r$.

37. A 200-gallon tank is full of a solution containing 25 pounds of concentrate. Starting at time $t = 0$, distilled water is admitted to the tank at a rate of 10 gallons per minute, and the well-stirred solution is withdrawn at the same rate.

(a) Find the amount of concentrate Q in the solution as a function of t.

(b) Find the time at which the amount of concentrate in the tank reaches 15 pounds.

(c) Find the quantity of the concentrate in the solution as $t \to \infty$.

38. A 200-gallon tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 pound of concentrate per gallon enters the tank at the rate of 5 gallons per minute, and the well-stirred mixture is withdrawn at the rate of 3 gallons per minute.

(a) At what time will the tank be full?

(b) At the time the tank is full, how many pounds of concentrate will it contain?

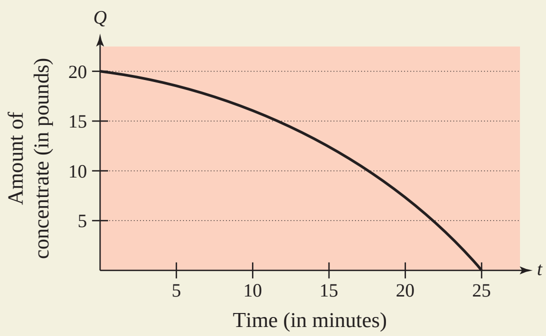
(c) Repeat parts (a) and (b), assuming that the solution entering the tank contains 1 pound of concentrate per gallon.

39. Using an Integrating Factor The expression $u(x)$ is an integrating factor for $y' + P(x)y = Q(x)$. Which of the following is equal to $u'(x)$? Verify your answer.

(a) $P(x)u(x)$ (b) $P'(x)u(x)$

(c) $Q(x)u(x)$ (d) $Q'(x)u(x)$

40. HOW DO YOU SEE IT? The graph shows the amount of concentrate Q (in pounds) in a solution in a tank at time t (in minutes) as a solution with concentrate enters the tank, is well stirred, and is withdrawn from the tank.



(a) How much concentrate is in the tank at time $t = 0$?

(b) Which is greater, the rate of solution into the tank, or the rate of solution withdrawn from the tank? Explain.

(c) At what time is there no concentrate in the tank? What does this mean?

WRITING ABOUT CONCEPTS

41. Standard Form Give the standard form of a first-order linear differential equation. What is its integrating factor?

42. First-Order What does the term "first-order" refer to in a first-order linear differential equation?

Auremar/Shutterstock.com

Matching **In Exercises 43–46, match the differential equation with its solution.**

Differential Equation	Solution
43. $y' - 2x = 0$	(a) $y = Ce^{x^2}$
44. $y' - 2y = 0$	(b) $y = -\frac{1}{2} + Ce^{x^2}$
45. $y' - 2xy = 0$	(c) $y = x^2 + C$
46. $y' - 2xy = x$	(d) $y = Ce^{2x}$

Slope Field **In Exercises 47–50, (a) use a graphing utility to graph the slope field for the differential equation, (b) find the particular solutions of the differential equation passing through the given points, and (c) use a graphing utility to graph the particular solutions on the slope field.**

Differential Equation	Points
47. $\frac{dy}{dx} - \frac{1}{x}y = x^2$	$(-2, 4), (2, 8)$
48. $\frac{dy}{dx} + 4x^3y = x^3$	$\left(0, \frac{7}{2}\right), \left(0, -\frac{1}{2}\right)$
49. $\frac{dy}{dx} + (\cot x)y = 2$	$(1, 1), (3, -1)$
50. $\frac{dy}{dx} + 2xy = xy^2$	$(0, 3), (0, 1)$

Solving a First-Order Linear Differential Equation **In Exercises 51–58, solve the first-order differential equation by any appropriate method.**

51. $\frac{dy}{dx} = \frac{e^{2x+y}}{e^{x-y}}$

52. $\frac{dy}{dx} = \frac{x-3}{y(y+4)}$

53. $y \cos x - \cos x + \frac{dy}{dx} = 0$

54. $y' = 2x\sqrt{1 - y^2}$

55. $(2y - e^x)\,dx + x\,dy = 0$

56. $(x + y)\,dx - x\,dy = 0$

57. $3(y - 4x^2)\,dx + x\,dy = 0$

58. $x\,dx + (y + e^y)(x^2 + 1)\,dy = 0$

Solving a Bernoulli Differential Equation **In Exercises 59–66, solve the Bernoulli differential equation. The *Bernoulli equation* is a well-known nonlinear equation of the form**

$$y' + P(x)y = Q(x)y^n$$

that can be reduced to a linear form by a substitution. The general solution of a Bernoulli equation is

$$y^{1-n}e^{\int(1-n)P(x)\,dx} = \int (1-n)Q(x)e^{\int(1-n)P(x)\,dx}\,dx + C.$$

59. $y' + 3x^2y = x^2y^3$

60. $y' + xy = xy^{-1}$

61. $y' + \left(\frac{1}{x}\right)y = xy^2$

62. $y' + \left(\frac{1}{x}\right)y = x\sqrt{y}$

63. $xy' + y = xy^3$

64. $y' - y = y^3$

65. $y' - y = e^x\sqrt[3]{y}$

66. $yy' - 2y^2 = e^x$

True or False? **In Exercises 67 and 68, determine whether the statement is true or false. If it is false, explain why or give an example that shows it is false.**

67. $y' + x\sqrt{y} = x^2$ is a first-order linear differential equation.

68. $y' + xy = e^xy$ is a first-order linear differential equation.

SECTION PROJECT

Weight Loss

A person's weight depends on both the number of calories consumed and the energy used. Moreover, the amount of energy used depends on a person's weight—the average amount of energy used by a person is 17.5 calories per pound per day. So, the more weight a person loses, the less energy a person uses (assuming that the person maintains a constant level of activity). An equation that can be used to model weight loss is

$$\frac{dw}{dt} = \frac{C}{3500} - \frac{17.5}{3500}w$$

where w is the person's weight (in pounds), t is the time in days, and C is the constant daily calorie consumption.

(a) Find the general solution of the differential equation.

(b) Consider a person who weighs 180 pounds and begins a diet of 2500 calories per day. How long will it take the person to lose 10 pounds? How long will it take the person to lose 35 pounds?

(c) Use a graphing utility to graph the solution. What is the "limiting" weight of the person?

(d) Repeat parts (b) and (c) for a person who weighs 200 pounds when the diet is started.

FOR FURTHER INFORMATION For more information on modeling weight loss, see the article "A Linear Diet Model" by Arthur C. Segal in *The College Mathematics Journal.*

6.6 Predator-Prey Differential Equations

- **Analyze predator-prey differential equations.**
- **Analyze competing-species differential equations.**

Predator-Prey Differential Equations

ALFRED LOTKA (1880–1949)
VITO VOLTERRA (1860–1940)

Although Alfred Lotka and Vito Volterra both worked on other problems, they are most known for their work on predator-prey equations. Lotka was also a statistician, and Volterra did work in the development of integral equations and functional analysis.

In the 1920s, mathematicians Alfred Lotka (1880–1949) and Vito Volterra (1860–1940) independently developed mathematical models to represent many of the different ways in which two species can interact with each other. Two common ways in which species interact with each other are as predator and prey, and as competing species.

Consider a predator-prey relationship involving foxes (predators) and rabbits (prey). Assume that the rabbits are the primary food source for the foxes, the rabbits have an unlimited food supply, and there is no threat to the rabbits other than from the foxes. Let x represent the number of rabbits, let y represent the number of foxes, and let t represent time. When there are no foxes, the rabbit population grows according to the exponential growth model

$$\frac{dx}{dt} = ax, \quad a > 0.$$

When there are foxes but no rabbits, the foxes have no food and their population decays according to the exponential decay model

$$\frac{dy}{dt} = -my, \quad m > 0.$$

When both foxes and rabbits are present, there is an interaction rate of *decline* for the rabbit population given by $-bxy$, and an interaction rate of *increase* in the fox population given by nxy, where $b, n > 0$. So, the rates of change of each population can be modeled by the following predator-prey system of differential equations.

$$\frac{dx}{dt} = ax - bxy \qquad \text{Rate of change of prey}$$

$$\frac{dy}{dt} = -my + nxy \qquad \text{Rate of change of predators}$$

These equations are called **predator-prey equations** or **Lotka-Volterra equations.** The equations are **autonomous** because the rates of change do not depend explicitly on time t.

In general, it is not possible to solve the predator-prey equations explicitly for x and y. However, you can use techniques such as Euler's Method to approximate solutions. Also, you can discover properties of the solutions by analyzing the differential equations.

EXAMPLE 1 Analyzing Predator-Prey Equations

Write the predator-prey equations for $a = 0.04$, $b = 0.002$, $m = 0.08$, and $n = 0.0004$. Then find the values of x and y for which $dx/dt = dy/dt = 0$.

Solution For $a = 0.04$, $b = 0.002$, $m = 0.08$, and $n = 0.0004$, the predator-prey equations are shown below.

$$\frac{dx}{dt} = 0.04x - 0.002xy \qquad \text{Rate of change of prey}$$

$$\frac{dy}{dt} = -0.08y + 0.0004xy \qquad \text{Rate of change of predators}$$

Solving $dx/dt = x(0.04 - 0.002y) = 0$ and $dy/dt = y(-0.08 + 0.0004x) = 0$, you can see that $dx/dt = dy/dt = 0$ when $(x, y) = (0, 0)$ and when $(x, y) = (200, 20)$.

There are two points of interest you should consider when analyzing predator-prey equations. Consider the predator-prey equations

$$\frac{dx}{dt} = ax - bxy \quad \text{and} \quad \frac{dy}{dt} = -my + nxy.$$

$\frac{dx}{dt} = 0$ when $x = 0$ or $y = \frac{a}{b}$, and $\frac{dy}{dt} = 0$ when $y = 0$ or $x = \frac{m}{n}$. So, at the points $(0, 0)$ and $\left(\frac{m}{n}, \frac{a}{b}\right)$, the prey and predator populations are constant. These points are called **critical points** or **equilibrium points** of the predator-prey equations.

EXAMPLE 2 Analyzing Predator-Prey Equations Graphically

Let the predator-prey equations from Example 1

$$\frac{dx}{dt} = 0.04x - 0.002xy \qquad \text{Rate of change of rabbit population}$$

and

$$\frac{dy}{dt} = -0.08y + 0.0004xy \qquad \text{Rate of change of fox population}$$

model a predator-prey relationship involving foxes and rabbits, where x is the number of rabbits and y is the number of foxes after t months. Use a graphing utility to graph the functions x and y when $0 \le t \le 240$ and the initial conditions are 200 rabbits and 10 foxes. What do you observe?

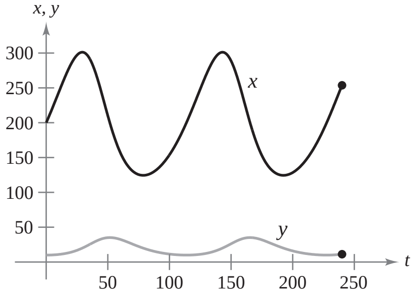


Figure 6.30

Solution The graphs of x and y are shown in Figure 6.30. Here are some observations.

- The rabbit and fox populations oscillate periodically between their respective minimum and maximum values.
- The rabbit population oscillates from about 125 rabbits to about 300 rabbits.
- The fox population oscillates from about 10 foxes to about 35 foxes.
- About 20 months after the rabbit population peaks, the fox population peaks.
- The period of each population appears to be about 115 months.

In Example 2, the graph shows the curves plotted together with time t along the horizontal axis. You can also use the predator-prey equations dy/dt and dx/dt to graph a slope field. The slope field is graphed using the x-axis to represent the prey and the y-axis to represent the predators.

EXAMPLE 3 Predator-Prey Equations and Slope Fields

Use a graphing utility to graph the slope field of the predator-prey equations given in Example 2.

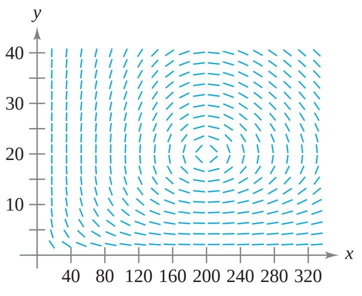


Figure 6.31

Solution The slope field is shown in Figure 6.31. The x-axis represents the rabbit population, and the y-axis represents the fox population.

TECHNOLOGY If you are using a graphing utility, you may need to rewrite the equations as a function of x:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-my + nxy}{ax - bxy}.$$

EXAMPLE 4 Graphing a Solution Curve

Use the predator-prey equations

$$\frac{dx}{dt} = 0.04x - 0.002xy \quad \text{and} \quad \frac{dy}{dt} = -0.08y + 0.0004xy$$

and the slope field from Example 3 to graph the solution curve using the initial conditions of 200 rabbits and 10 foxes. Describe the changes in the populations as you trace the solution curve.

Solution The graph of the solution is a closed curve, as shown in Figures 6.32 and 6.33.

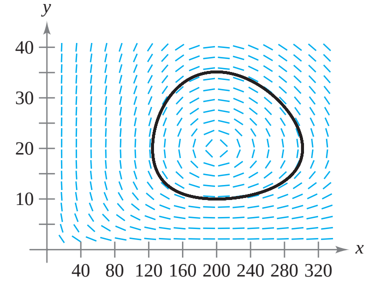


Figure 6.32

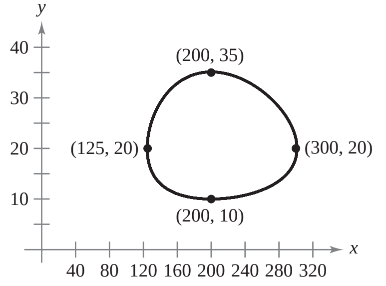


Figure 6.33

At $(200, 10)$, $dy/dt = 0$ and $dx/dt = 4$. So, the rabbit population is increasing at $(200, 10)$. This means that you should trace the curve counterclockwise as t increases. As you trace the curve, note the changes listed in Figure 6.34.

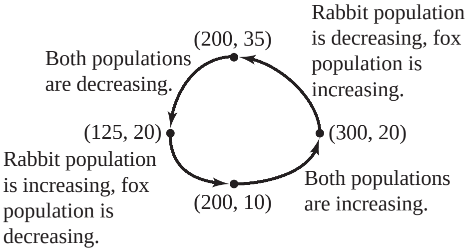


Figure 6.34

Although it is generally not possible to solve predator-prey equations explicitly for x and y, you can separate variables to derive an implicit solution. Begin by writing the equations dy/dt and dx/dt as a function of x.

$$\frac{dy}{dx} = \frac{y(-m + nx)}{x(a - by)} \qquad \text{Factor numerator and denominator.}$$

$$x(a - by)\,dy = y(-m + nx)\,dx \qquad \text{Differential form}$$

$$\frac{a - by}{y}\,dy = \frac{-m + nx}{x}\,dx \qquad \text{Separate variables.}$$

$$\int \frac{a - by}{y}\,dy = \int \frac{-m + nx}{x}\,dx \qquad \text{Integrate.}$$

$$a \ln y - by = -m \ln x + nx + C \qquad \text{Assume } x \text{ and } y \text{ are positive.}$$

$$a \ln y + m \ln x - by - nx = C \qquad \text{General solution}$$

The constant C is determined by the initial conditions.

REMARK The general solution

$$a \ln y + m \ln x - by - nx = C$$

can be rewritten as

$$\ln(y^a x^m) = C + by + nx$$

or as $y^a x^m = C_1 e^{by + nx}$.

Competing Species

Consider two species that compete with each other for the food available in their common environment. Assume that their populations are given by x and y at time t. When there is no interaction or competition between the species, the populations x and y each experience logistic growth. So, the populations of the first species x and the second species y can be modeled by the following differential equations.

$$\frac{dx}{dt} = ax - bx^2 \qquad \text{Rate of change of first species without interaction}$$

$$\frac{dy}{dt} = my - ny^2 \qquad \text{Rate of change of second species without interaction}$$

When the species interact, their competition for resources causes a rate of decline in each population proportional to the product xy. Using a negative interaction factor leads to the following **competing-species equations** (where a, b, c, m, n, and p are positive constants).

$$\frac{dx}{dt} = ax - bx^2 - cxy \qquad \text{Rate of change of first species with interaction}$$

$$\frac{dy}{dt} = my - ny^2 - pxy \qquad \text{Rate of change of second species with interaction}$$

In this text it is assumed that competing-species equations have four critical points, as shown in Example 5.

EXAMPLE 5 Deriving the Critical Points

Show that the critical points of the competing-species equations

$$\frac{dx}{dt} = ax - bx^2 - cxy \quad \text{and} \quad \frac{dy}{dt} = my - ny^2 - pxy$$

are $(0, 0)$, $\left(0, \frac{m}{n}\right)$, $\left(\frac{a}{b}, 0\right)$, and $\left(\frac{an - mc}{bn - cp}, \frac{bm - ap}{bn - cp}\right)$.

Solution Set dx/dt and dy/dt equal to 0 and then factor to obtain the following system of equations.

$$x(a - bx - cy) = 0 \qquad \text{Set } dx/dt \text{ equal to 0 and factor out } x.$$

$$y(m - ny - px) = 0 \qquad \text{Set } dy/dt \text{ equal to 0 and factor out } y.$$

If $x = 0$, then $y = 0$ or $y = m/n$. If $y = 0$, then $x = 0$ or $x = a/b$. So three of the critical points are $(0, 0)$, $(0, m/n)$, and $(a/b, 0)$. At each of these critical points, one of the populations is 0. These points represent the possibility that both species cannot coexist.

The fourth critical point is obtained by solving the system

$$a - bx - cy = 0$$

$$m - ny - px = 0.$$

The solution of this system is

$$(x, y) = \left(\frac{an - mc}{bn - cp}, \frac{bm - ap}{bn - cp}\right).$$

Assuming this point exists and lies in Quadrant I of the xy-plane, the point represents the possibility that both species can coexist.

EXAMPLE 6 Competing Species: One Species Survives

Consider the competing-species equations given by

$$\frac{dx}{dt} = 10x - x^2 - 2xy \qquad \text{and} \qquad \frac{dy}{dt} = 10y - y^2 - 2xy.$$

a. Find the critical points.

b. Use a graphing utility to graph the solution of the equations when $0 \le t \le 3$ and the initial conditions are $x(0) = 10$ and $y(0) = 15$. What do you observe?

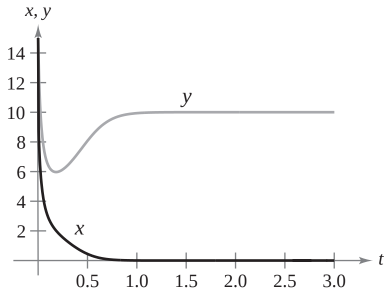


Figure 6.35

Solution

a. Note that $a = 10$, $b = 1$, $c = 2$, $m = 10$, $n = 1$, and $p = 2$. So, the critical points are $(0, 0)$, $(0, 10)$, $(10, 0)$, and $\left(\frac{10 - 20}{1 - 4}, \frac{10 - 20}{1 - 4}\right) = \left(\frac{10}{3}, \frac{10}{3}\right)$.

b. The solution of the competing-species equations is shown in Figure 6.35. From the graph, it appears that one species survives. The population of the surviving species, represented by the graph of y, appears to remain constant at 10.

EXAMPLE 7 Competing Species: Both Species Survive

Consider the competing-species equations given by

$$\frac{dx}{dt} = 10x - 3x^2 - xy \qquad \text{and} \qquad \frac{dy}{dt} = 14y - 3y^2 - xy.$$

a. Find the critical points.

b. Use a graphing utility to graph the solution of the equations when $0 \le t \le 15$ and the initial conditions are $x(0) = 10$ and $y(0) = 15$. What do you observe?

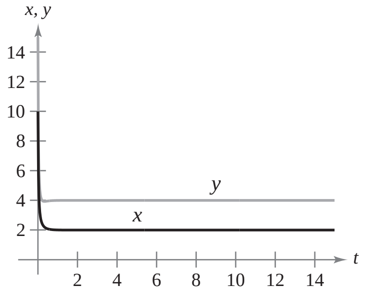


Figure 6.36

Solution

a. Note that $a = 10$, $b = 3$, $c = 1$, $m = 14$, $n = 3$, and $p = 1$. So, the critical points are $(0, 0)$, $\left(0, \frac{14}{3}\right)$, $\left(\frac{10}{3}, 0\right)$, and $\left(\frac{30 - 14}{9 - 1}, \frac{42 - 10}{9 - 1}\right) = (2, 4)$.

b. The solution of the competing-species equations is shown in Figure 6.36. From the graph, it appears that both species survive. The population represented by y appears to remain constant at 4. The population represented by x appears to remain constant at 2.

Examples 6 and 7 imply general conclusions about competing-species equations that have precisely four critical points. In general, it can be shown that when $bn > cp$, both species survive. When $bn < cp$, one species will survive and the other will not.

You can also use slope fields to analyze solutions of competing-species equations, as shown in Figures 6.37 (Example 6) and 6.38 (Example 7).

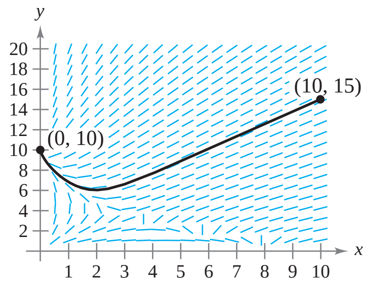


Figure 6.37

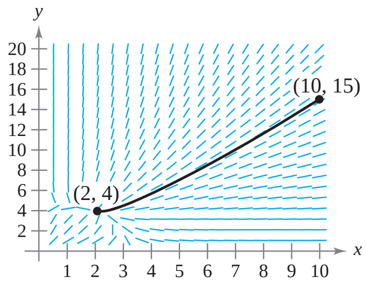


Figure 6.38

6.6 Exercises

See **CalcChat.com** for tutorial help and worked-out solutions to odd-numbered exercises.

Analyzing Predator-Prey Equations **In Exercises 1–4, use the given values to write the predator-prey equations $dx/dt = ax - bxy$ and $dy/dt = -my + nxy$. Then find the values of x and y for which $dx/dt = dy/dt = 0$.**

1. $a = 0.9,\ b = 0.05,\ m = 0.6,\ n = 0.008$

2. $a = 0.75,\ b = 0.006,\ m = 0.9,\ n = 0.003$

3. $a = 0.5,\ b = 0.01,\ m = 0.49,\ n = 0.007$

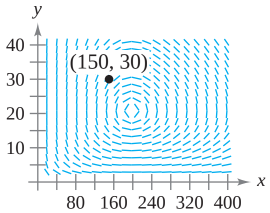
4. $a = 1.2,\ b = 0.04,\ m = 1.2,\ n = 0.02$

Predator-Prey Equations and Slope Fields **In Exercises 5 and 6, predator-prey equations, a point, and a slope field are given. (a) Sketch a solution of the predator-prey equations on the slope field that passes through the given point. (b) Use a graphing utility to graph the solution. Compare the result with the sketch in part (a). To print an enlarged copy of the graph, go to *MathGraphs.com*.**

5. $\dfrac{dx}{dt} = 0.04x - 0.002xy$

$\dfrac{dy}{dt} = -0.08y + 0.0004xy$

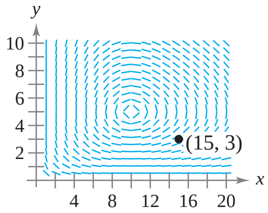
$(150, 30)$



6. $\dfrac{dx}{dt} = 0.03x - 0.006xy$

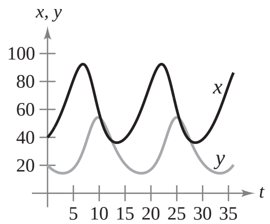
$\dfrac{dy}{dt} = -0.04y + 0.004xy$

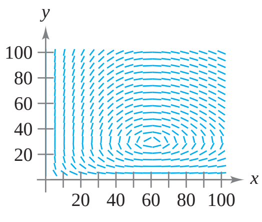
$(15, 3)$



Predator-Prey Equations and Slope Fields **In Exercises 7 and 8, two graphs are given. The first is a graph of the functions x and y of a set of predator-prey equations, where x is the number of prey and y is the number of predators at time t. The second graph is the corresponding slope field of the predator-prey equations. (a) Identify the initial conditions. (b) Sketch a solution of the predator-prey equations on the slope field that passes through the initial conditions. To print an enlarged copy of the graph, go to *MathGraphs.com*.**

7.





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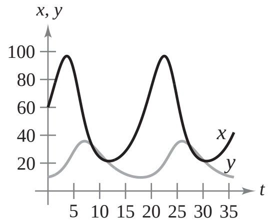
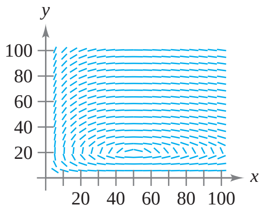




Rabbits and Foxes **In Exercises 9–12, consider a predator-prey relationship involving foxes (predators) and rabbits (prey). Let x represent the number of rabbits, let y represent the number of foxes, and let t represent the time in months. Assume that the following predator-prey equations model the rates of change of each population.**

$\dfrac{dx}{dt} = 0.8x - 0.04xy$ Rate of change of prey population

$\dfrac{dy}{dt} = -0.3y + 0.006xy$ Rate of change of predator population

When $t = 0$, $x = 55$ and $y = 10$.

9. Find the critical points of the predator-prey equations.

10. Use a graphing utility to graph the functions x and y when $0 \le t \le 36$. Describe the behavior of each solution as t increases.

11. Use a graphing utility to graph a slope field of the predator-prey equations when $0 \le x \le 150$ and $0 \le y \le 50$.

12. Use the predator-prey equations and the slope field in Exercise 11 to graph the solution curve using the initial conditions. Describe the changes in the rabbit and fox populations as you trace the solution curve.

Prairie Dogs and Black-Footed Ferrets **In Exercises 13–16, consider a predator-prey relationship involving black-footed ferrets (predators) and prairie dogs (prey). Let x represent the number of prairie dogs, let y represent the number of black-footed ferrets, and let t represent the time in months. Assume that the following predator-prey equations model the rates of change of each population.**

$\dfrac{dx}{dt} = 0.1x - 0.00008xy$ Rate of change of prey population

$\dfrac{dy}{dt} = -0.4y + 0.00004xy$ Rate of change of predator population

When $t = 0$, $x = 4000$ and $y = 1000$.

13. Find the critical points of the predator-prey equations.

14. Use a graphing utility to graph the functions x and y when $0 \le t \le 240$. Describe the behavior of each solution as t increases.

15. Use a graphing utility to graph a slope field of the predator-prey equations when $0 \le x \le 25{,}000$ and $0 \le y \le 5000$.

16. Use the predator-prey equations and the slope field in Exercise 15 to graph the solution curve using the initial conditions. Describe the changes in the prairie dog and black-footed ferret populations as you trace the solution curve.

17. Critical Point as the Initial Condition In Exercise 9, you found the critical points of the predator-prey system. Assume that the critical point given by $\left(\dfrac{m}{n}, \dfrac{a}{b}\right)$ is the initial condition and repeat Exercises 10–12. Compare the results.

18. Critical Point as the Initial Condition In Exercise 13, you found the critical points of the predator-prey system. Assume that the critical point given by $(m/n, a/b)$ is the initial condition and repeat Exercises 14–16. Compare the results.

Analyzing Competing-Species Equations **In Exercises 19–22, use the given values to write the competing-species equations $dx/dt = ax - bx^2 - cxy$ and $dy/dt = my - ny^2 - pxy$. Then find the values of x and y for which $dx/dt = dy/dt = 0$.**

19. $a = 2, b = 3, c = 2, m = 2, n = 3, p = 2$

20. $a = 1, b = 0.5, c = 0.5, m = 2.5, n = 2, p = 0.5$

21. $a = 0.15, b = 0.6, c = 0.75, m = 0.15, n = 1.2, p = 0.45$

22. $a = 0.025, b = 0.1, c = 0.2, m = 0.3, n = 0.45, p = 0.1$

Bass and Trout **In Exercises 23 and 24, consider a competing-species relationship involving bass and trout. Assume the bass and trout compete for the same resources. Let x represent the number of bass (in thousands), let y represent the number of trout (in thousands), and let t represent the time in months. Assume that the following competing-species equations model the rates of change of the two populations.**

$$\frac{dx}{dt} = 0.8x - 0.4x^2 - 0.1xy \qquad \text{Rate of change of bass population}$$

$$\frac{dy}{dt} = 0.3y - 0.6y^2 - 0.1xy \qquad \text{Rate of change of trout population}$$

When $t = 0$, $x = 9$ and $y = 5$.

23. Find the critical points of the competing-species equations.

24. Use a graphing utility to graph the functions x and y when $0 \le t \le 36$. Describe the behavior of each solution as t increases.

Bass and Trout **In Exercises 25 and 26, consider a competing-species relationship involving bass and trout. Assume the bass and trout compete for the same resources. Let x represent the number of bass (in thousands), let y represent the number of trout (in thousands), and let t represent the time in months. Assume that the following competing-species equations model the rates of change of the two populations.**

$$\frac{dx}{dt} = 0.8x - 0.4x^2 - xy \qquad \text{Rate of change of bass population}$$

$$\frac{dy}{dt} = 0.3y - 0.6y^2 - xy \qquad \text{Rate of change of trout population}$$

When $t = 0$, $x = 7$ and $y = 6$.

25. Find the critical points of the competing-species equations.

26. Use a graphing utility to graph the functions x and y when $0 \le t \le 36$. Describe the behavior of each solution as t increases.

27. Critical Point as the Initial Condition In Exercise 23, you found the critical points of the competing-species system. Assume that the critical point given by $\left(\frac{an - mc}{bn - cp}, \frac{bm - ap}{bn - cp}\right)$ is the initial condition and repeat Exercise 24. Compare the results.

28. Critical Point as the Initial Condition In Exercise 23, you found the critical points of the competing-species system. Assume that the critical point given by $(0, m/n)$ is the initial condition and repeat Exercise 24. Compare the results.

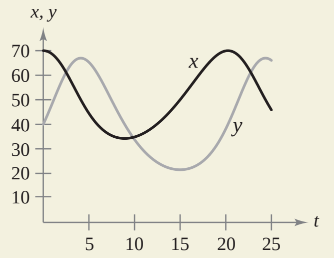
WRITING ABOUT CONCEPTS

29. Separation of Variables Are predator-prey equations separable? Explain.

30. Determining Initial Values Given a set of predator-prey equations, describe how to determine initial values so that both populations remain constant for all $t \ge 0$.

31. Determining Initial Values Given a set of competing-species equations, describe how to determine initial values so that both populations remain constant for all $t > 0$.

32. HOW DO YOU SEE IT? The populations of two species x and y are shown in the figure. Sketch the graph of the solution curve by hand for $0 \le t \le 20$.



33. Revising the Predator-Prey Equations Consider a predator-prey relationship with x prey and y predators at time t. Assume both predator and prey are present. Then the rates of change of the two populations can be modeled by the following revised predator-prey system of differential equations.

$$\frac{dx}{dt} = ax\left(1 - \frac{x}{L}\right) - bxy \qquad \text{Rate of change of prey population}$$

$$\frac{dy}{dt} = -my + nxy \qquad \text{Rate of change of predator population}$$

(a) When there are no predators, the prey population will grow according to what model?

(b) Write the revised predator-prey equations for $a = 0.4$, $L = 100$, $b = 0.01$, $m = 0.3$, and $n = 0.005$. Find the critical numbers.

(c) Use a graphing utility to graph the functions x and y of the revised predator-prey equations when $0 \le t \le 72$ and the initial conditions are $x(0) = 40$ and $y(0) = 80$. Describe the behavior of each solution as t increases.

(d) Use a graphing utility to graph a slope field of the revised predator-prey equations when $0 \le x \le 100$ and $0 \le y \le 80$.

(e) Use the predator-prey equations and the slope field in part (d) to graph the solution curve using the initial conditions in part (c). Describe the changes in the prey and predator populations as you trace the solution curve.

Review Exercises

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

1. **Determining a Solution** Determine whether the function $y = x^3$ is a solution of the differential equation $2xy' + 4y = 10x^3$.

2. **Determining a Solution** Determine whether the function $y = 2 \sin 2x$ is a solution of the differential equation $y''' - 8y = 0$.

Finding a General Solution **In Exercises 3–8, use integration to find a general solution of the differential equation.**

3. $\dfrac{dy}{dx} = 4x^2 + 7$

4. $\dfrac{dy}{dx} = 3x^3 - 8x$

5. $\dfrac{dy}{dx} = \cos 2x$

6. $\dfrac{dy}{dx} = 2 \sin x$

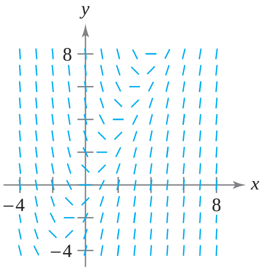
7. $\dfrac{dy}{dx} = e^{2-x}$

8. $\dfrac{dy}{dx} = 2e^{3x}$

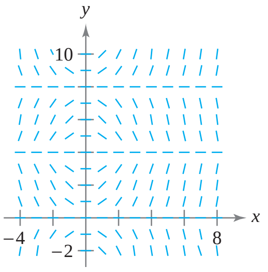
Slope Field **In Exercises 9 and 10, a differential equation and its slope field are given. Complete the table by determining the slopes (if possible) in the slope field at the given points.**

x	-4	-2	0	2	4	8
y	2	0	4	4	6	8
dy/dx						

9. $\dfrac{dy}{dx} = 2x - y$



10. $\dfrac{dy}{dx} = x \sin\left(\dfrac{\pi y}{4}\right)$



Slope Field **In Exercises 11 and 12, (a) sketch the slope field for the differential equation, and (b) use the slope field to sketch the solution that passes through the given point. Use a graphing utility to verify your results. To print a blank graph, go to *MathGraphs.com*.**

11. $y' = 2x^2 - x, \quad (0, 2)$

12. $y' = y + 4x, \quad (-1, 1)$

Euler's Method **In Exercises 13 and 14, use Euler's Method to make a table of values for the approximate solution of the differential equation with the specified initial value. Use n steps of size h.**

13. $y' = x - y, \quad y(0) = 4, \quad n = 10, \quad h = 0.05$

14. $y' = 5x - 2y, \quad y(0) = 2, \quad n = 10, \quad h = 0.1$

Solving a Differential Equation **In Exercises 15–20, solve the differential equation.**

15. $\dfrac{dy}{dx} = 2x - 5x^2$

16. $\dfrac{dy}{dx} = y + 8$

17. $\dfrac{dy}{dx} = (3 + y)^2$

18. $\dfrac{dy}{dx} = 10\sqrt{y}$

19. $(2 + x)y' - xy = 0$

20. $xy' - (x + 1)y = 0$

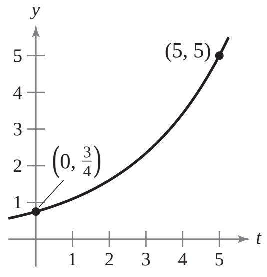
Writing and Solving a Differential Equation **In Exercises 21 and 22, write and solve the differential equation that models the verbal statement.**

21. The rate of change of y with respect to t is inversely proportional to the cube of t.

22. The rate of change of y with respect to t is proportional to $50 - t$.

Finding an Exponential Function **In Exercises 23–26, find the exponential function $y = Ce^{kt}$ that passes through the two points.**

23.



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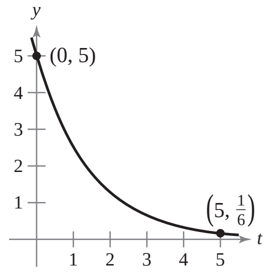

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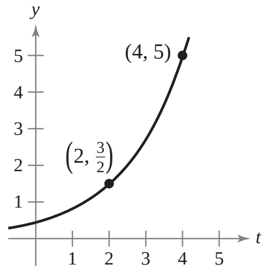

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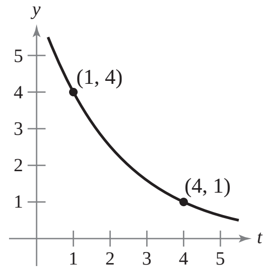


27. **Air Pressure** Under ideal conditions, air pressure decreases continuously with the height above sea level at a rate proportional to the pressure at that height. The barometer reads 30 inches at sea level and 15 inches at 18,000 feet. Find the barometric pressure at 35,000 feet.

28. **Radioactive Decay** Radioactive radium has a half-life of approximately 1599 years. The initial quantity is 15 grams. How much remains after 750 years?

29. **Population Growth** A population grows continuously at a rate of 1.85%. How long will it take the population to double?

30. Compound Interest Find the balance in an account when $1000 is deposited for 8 years at an interest rate of 4% compounded continuously.

31. Sales The sales S (in thousands of units) of a new product after it has been on the market for t years is given by

$$S = Ce^{k/t}.$$

(a) Find S as a function of t when 5000 units have been sold after 1 year and the saturation point for the market is 30,000 units (that is, $\lim_{t\to\infty} S = 30$).

(b) How many units will have been sold after 5 years?

32. Sales The sales S (in thousands of units) of a new product after it has been on the market for t years is given by

$$S = 25(1 - e^{kt}).$$

(a) Find S as a function of t when 4000 units have been sold after 1 year.

(b) How many units will saturate this market?

(c) How many units will have been sold after 5 years?

Finding a General Solution Using Separation of Variables **In Exercises 33–36, find the general solution of the differential equation.**

33. $\dfrac{dy}{dx} = \dfrac{5x}{y}$ **34.** $\dfrac{dy}{dx} = \dfrac{x^3}{2y^2}$

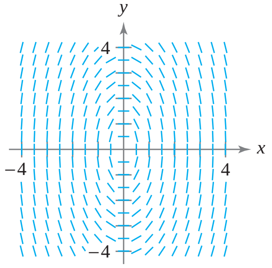
35. $y' - 16xy = 0$ **36.** $y' - e^y \sin x = 0$

Finding a Particular Solution Using Separation of Variables **In Exercises 37–40, find the particular solution that satisfies the initial condition.**

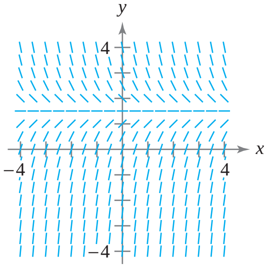
	Differential Equation	Initial Condition
37.	$y^3y' - 3x = 0$	$y(2) = 2$
38.	$yy' - 5e^{2x} = 0$	$y(0) = -3$
39.	$y^3(x^4 + 1)y' - x^3(y^4 + 1) = 0$	$y(0) = 1$
40.	$yy' - x \cos x^2 = 0$	$y(0) = -2$

Slope Field **In Exercises 41 and 42, sketch a few solutions of the differential equation on the slope field and then find the general solution analytically. To print an enlarged copy of the graph, go to *MathGraphs.com.***

41. $\dfrac{dy}{dx} = -\dfrac{4x}{y}$



42. $\dfrac{dy}{dx} = 3 - 2y$



Using a Logistic Equation **In Exercises 43 and 44, the logistic equation models the growth of a population. Use the equation to (a) find the value of k, (b) find the carrying capacity, (c) find the initial population, (d) determine when the population will reach 50% of its carrying capacity, and (e) write a logistic differential equation that has the solution $P(t)$.**

43. $P(t) = \dfrac{5250}{1 + 34e^{-0.55t}}$

44. $P(t) = \dfrac{4800}{1 + 14e^{-0.15t}}$

Solving a Logistic Differential Equation **In Exercises 45 and 46, find the logistic equation that passes through the given point.**

45. $\dfrac{dy}{dt} = y\left(1 - \dfrac{y}{80}\right), \quad (0, 8)$

46. $\dfrac{dy}{dt} = 1.76y\left(1 - \dfrac{y}{8}\right), \quad (0, 3)$

47. Environment A conservation department releases 1200 brook trout into a lake. It is estimated that the carrying capacity of the lake for the species is 20,400. After the first year, there are 2000 brook trout in the lake.

(a) Write a logistic equation that models the number of brook trout in the lake.

(b) Find the number of brook trout in the lake after 8 years.

(c) When will the number of brook trout reach 10,000?

48. Environment Write a logistic differential equation that models the growth rate of the brook trout population in Exercise 47. Then repeat part (b) using Euler's Method with a step size of $h = 1$. Compare the approximation with the exact answer.

49. Sales Growth The rate of change in sales S (in thousands of units) of a new product is proportional to $L - S$. L (in thousands of units) is the estimated maximum level of sales, and $S = 0$ when $t = 0$. Write and solve the differential equation for this sales model.

50. Sales Growth Use the result of Exercise 49 to write S as a function of t for (a) $L = 100$, $S = 25$ when $t = 2$, and (b) $L = 500$, $S = 50$ when $t = 1$.

Learning Theory **In Exercises 51 and 52, assume that the rate of change in the proportion P of correct responses after n trials is proportional to the product of P and $L - P$, where L is the limiting proportion of correct responses.**

51. Write and solve the differential equation for this learning theory model.

52. Use the solution of Exercise 51 to write P as a function of n, and then use a graphing utility to graph the solution.

(a) $L = 1.00$
$P = 0.50$ when $n = 0$
$P = 0.85$ when $n = 4$

(b) $L = 0.80$
$P = 0.25$ when $n = 0$
$P = 0.60$ when $n = 10$

Slope Field **In Exercises 53–56, (a) sketch an approximate solution of the differential equation satisfying the initial condition by hand on the slope field, (b) find the particular solution that satisfies the initial condition, and (c) use a graphing utility to graph the particular solution. Compare the graph with the hand-drawn graph in part (a). To print an enlarged copy of the graph, go to *MathGraphs.com*.**

	Differential Equation	**Initial Condition**
53.	$\dfrac{dy}{dx} = e^{x/2} - y$	$(0, -1)$
54.	$y' + 2y = \sin x$	$(0, 4)$
55.	$y' = \csc x + y \cot x$	$(1, 1)$
56.	$y' = \csc x - y \cot x$	$(1, 2)$

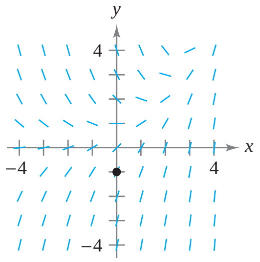


Figure for 53

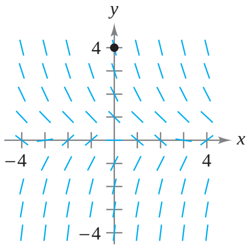


Figure for 54

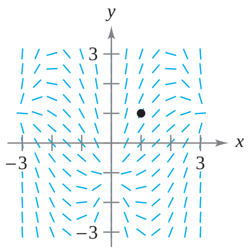


Figure for 55

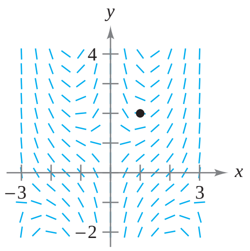


Figure for 56

Solving a First-Order Linear Differential Equation **In Exercises 57–64, solve the first-order linear differential equation.**

57. $y' - y = 10$

58. $e^x y' + 4e^x y = 1$

59. $4y' = e^{x/4} + y$

60. $\dfrac{dy}{dx} - \dfrac{5y}{x^2} = \dfrac{1}{x^2}$

61. $(x - 2)y' + y = 1$

62. $(x + 3)y' + 2y = 2(x + 3)^2$

63. $y' + 5y = e^{5x}$

64. $xy' - ay = bx^4$

Finding a Particular Solution **In Exercises 65 and 66, find the particular solution of the differential equation that satisfies the initial condition.**

	Differential Equation	**Initial Condition**
65.	$y' + 5y = e^{5x}$	$y(0) = 3$
66.	$y' - \left(\dfrac{3}{x}\right)y = 2x^3$	$y(1) = 1$

Writing **In Exercises 67–69, write an example of the given differential equation. Then solve your equation.**

67. Homogeneous differential equation

68. Logistic differential equation

69. First-order linear differential equation

70. Investment Let $A(t)$ be the amount in a fund earning interest at an annual rate r compounded continuously. When a continuous cash flow of P dollars per year is withdrawn from the fund, the rate of change of A is given by the differential equation

$$\frac{dA}{dt} = rA - P$$

where $A = A_0$ when $t = 0$. Solve this differential equation for A as a function of t.

71. Investment A retired couple plans to withdraw P dollars per year from a retirement account of \$500,000 earning 10% interest compounded continuously. Use the result of Exercise 70 and a graphing utility to graph the function A for each of the following continuous annual cash flows. Use the graphs to describe what happens to the balance in the fund for each case.

(a) $P = \$40{,}000$

(b) $P = \$50{,}000$

(c) $P = \$60{,}000$

72. Investment Use the result of Exercise 70 to find the time necessary to deplete a fund earning 14% interest compounded continuously when $A_0 = \$1{,}000{,}000$ and $P = \$200{,}000$.

Analyzing Predator-Prey Equations **In Exercises 73 and 74, (a) use the given values to write a set of predator-prey equations, (b) find the values of x and y for which $x' = y' = 0$, and (c) use a graphing utility to graph the solutions x and y of the predator-prey equations for the given time frame. Describe the behavior of each solution as t increases.**

73. Constants: $a = 0.3$, $b = 0.02$, $m = 0.4$, $n = 0.01$
Initial condition: $(20, 20)$
Time frame: $0 \le t \le 36$

74. Constants: $a = 0.4$, $b = 0.04$, $m = 0.6$, $n = 0.02$
Initial condition: $(30, 15)$
Time frame: $0 \le t \le 24$

Analyzing Competing-Species Equations **In Exercises 75 and 76, (a) use the given values to write a set of competing-species equations, (b) find the values of x and y for which $x' = y' = 0$, and (c) use a graphing utility to graph the solutions x and y of the competing-species equations for the given time frame. Describe the behavior of each solution as t increases.**

75. Constants: $a = 3$, $b = 1$, $c = 1$, $m = 2$, $n = 1$, $p = 0.5$
Initial condition: $(3, 2)$
Time frame: $0 \le t \le 6$

76. Constants: $a = 15$, $b = 2$, $c = 4$, $m = 17$, $n = 2$, $p = 4$
Initial condition: $(9, 10)$
Time frame: $0 \le t \le 4$

P.S. Problem Solving

See CalcChat.com for tutorial help and worked-out solutions to odd-numbered exercises.

1. Doomsday Equation The differential equation

$$\frac{dy}{dt} = ky^{1+\varepsilon}$$

where k and ε are positive constants, is called the **doomsday equation.**

(a) Solve the doomsday equation

$$\frac{dy}{dt} = y^{1.01}$$

given that $y(0) = 1$. Find the time T at which

$$\lim_{t \to T^-} y(t) = \infty.$$

(b) Solve the doomsday equation

$$\frac{dy}{dt} = ky^{1+\varepsilon}$$

given that $y(0) = y_0$. Explain why this equation is called the doomsday equation.

2. Sales Let S represent sales of a new product (in thousands of units), let L represent the maximum level of sales (in thousands of units), and let t represent time (in months). The rate of change of S with respect to t varies jointly as the product of S and $L - S$.

(a) Write the differential equation for the sales model using these conditions.

When $t = 0$: $L = 100$, $S = 10$

When $t = 1$: $S = 20$

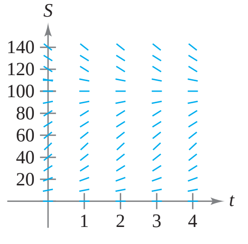
Verify that

$$S = \frac{L}{1 + Ce^{-kt}}.$$

(b) At what time is the growth in sales increasing most rapidly?

(c) Use a graphing utility to graph the sales function.

(d) Sketch the solution from part (a) on the slope field shown in the figure below. To print an enlarged copy of the graph, go to *MathGraphs.com.*



(e) Assume the estimated maximum level of sales is correct. Use the slope field to describe the shape of the solution curves for sales when, at some period of time, sales exceed L.

3. Modified Euler's Method Another numerical approach to approximating the particular solution of the differential equation $y' = F(x, y)$ is shown below.

$$x_n = x_{n-1} + h$$

$$y_n = y_{n-1} + hf\left(x_{n-1} + \frac{h}{2}, y_{n-1} + \frac{h}{2} f(x_{n-1}, y_{n-1})\right)$$

This approach is called **modified Euler's Method.**

(a) Use this method to approximate the solution of the differential equation $y' = x - y$ passing through the point $(0, 1)$. Use a step size of $h = 0.1$.

(b) Use a graphing utility to graph the exact solution and the approximations found using Euler's Method and modified Euler's Method (see Example 6, page 384). Compare the first 10 approximations found using modified Euler's Method to those found using Euler's Method and to the exact solution $y = x - 1 + 2e^{-x}$. Which approximation appears to be more accurate?

4. Error Using the Product Rule Although it is true for some functions f and g, a common mistake in calculus is to believe that the Product Rule for derivatives is $(fg)' = f'g'$.

(a) Given $g(x) = x$, find f such that $(fg)' = f'g'$.

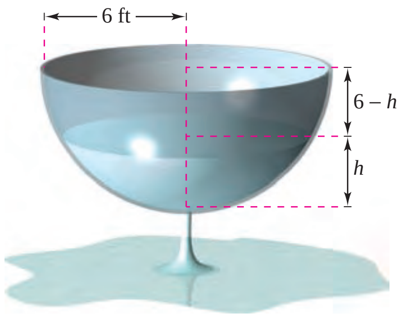
(b) Given an arbitrary function g, find a function f such that $(fg)' = f'g'$.

(c) Describe what happens when $g(x) = e^x$.

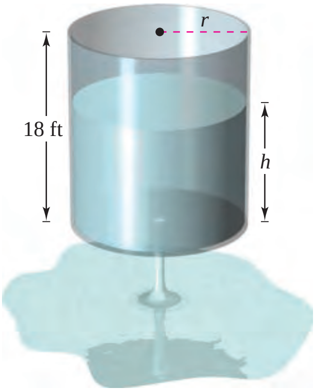
5. Torricelli's Law **Torricelli's Law** states that water will flow from an opening at the bottom of a tank with the same speed that it would attain falling from the surface of the water to the opening. One of the forms of Torricelli's Law is

$$A(h)\frac{dh}{dt} = -k\sqrt{2gh}$$

where h is the height of the water in the tank, k is the area of the opening at the bottom of the tank, $A(h)$ is the horizontal cross-sectional area at height h, and g is the acceleration due to gravity ($g \approx 32$ feet per second per second). A hemispherical water tank has a radius of 6 feet. When the tank is full, a circular valve with a radius of 1 inch is opened at the bottom, as shown in the figure. How long will it take for the tank to drain completely?



6. Torricelli's Law The cylindrical water tank shown in the figure has a height of 18 feet. When the tank is full, a circular valve is opened at the bottom of the tank. After 30 minutes, the depth of the water is 12 feet.



(a) Using Torricelli's Law, how long will it take for the tank to drain completely?

(b) What is the depth of the water in the tank after 1 hour?

7. Torricelli's Law A tank similar to the one in Exercise 6 has a height of 20 feet and a radius of 8 feet, and the valve is circular with a radius of 2 inches. The tank is full when the valve is opened. How long will it take for the tank to drain completely?

8. Rewriting the Logistic Equation Show that the logistic equation

$$y = \frac{L}{1 + be^{-kt}}$$

can be written as

$$y = \frac{1}{2}L\left[1 + \tanh\left(\frac{1}{2}k\left(t - \frac{\ln b}{k}\right)\right)\right].$$

What can you conclude about the graph of the logistic equation?

9. Biomass Biomass is a measure of the amount of living matter in an ecosystem. The biomass $s(t)$ in a given ecosystem increases at a rate of about 3.5 tons per year, and decreases by about 1.9% per year. This situation can be modeled by the differential equation

$$\frac{ds}{dt} = 3.5 - 0.019s.$$

(a) Solve the differential equation.

(b) Use a graphing utility to graph the slope field for the differential equation. What do you notice?

(c) Explain what happens as $t \to \infty$.

10. Finding a Function Consider a function f such that

$$f(0) = 1, \quad f'(0) = 1, \quad \text{and} \quad f(a + b) = f(a)(b)$$

where a and b are real numbers. For all values of x, show that $f'(x) = f(x)$ and conclude that $f(x) = e^x$.

Medical Science **In Exercises 11–13, a medical researcher wants to determine the concentration C (in moles per liter) of a tracer drug injected into a moving fluid. Solve this problem by considering a single-compartment dilution model (see figure). Assume that the fluid is continuously mixed and that the volume of the fluid in the compartment is constant.**

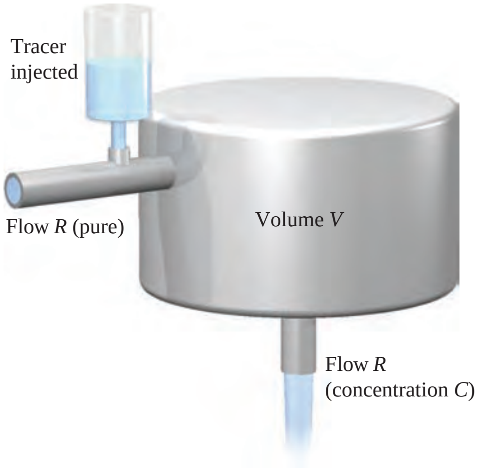


Figure for 11–13

11. If the tracer is injected instantaneously at time $t = 0$, then the concentration of the fluid in the compartment begins diluting according to the differential equation

$$\frac{dC}{dt} = \left(-\frac{R}{V}\right)C$$

where $C = C_0$ when $t = 0$.

(a) Solve this differential equation to find the concentration C as a function of time t.

(b) Find the limit of C as $t \to \infty$.

12. Use the solution of the differential equation in Exercise 11 to find the concentration C as a function of time t, and use a graphing utility to graph the function.

(a) $V = 2$ liters, $R = 0.5$ liter per minute, and $C_0 = 0.6$ mole per liter

(b) $V = 2$ liters, $R = 1.5$ liters per minute, and $C_0 = 0.6$ mole per liter

13. In Exercises 11 and 12, it was assumed that there was a single initial injection of the tracer drug into the compartment. Now consider the case in which the tracer is continuously injected (beginning at $t = 0$) at the rate of Q moles per minute. Considering Q to be negligible compared with R, use the differential equation

$$\frac{dC}{dt} = \frac{Q}{V} - \left(\frac{R}{V}\right)C$$

where $C = 0$ when $t = 0$.

(a) Solve this differential equation to find the concentration C as a function of time t.

(b) Find the limit of C as $t \to \infty$.