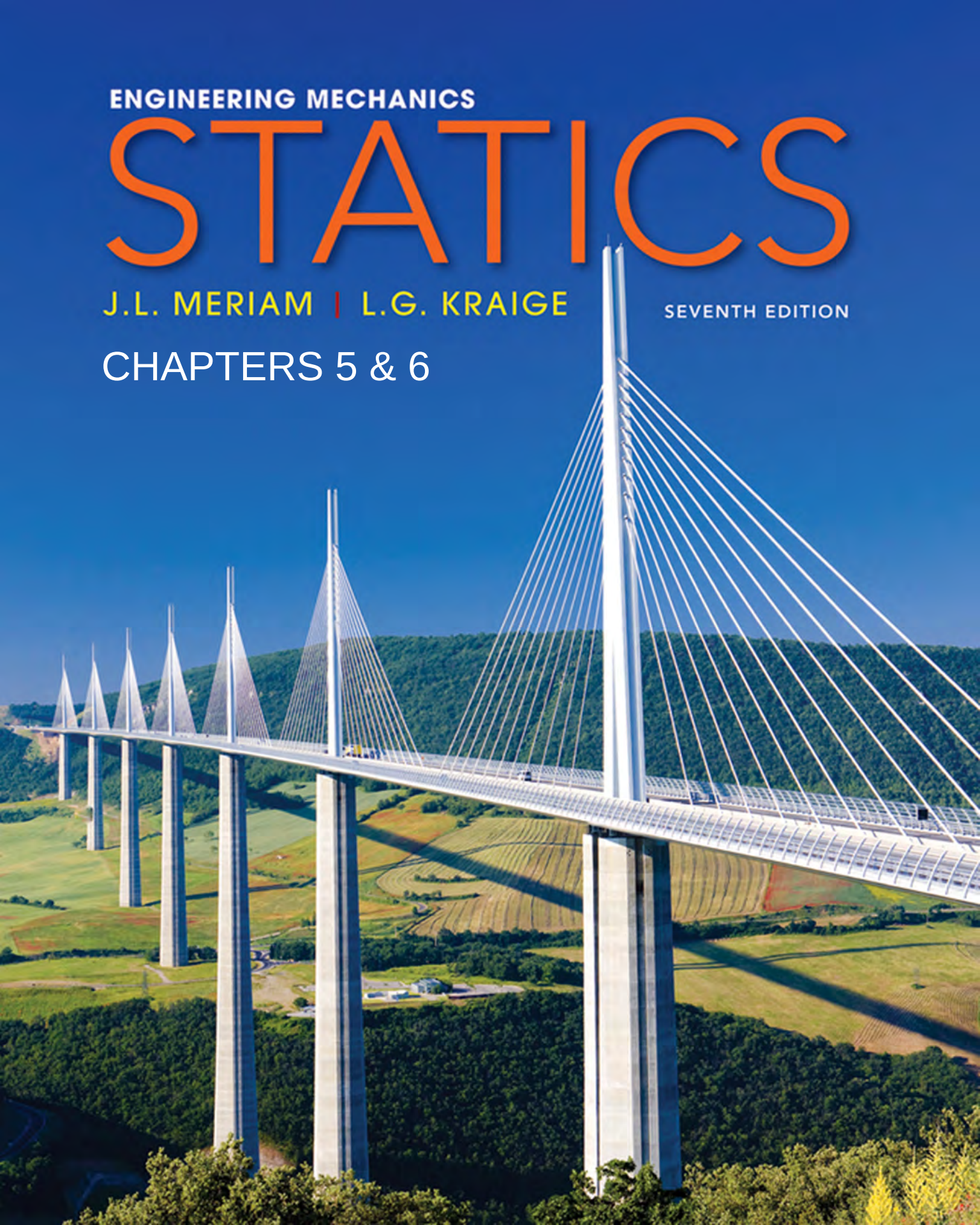
ENGINEERING MECHANICS

STATICS

J.L. MERIAM | L.G. KRAIGE

SEVENTH EDITION

CHAPTERS 5 & 6



Conversion Factors

U.S. Customary Units to SI Units

To convert from	To	Multiply by
(*Acceleration*)		
foot/second^2 (ft/sec^2)	meter/second^2 (m/s^2)	3.048×10^{-1}*
inch/second^2 (in./sec^2)	meter/second^2 (m/s^2)	2.54×10^{-2}*
(*Area*)		
foot^2 (ft^2)	meter^2 (m^2)	9.2903×10^{-2}
inch^2 (in.^{2})	meter^2 (m^2)	6.4516×10^{-4}*
(*Density*)		
pound mass/inch^3 (lbm/in.^{3})	kilogram/meter^3 (kg/m^3)	2.7680×10^4
pound mass/foot^3 (lbm/ft^3)	kilogram/meter^3 (kg/m^3)	1.6018×10
(*Force*)		
kip (1000 lb)	newton (N)	4.4482×10^3
pound force (lb)	newton (N)	4.4482
(*Length*)		
foot (ft)	meter (m)	3.048×10^{-1}*
inch (in.)	meter (m)	2.54×10^{-2}*
mile (mi), (U.S. statute)	meter (m)	1.6093×10^3
mile (mi), (international nautical)	meter (m)	1.852×10^3*
(*Mass*)		
pound mass (lbm)	kilogram (kg)	4.5359×10^{-1}
slug (lb-sec^2/ft)	kilogram (kg)	1.4594×10
ton (2000 lbm)	kilogram (kg)	9.0718×10^2
(*Moment of force*)		
pound-foot (lb-ft)	newton-meter (N · m)	1.3558
pound-inch (lb-in.)	newton-meter (N · m)	0.1129 8
(*Moment of inertia, area*)		
inch^4	meter^4 (m^4)	41.623×10^{-8}
(*Moment of inertia, mass*)		
pound-foot-second^2 (lb-ft-sec^2)	kilogram-meter^2 (kg · m^2)	1.3558
(*Momentum, linear*)		
pound-second (lb-sec)	kilogram-meter/second (kg · m/s)	4.4482
(*Momentum, angular*)		
pound-foot-second (lb-ft-sec)	newton-meter-second (kg · m^2/s)	1.3558
(*Power*)		
foot-pound/minute (ft-lb/min)	watt (W)	2.2597×10^{-2}
horsepower (550 ft-lb/sec)	watt (W)	7.4570×10^2
(*Pressure, stress*)		
atmosphere (std)(14.7 lb/in.^{2})	newton/meter^2 (N/m^2 or Pa)	1.0133×10^5
pound/foot^2 (lb/ft^2)	newton/meter^2 (N/m^2 or Pa)	4.7880×10
pound/inch^2 (lb/in.^{2} or psi)	newton/meter^2 (N/m^2 or Pa)	6.8948×10^3
(*Spring constant*)		
pound/inch (lb/in.)	newton/meter (N/m)	1.7513×10^2
(*Velocity*)		
foot/second (ft/sec)	meter/second (m/s)	3.048×10^{-1}*
knot (nautical mi/hr)	meter/second (m/s)	5.1444×10^{-1}
mile/hour (mi/hr)	meter/second (m/s)	4.4704×10^{-1}*
mile/hour (mi/hr)	kilometer/hour (km/h)	1.6093
(*Volume*)		
foot^3 (ft^3)	meter^3 (m^3)	2.8317×10^{-2}
inch^3 (in.^{3})	meter^3 (m^3)	1.6387×10^{-5}
(*Work, Energy*)		
British thermal unit (BTU)	joule (J)	1.0551×10^3
foot-pound force (ft-lb)	joule (J)	1.3558
kilowatt-hour (kw-h)	joule (J)	3.60×10^6*

*Exact value

SI Units Used in Mechanics

Quantity	Unit	SI Symbol
(*Base Units*)		
Length	meter*	m
Mass	kilogram	kg
Time	second	s
(*Derived Units*)		
Acceleration, linear	meter/second^2	m/s^2
Acceleration, angular	radian/second^2	rad/s^2
Area	meter^2	m^2
Density	kilogram/meter^3	kg/m^3
Force	newton	N (= kg · m/s^2)
Frequency	hertz	Hz (= 1/s)
Impulse, linear	newton-second	N · s
Impulse, angular	newton-meter-second	N · m · s
Moment of force	newton-meter	N · m
Moment of inertia, area	meter^4	m^4
Moment of inertia, mass	kilogram-meter^2	kg · m^2
Momentum, linear	kilogram-meter/second	kg · m/s (= N · s)
Momentum, angular	kilogram-meter^2/second	kg · m^2/s (= N · m · s)
Power	watt	W (= J/s = N · m/s)
Pressure, stress	pascal	Pa (= N/m^2)
Product of inertia, area	meter^4	m^4
Product of inertia, mass	kilogram-meter^2	kg · m^2
Spring constant	newton/meter	N/m
Velocity, linear	meter/second	m/s
Velocity, angular	radian/second	rad/s
Volume	meter^3	m^3
Work, energy	joule	J (= N · m)
(*Supplementary and Other Acceptable Units*)		
Distance (navigation)	nautical mile	(= 1,852 km)
Mass	ton (metric)	t (= 1000 kg)
Plane angle	degrees (decimal)	°
Plane angle	radian	—
Speed	knot	(1.852 km/h)
Time	day	d
Time	hour	h
Time	minute	min

*Also spelled *metre*.

SI Unit Prefixes

Multiplication Factor	Prefix	Symbol
1 000 000 000 000 = 10^{12}	tera	T
1 000 000 000 = 10^{9}	giga	G
1 000 000 = 10^{6}	mega	M
1 000 = 10^{3}	kilo	k
100 = 10^{2}	hecto	h
10 = 10	deka	da
0.1 = 10^{-1}	deci	d
0.01 = 10^{-2}	centi	c
0.001 = 10^{-3}	milli	m
0.000 001 = 10^{-6}	micro	μ
0.000 000 001 = 10^{-9}	nano	n
0.000 000 000 001 = 10^{-12}	pico	p

Selected Rules for Writing Metric Quantities

1. (a) Use prefixes to keep numerical values generally between 0.1 and 1000.
   (b) Use of the prefixes hecto, deka, deci, and centi should generally be avoided except for certain areas or volumes where the numbers would be awkward otherwise.
   (c) Use prefixes only in the numerator of unit combinations. The one exception is the base unit kilogram. (*Example:* write kN/m not N/mm; J/kg not mJ/g)
   (d) Avoid double prefixes. (*Example:* write GN not kMN)
2. Unit designations
   (a) Use a dot for multiplication of units. (*Example:* write N · m not Nm)
   (b) Avoid ambiguous double solidus. (*Example:* write N/m^2 not N/m/m)
   (c) Exponents refer to entire unit. (*Example:* mm^2 means (mm)^2)
3. Number grouping
   Use a space rather than a comma to separate numbers in groups of three, counting from the decimal point in both directions. *Example:* 4 607 321.048 72) Space may be omitted for numbers of four digits. (*Example:* 4296 or 0.0476)

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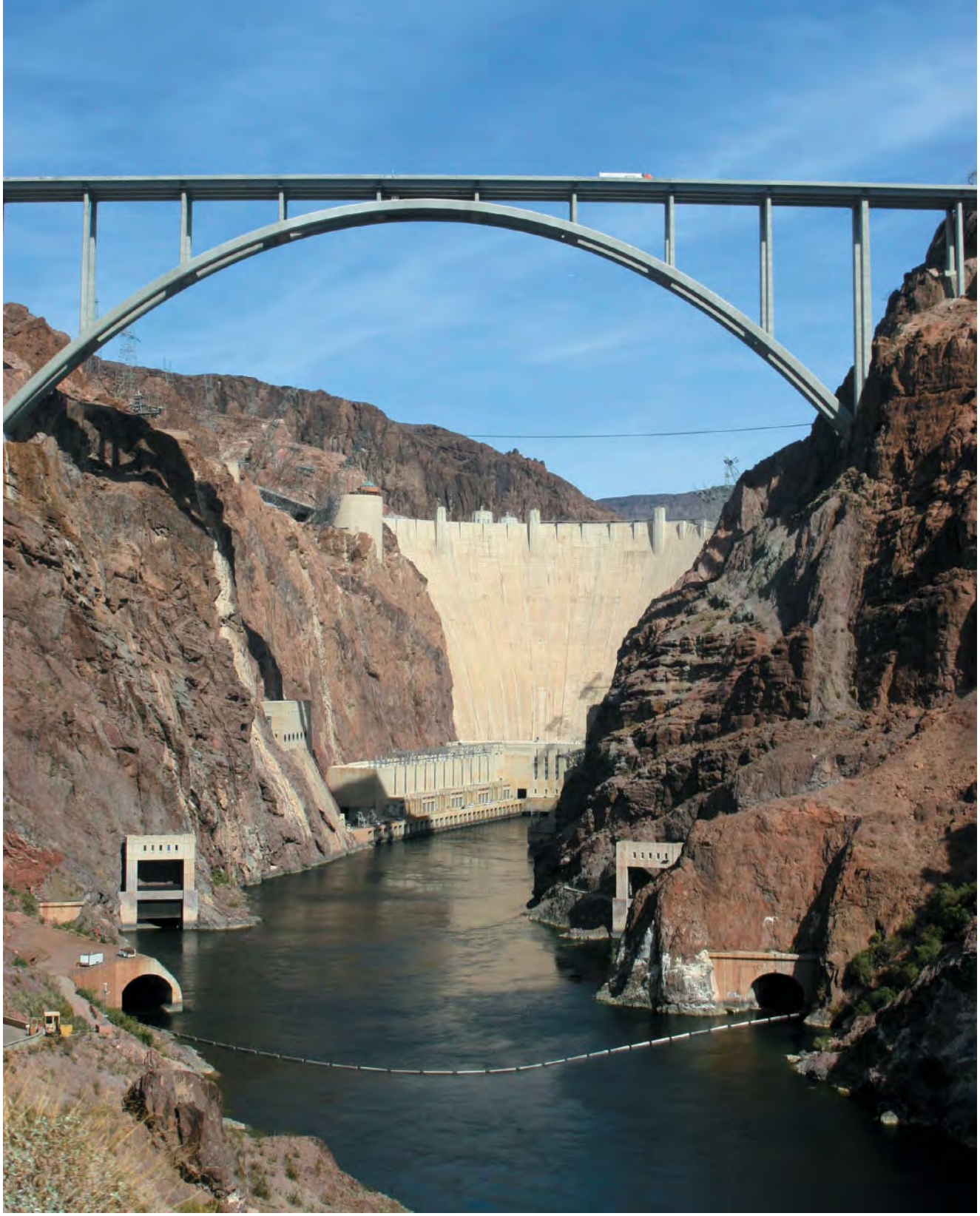

When forces are continuously distributed over a region of a structure, the cumulative effect of this distribution must be determined. In the case of Hoover Dam, the water pressure is distributed over the area of the dam and varies greatly with the depth of the water. For the Mike O'Callaghan–Pat Tillman Memorial Bridge, the weights of the roadway and any supported vehicles are distributed along the length of the roadway.

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Distributed Forces

5

CHAPTER OUTLINE

5/1 Introduction

Section A Centers of Mass and Centroids

5/2 Center of Mass

5/3 Centroids of Lines, Areas, and Volumes

5/4 Composite Bodies and Figures; Approximations

5/5 Theorems of Pappus

Section B Special Topics

5/6 Beams—External Effects

5/7 Beams—Internal Effects

5/8 Flexible Cables

5/9 Fluid Statics

5/10 Chapter Review

5/1 Introduction

In the previous chapters we treated all forces as concentrated along their lines of action and at their points of application. This treatment provided a reasonable model for those forces. Actually, "concentrated" forces do not exist in the exact sense, since every external force applied mechanically to a body is distributed over a finite contact area, however small.

The force exerted by the pavement on an automobile tire, for instance, is applied to the tire over its entire area of contact, Fig. 5/1*a*, which may be appreciable if the tire is soft. When analyzing the forces acting on the car as a whole, if the dimension b of the contact area is negligible compared with the other pertinent dimensions, such as the distance between wheels, then we may replace the actual distributed contact forces by their resultant R treated as a concentrated force. Even the force of contact between a hardened steel ball and its race in a loaded ball bearing, Fig. 5/1*b*, is applied over a finite though extremely small contact area. The forces applied to a two-force member of a truss, Fig. 5/1*c*, are applied over an actual area of contact of the pin against the hole and internally across the cut section as shown. In these and other similar examples we may treat the forces as concentrated when analyzing their external effects on bodies as a whole.

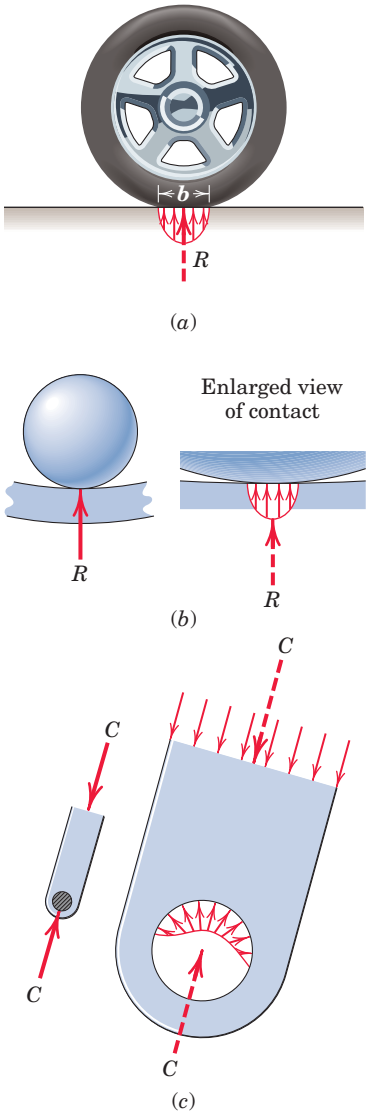


Figure 5/1

If, on the other hand, we want to find the distribution of *internal* forces in the material of the body near the contact location, where the internal stresses and strains may be appreciable, then we must not treat the load as concentrated but must consider the actual distribution. This problem will not be discussed here because it requires a knowledge of the properties of the material and belongs in more advanced treatments of the mechanics of materials and the theories of elasticity and plasticity.

When forces are applied over a region whose dimensions are not negligible compared with other pertinent dimensions, then we must account for the actual manner in which the force is distributed. We do this by summing the effects of the distributed force over the entire region using mathematical integration. This requires that we know the intensity of the force at any location. There are three categories of such problems.

(1) Line Distribution. When a force is distributed along a line, as in the continuous vertical load supported by a suspended cable, Fig. 5/2*a*, the intensity w of the loading is expressed as force per unit length of line, newtons per meter (N/m) or pounds per foot (lb/ft).

(2) Area Distribution. When a force is distributed over an area, as with the hydraulic pressure of water against the inner face of a section of dam, Fig. 5/2*b*, the intensity is expressed as force per unit area. This intensity is called *pressure* for the action of fluid forces and *stress* for the internal distribution of forces in solids. The basic unit for pressure or stress in SI is the newton per square meter (N/m^2), which is also called the *pascal* (Pa). This unit, however, is too small for most applications (6895 Pa = 1 $lb/in.^2$). The kilopascal (kPa), which equals 10^3 Pa, is more commonly used for fluid pressure, and the megapascal, which equals 10^6 Pa, is used for stress. In the U.S. customary system of units, both fluid pressure and mechanical stress are commonly expressed in pounds per square inch ($lb/in.^2$).

(3) Volume Distribution. A force which is distributed over the volume of a body is called a *body force*. The most common body force is the force of gravitational attraction, which acts on all elements of mass in a body. The determination of the forces on the supports of the heavy cantilevered structure in Fig. 5/2*c*, for example, would require accounting for the distribution of gravitational force throughout the structure. The intensity of gravitational force is the *specific weight* ρg, where ρ is the density (mass per unit volume) and g is the acceleration due to gravity. The units for ρg are $(kg/m^3)(m/s^2) = N/m^3$ in SI units and lb/ft^3 or $lb/in.^3$ in the U.S. customary system.

The body force due to the gravitational attraction of the earth (weight) is by far the most commonly encountered distributed force. Section A of this chapter treats the determination of the point in a body through which the resultant gravitational force acts, and discusses the associated geometric properties of lines, areas, and volumes. Section B treats distributed forces which act on and in beams and flexible cables and distributed forces which fluids exert on exposed surfaces.

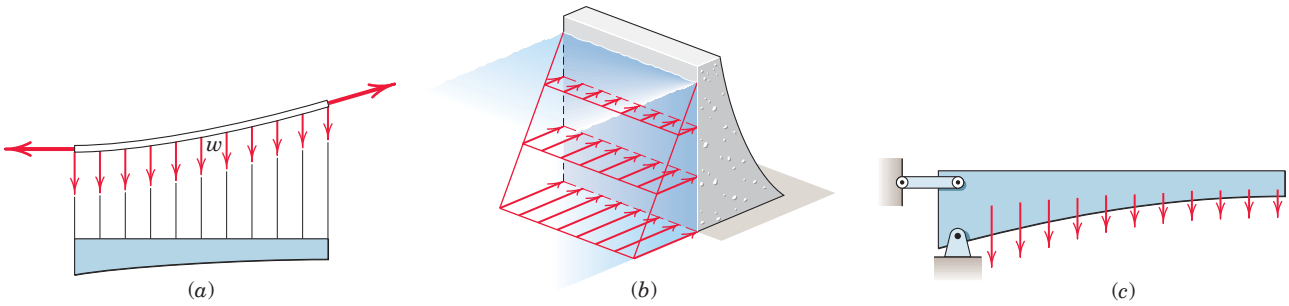


Figure 5/2

SECTION A CENTERS OF MASS AND CENTROIDS

5/2 Center of Mass

Consider a three-dimensional body of any size and shape, having a mass m. If we suspend the body, as shown in Fig. 5/3, from any point such as A, the body will be in equilibrium under the action of the tension in the cord and the resultant W of the gravitational forces acting on all particles of the body. This resultant is clearly collinear with the cord. Assume that we mark its position by drilling a hypothetical hole of negligible size along its line of action. We repeat the experiment by suspending the body from other points such as B and C, and in each instance we mark the line of action of the resultant force. For all practical purposes these lines of action will be concurrent at a single point G, which is called the *center of gravity* of the body.

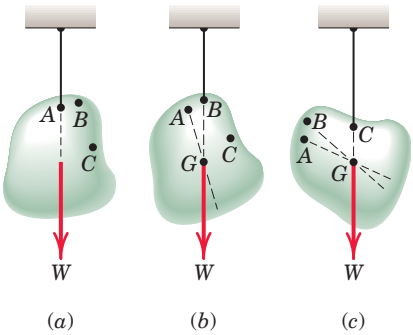


Figure 5/3

An exact analysis, however, would account for the slightly differing directions of the gravity forces for the various particles of the body, because those forces converge toward the center of attraction of the earth. Also, because the particles are at different distances from the earth, the intensity of the force field of the earth is not exactly constant over the body. As a result, the lines of action of the gravity-force resultants in the experiments just described will not be quite concurrent, and therefore no unique center of gravity exists in the exact sense. This is of no practical importance as long as we deal with bodies whose dimensions are small compared with those of the earth. We therefore assume a uniform and parallel force field due to the gravitational attraction of the earth, and this assumption results in the concept of a unique center of gravity.

Determining the Center of Gravity

To determine mathematically the location of the center of gravity of any body, Fig. 5/4*a*, we apply the *principle of moments* (see Art. 2/6) to the parallel system of gravitational forces. The moment of the resultant gravitational force W about any axis equals the sum of the moments

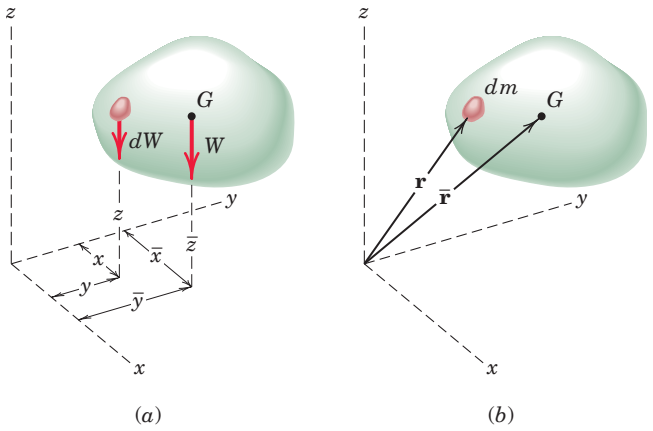


Figure 5/4

about the same axis of the gravitational forces dW acting on all particles treated as infinitesimal elements of the body. The resultant of the gravitational forces acting on all elements is the weight of the body and is given by the sum $W = \int dW$. If we apply the moment principle about the y-axis, for example, the moment about this axis of the elemental weight is $x\,dW$, and the sum of these moments for all elements of the body is $\int x\,dW$. This sum of moments must equal $W\bar{x}$, the moment of the sum. Thus, $\bar{x}W = \int x\,dW$.

With similar expressions for the other two components, we may express the coordinates of the center of gravity G as

$$\bar{x} = \frac{\int x\,dW}{W} \qquad \bar{y} = \frac{\int y\,dW}{W} \qquad \bar{z} = \frac{\int z\,dW}{W} \tag{5/1a}$$

To visualize the physical moments of the gravity forces appearing in the third equation, we may reorient the body and attached axes so that the z-axis is horizontal. It is essential to recognize that the numerator of each of these expressions represents the *sum of the moments*, whereas the product of W and the corresponding coordinate of G represents the *moment of the sum*. This moment principle finds repeated use throughout mechanics.

With the substitution of $W = mg$ and $dW = g\,dm$, the expressions for the coordinates of the center of gravity become

$$\bar{x} = \frac{\int x\,dm}{m} \qquad \bar{y} = \frac{\int y\,dm}{m} \qquad \bar{z} = \frac{\int z\,dm}{m} \tag{5/1b}$$

Equations 5/1*b* may be expressed in vector form with the aid of Fig. 5/4*b*, in which the elemental mass and the mass center G are located by

their respective position vectors $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\bar{\mathbf{r}} = \bar{x}\mathbf{i} + \bar{y}\mathbf{j} + \bar{z}\mathbf{k}$. Thus, Eqs. 5/1*b* are the components of the single vector equation

$$\bar{\mathbf{r}} = \frac{\int \mathbf{r}\, dm}{m} \tag{5/2}$$

The density ρ of a body is its mass per unit volume. Thus, the mass of a differential element of volume dV becomes $dm = \rho\, dV$. If ρ is not constant throughout the body but can be expressed as a function of the coordinates of the body, we must account for this variation when calculating the numerators and denominators of Eqs. 5/1*b*. We may then write these expressions as

$$\bar{x} = \frac{\int x\rho\, dV}{\int \rho\, dV} \qquad \bar{y} = \frac{\int y\rho\, dV}{\int \rho\, dV} \qquad \bar{z} = \frac{\int z\rho\, dV}{\int \rho\, dV} \tag{5/3}$$

Center of Mass versus Center of Gravity

Equations 5/1*b*, 5/2, and 5/3 are independent of gravitational effects since g no longer appears. They therefore define a unique point in the body which is a function solely of the distribution of mass. This point is called the *center of mass*, and clearly it coincides with the center of gravity as long as the gravity field is treated as uniform and parallel.

It is meaningless to speak of the center of gravity of a body which is removed from the gravitational field of the earth, since no gravitational forces would act on it. The body would, however, still have its unique center of mass. We will usually refer henceforth to the center of mass rather than to the center of gravity. Also, the center of mass has a special significance in calculating the dynamic response of a body to unbalanced forces. This class of problems is discussed at length in *Vol. 2 Dynamics*.

In most problems the calculation of the position of the center of mass may be simplified by an intelligent choice of reference axes. In general the axes should be placed so as to simplify the equations of the boundaries as much as possible. Thus, polar coordinates will be useful for bodies with circular boundaries.

Another important clue may be taken from considerations of symmetry. Whenever there exists a line or plane of symmetry in a homogeneous body, a coordinate axis or plane should be chosen to coincide with this line or plane. The center of mass will always lie on such a line or plane, since the moments due to symmetrically located elements will always cancel, and the body may be considered composed of pairs of these elements. Thus, the center of mass G of the homogeneous right-circular cone of Fig. 5/5*a* will lie somewhere on its central axis, which is a line of symmetry. The center of mass of the half right-circular cone lies on its plane of symmetry, Fig. 5/5*b*. The center of mass of the half ring in Fig. 5/5*c* lies in both of its planes of symmetry and therefore is situated on

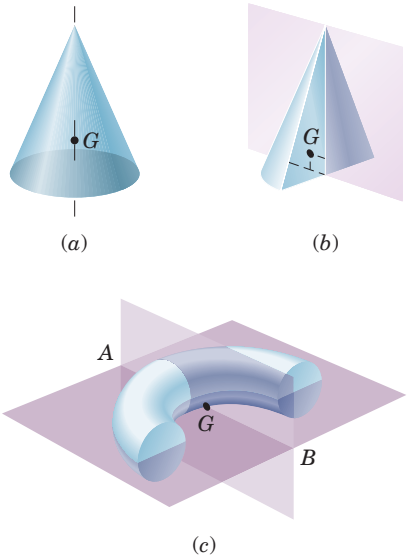


Figure 5/5

line AB. It is easiest to find the location of G by using symmetry when it exists.

5/3 Centroids of Lines, Areas, and Volumes

When the density ρ of a body is uniform throughout, it will be a constant factor in both the numerators and denominators of Eqs. 5/3 and will therefore cancel. The remaining expressions define a purely geometrical property of the body, since any reference to its mass properties has disappeared. The term *centroid* is used when the calculation concerns a geometrical shape only. When speaking of an actual physical body, we use the term *center of mass*. If the density is uniform throughout the body, the positions of the centroid and center of mass are identical, whereas if the density varies, these two points will, in general, not coincide.

The calculation of centroids falls within three distinct categories, depending on whether we can model the shape of the body involved as a line, an area, or a volume.

(1) Lines. For a slender rod or wire of length L, cross-sectional area A, and density ρ, Fig. 5/6, the body approximates a line segment, and $dm = \rho A\, dL$. If ρ and A are constant over the length of the rod, the coordinates of the center of mass also become the coordinates of the centroid C of the line segment, which, from Eqs. 5/1*b*, may be written

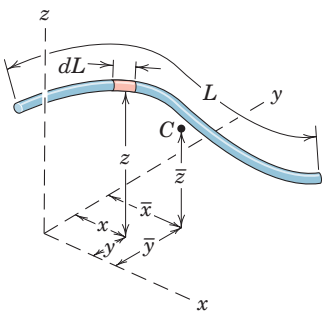


Figure 5/6

$$\bar{x} = \frac{\int x\, dL}{L} \qquad \bar{y} = \frac{\int y\, dL}{L} \qquad \bar{z} = \frac{\int z\, dL}{L} \tag{5/4}$$

Note that, in general, the centroid C will not lie on the line. If the rod lies on a single plane, such as the x-y plane, only two coordinates need to be calculated.

(2) Areas. When a body of density ρ has a small but constant thickness t, we can model it as a surface area A, Fig. 5/7. The mass of an element becomes $dm = \rho t\, dA$. Again, if ρ and t are constant over the entire area, the coordinates of the center of mass of the body also become the coordinates of the centroid C of the surface area, and from Eqs. 5/1*b* the coordinates may be written

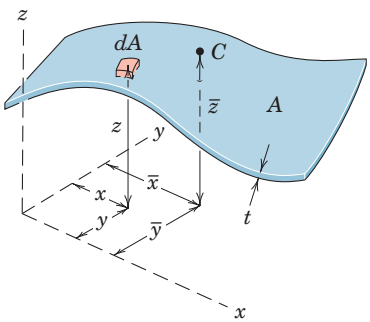


Figure 5/7

$$\bar{x} = \frac{\int x\, dA}{A} \qquad \bar{y} = \frac{\int y\, dA}{A} \qquad \bar{z} = \frac{\int z\, dA}{A} \tag{5/5}$$

The numerators in Eqs. 5/5 are called the *first moments of area*.* If the surface is curved, as illustrated in Fig. 5/7 with the shell segment, all three coordinates will be involved. The centroid C for the curved surface will in general not lie on the surface. If the area is a flat surface in,

*Second moments of areas (moments of first moments) appear later in our discussion of area moments of inertia in Appendix A.

say, the x-y plane, only the coordinates of C in that plane need to be calculated.

(3) Volumes. For a general body of volume V and density ρ, the element has a mass $dm = \rho\, dV$. The density ρ cancels if it is constant over the entire volume, and the coordinates of the center of mass also become the coordinates of the centroid C of the body. From Eqs. 5/3 or 5/1b they become

$$\bar{x} = \frac{\int x\, dV}{V} \qquad \bar{y} = \frac{\int y\, dV}{V} \qquad \bar{z} = \frac{\int z\, dV}{V} \tag{5/6}$$

KEY CONCEPTS

Choice of Element for Integration

The principal difficulty with a theory often lies not in its concepts but in the procedures for applying it. With mass centers and centroids the concept of the moment principle is simple enough; the difficult steps are the choice of the differential element and setting up the integrals. The following five guidelines will be useful.

(1) Order of Element. Whenever possible, a first-order differential element should be selected in preference to a higher-order element so that only one integration will be required to cover the entire figure. Thus, in Fig. 5/8a a first-order horizontal strip of area $dA = l\, dy$ will require only one integration with respect to y to cover the entire figure. The second-order element $dx\, dy$ will require two integrations, first with respect to x and second with respect to y, to cover the figure. As a further example, for the solid cone in Fig. 5/8b we choose a first-order element in the form of a circular slice of volume $dV = \pi r^2\, dy$. This choice requires only one integration, and thus is preferable to choosing a third-order element $dV = dx\, dy\, dz$, which would require three awkward integrations.

(2) Continuity. Whenever possible, we choose an element which can be integrated in one continuous operation to cover the figure. Thus, the horizontal strip in Fig. 5/8a would be preferable to the vertical strip in Fig. 5/9, which, if used, would require two separate integrals because of the discontinuity in the expression for the height of the strip at $x = x_1$.

(3) Discarding Higher-Order Terms. Higher-order terms may always be dropped compared with lower-order terms (see Art. 1/7). Thus, the vertical strip of area under the curve in Fig. 5/10 is given by the

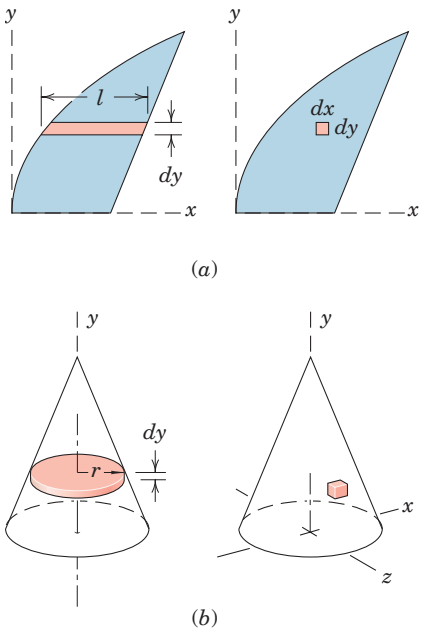


Figure 5/8

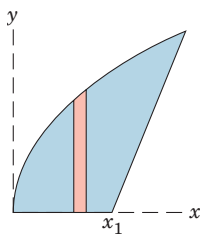


Figure 5/9

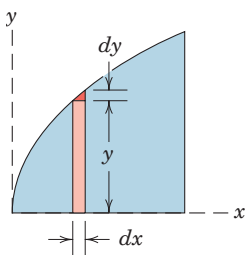


Figure 5/10

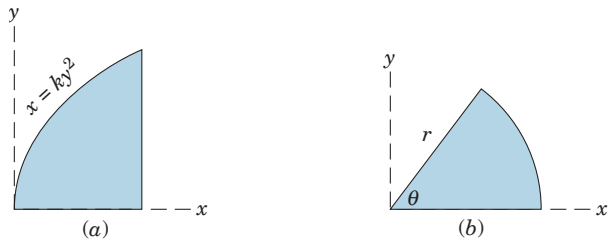


Figure 5/11

first-order term $dA = y\,dx$, and the second-order triangular area $\frac{1}{2}dx\,dy$ is discarded. In the limit, of course, there is no error.

(4) Choice of Coordinates. As a general rule, we choose the coordinate system which best matches the boundaries of the figure. Thus, the boundaries of the area in Fig. 5/11*a* are most easily described in rectangular coordinates, whereas the boundaries of the circular sector of Fig. 5/11*b* are best suited to polar coordinates.

(5) Centroidal Coordinate of Element. When a first- or second-order differential element is chosen, it is essential to use the *coordinate of the centroid of the element* for the moment arm in expressing the moment of the differential element. Thus, for the horizontal strip of area in Fig. 5/12*a*, the moment of dA about the y-axis is $x_c\,dA$, where x_c is the x-coordinate of the centroid C of the element. Note that x_c is *not* the x which describes either boundary of the area. In the y-direction for this element the moment arm y_c of the centroid of the element is the same, in the limit, as the y-coordinates of the two boundaries.

As a second example, consider the solid half-cone of Fig. 5/12*b* with the semicircular slice of differential thickness as the element of volume. The moment arm for the element in the x-direction is the distance x_c to the centroid of the face of the element and not the x-distance to the boundary of the element. On the other hand, in the z-direction the moment arm z_c of the centroid of the element is the same as the z-coordinate of the element.

With these examples in mind, we rewrite Eqs. 5/5 and 5/6 in the form

$$\bar{x} = \frac{\int x_c\,dA}{A} \qquad \bar{y} = \frac{\int y_c\,dA}{A} \qquad \bar{z} = \frac{\int z_c\,dA}{A} \tag{5/5a}$$

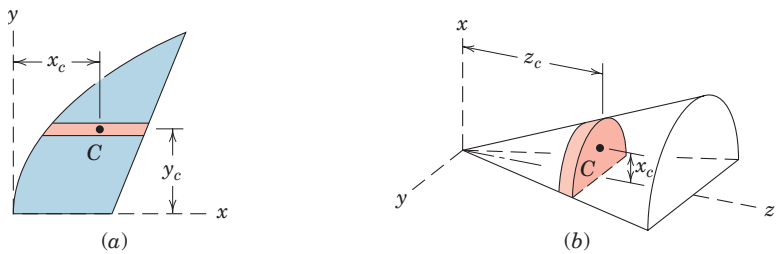


Figure 5/12

and

$$\bar{x} = \frac{\int x_c \, dV}{V} \qquad \bar{y} = \frac{\int y_c \, dV}{V} \qquad \bar{z} = \frac{\int z_c \, dV}{V} \tag{5/6a}$$

It is *essential* to recognize that the subscript c serves as a reminder that the moment arms appearing in the numerators of the integral expressions for moments are *always* the coordinates of the *centroids* of the particular elements chosen.

At this point you should be certain to understand clearly the principle of moments, which was introduced in Art. 2/4. You should recognize the physical meaning of this principle as it is applied to the system of parallel weight forces depicted in Fig. 5/4*a*. Keep in mind the equivalence between the moment of the resultant weight W and the sum (integral) of the moments of the elemental weights dW, to avoid mistakes in setting up the necessary mathematics. Recognition of the principle of moments will help in obtaining the correct expression for the moment arm x_c, y_c, or z_c of the centroid of the chosen differential element.

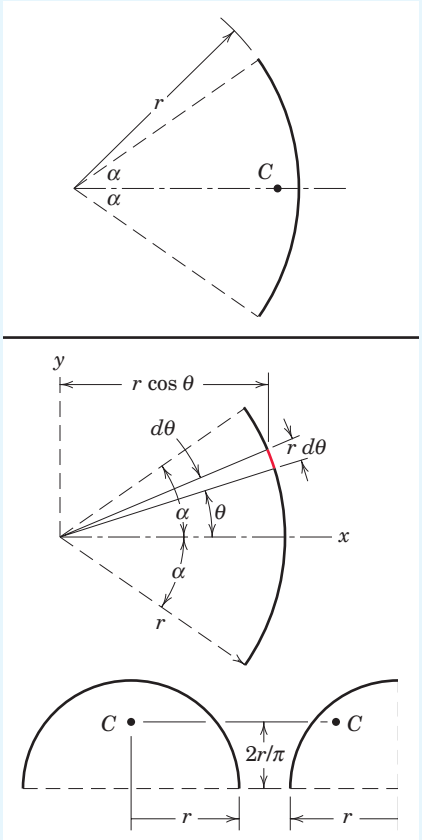
Keeping in mind the physical picture of the principle of moments, we will recognize that Eqs. 5/4, 5/5, and 5/6, which are geometric relationships, are descriptive also of homogeneous physical bodies, because the density ρ cancels. If the density of the body in question is not constant but varies throughout the body as some function of the coordinates, then it will not cancel from the numerator and denominator of the mass-center expressions. In this event, we must use Eqs. 5/3 as explained earlier.

Sample Problems 5/1 through 5/5 which follow have been carefully chosen to illustrate the application of Eqs. 5/4, 5/5, and 5/6 for calculating the location of the centroid for line segments (slender rods), areas (thin flat plates), and volumes (homogeneous solids). The five integration considerations listed above are illustrated in detail in these sample problems.

Section C/10 of Appendix C contains a table of integrals which includes those needed for the problems in this and subsequent chapters. A summary of the centroidal coordinates for some of the commonly used shapes is given in Tables D/3 and D/4, Appendix D.

SAMPLE PROBLEM 5/1

Centroid of a circular arc. Locate the centroid of a circular arc as shown in the figure.



Solution. Choosing the axis of symmetry as the x-axis makes $\bar{y} = 0$. A differential element of arc has the length $dL = r\,d\theta$ expressed in polar coordinates, and the x-coordinate of the element is $r\cos\theta$. (1)

Applying the first of Eqs. 5/4 and substituting $L = 2\alpha r$ give

$$[L\bar{x} = \int x\,dL] \qquad (2\alpha r)\bar{x} = \int_{-\alpha}^{\alpha} (r\cos\theta)\, r\,d\theta$$

$$2\alpha r\bar{x} = 2r^2 \sin\alpha$$

$$\bar{x} = \frac{r\sin\alpha}{\alpha} \qquad \textit{Ans.}$$

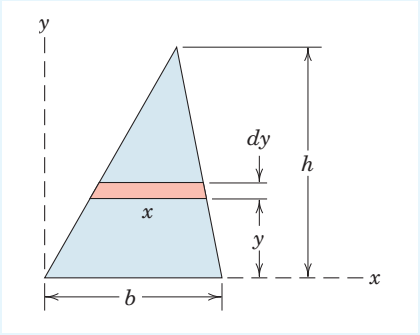
For a semicircular arc $2\alpha = \pi$, which gives $\bar{x} = 2r/\pi$. By symmetry we see immediately that this result also applies to the quarter-circular arc when the measurement is made as shown.

Helpful Hint

(1) It should be perfectly evident that polar coordinates are preferable to rectangular coordinates to express the length of a circular arc.

SAMPLE PROBLEM 5/2

Centroid of a triangular area. Determine the distance $\bar{h}$ from the base of a triangle of altitude h to the centroid of its area.



Solution. The x-axis is taken to coincide with the base. A differential strip of area $dA = x\,dy$ is chosen. By similar triangles $x/(h - y) = b/h$. Applying the second of Eqs. 5/5a gives (1)

$$[A\bar{y} = \int y_c\,dA] \qquad \frac{bh}{2}\bar{y} = \int_0^h y\,\frac{b(h-y)}{h}\,dy = \frac{bh^2}{6}$$

and

$$\bar{y} = \frac{h}{3} \qquad \textit{Ans.}$$

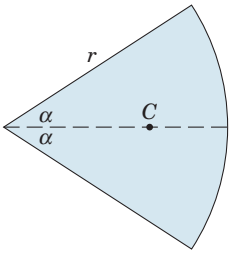
This same result holds with respect to either of the other two sides of the triangle considered a new base with corresponding new altitude. Thus, the centroid lies at the intersection of the medians, since the distance of this point from any side is one-third the altitude of the triangle with that side considered the base.

Helpful Hint

(1) We save one integration here by using the first-order element of area. Recognize that dA must be expressed in terms of the integration variable y; hence, $x = f(y)$ is required.

SAMPLE PROBLEM 5/3

Centroid of the area of a circular sector. Locate the centroid of the area of a circular sector with respect to its vertex.



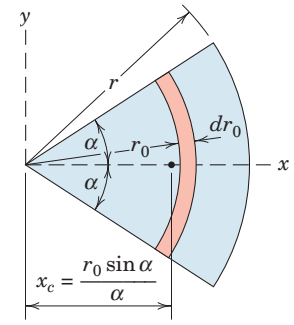
Solution I. The x-axis is chosen as the axis of symmetry, and $\bar{y}$ is therefore automatically zero. We may cover the area by moving an element in the form of a partial circular ring, as shown in the figure, from the center to the outer periphery. The radius of the ring is r_0 and its thickness is dr_0, so that its area is
1 $dA = 2r_0\alpha\, dr_0$.

The x-coordinate to the centroid of the element from Sample Problem 5/1 is
2 $x_c = r_0 \sin\alpha/\alpha$, where r_0 replaces r in the formula. Thus, the first of Eqs. 5/5a gives

$$[A\bar{x} = \int x_c\, dA] \qquad \frac{2\alpha}{2\pi}(\pi r^2)\bar{x} = \int_0^r \left(\frac{r_0 \sin\alpha}{\alpha}\right)(2r_0\alpha\, dr_0)$$

$$r^2\alpha\bar{x} = \frac{2}{3}r^3 \sin\alpha$$

$$\bar{x} = \frac{2}{3}\frac{r\sin\alpha}{\alpha} \qquad \textit{Ans.}$$



Solution I

Solution II. The area may also be covered by swinging a triangle of differential area about the vertex and through the total angle of the sector. This triangle, shown in the illustration, has an area $dA = (r/2)(r\, d\theta)$, where higher-order terms are neglected. From Sample Problem 5/2 the centroid of the triangular element of area is two-thirds of its altitude from its vertex, so that the x-coordinate to the centroid of the element is $x_c = \frac{2}{3}r\cos\theta$. Applying the first of Eqs. 5/5a gives

$$[A\bar{x} = \int x_c\, dA] \qquad (r^2\alpha)\bar{x} = \int_{-\alpha}^{\alpha} (\tfrac{2}{3}r\cos\theta)(\tfrac{1}{2}r^2\, d\theta)$$

$$r^2\alpha\bar{x} = \frac{2}{3}r^3 \sin\alpha$$

and as before

$$\bar{x} = \frac{2}{3}\frac{r\sin\alpha}{\alpha} \qquad \textit{Ans.}$$

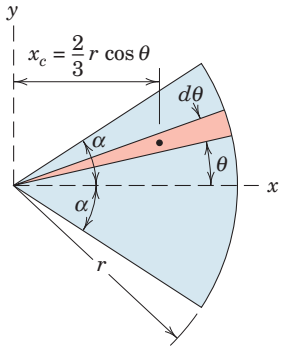
For a semicircular area $2\alpha = \pi$, which gives $\bar{x} = 4r/3\pi$. By symmetry we see immediately that this result also applies to the quarter-circular area where the measurement is made as shown.

It should be noted that, if we had chosen a second-order element $r_0\, dr_0\, d\theta$, one integration with respect to θ would yield the ring with which *Solution I* began. On the other hand, integration with respect to r_0 initially would give the triangular element with which *Solution II* began.

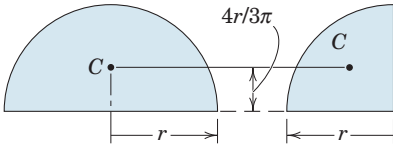
Helpful Hints

1 Note carefully that we must distinguish between the variable r_0 and the constant r.

2 Be careful not to use r_0 as the centroidal coordinate for the element.

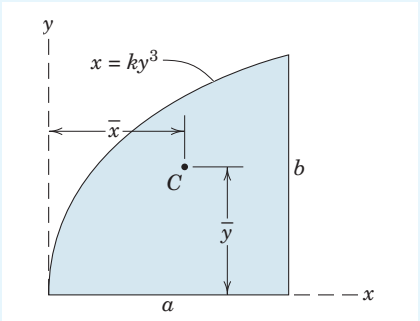


Solution II



SAMPLE PROBLEM 5/4

Locate the centroid of the area under the curve $x = ky^3$ from $x = 0$ to $x = a$.



Solution I. A vertical element of area $dA = y\,dx$ is chosen as shown in the figure. The x-coordinate of the centroid is found from the first of Eqs. 5/5a. Thus,

(1) $[A\bar{x} = \int x_c\,dA] \qquad \bar{x}\int_0^a y\,dx = \int_0^a xy\,dx$

Substituting $y = (x/k)^{1/3}$ and $k = a/b^3$ and integrating give

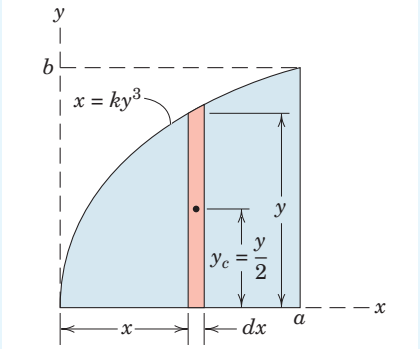
$$\frac{3ab}{4}\bar{x} = \frac{3a^2b}{7} \qquad \bar{x} = \tfrac{4}{7}a \qquad \textit{Ans.}$$

In the solution for $\bar{y}$ from the second of Eqs. 5/5a, the coordinate to the centroid of the rectangular element is $y_c = y/2$, where y is the height of the strip governed by the equation of the curve $x = ky^3$. Thus, the moment principle becomes

$$[A\bar{y} = \int y_c\,dA] \qquad \frac{3ab}{4}\bar{y} = \int_0^a \left(\frac{y}{2}\right)y\,dx$$

Substituting $y = b(x/a)^{1/3}$ and integrating give

$$\frac{3ab}{4}\bar{y} = \frac{3ab^2}{10} \qquad \bar{y} = \tfrac{2}{5}b \qquad \textit{Ans.}$$



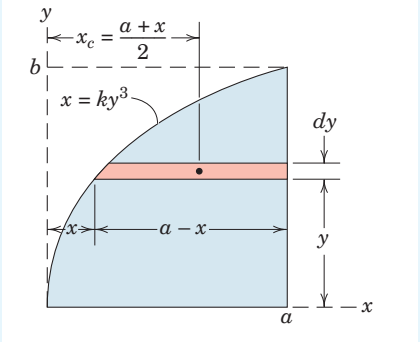
Solution II. The horizontal element of area shown in the lower figure may be employed in place of the vertical element. The x-coordinate to the centroid of the rectangular element is seen to be $x_c = x + \frac{1}{2}(a - x) = (a + x)/2$, which is simply the average of the coordinates a and x of the ends of the strip. Hence,

$$[A\bar{x} = \int x_c\,dA] \qquad \bar{x}\int_0^b (a - x)\,dy = \int_0^b \left(\frac{a + x}{2}\right)(a - x)\,dy$$

The value of $\bar{y}$ is found from

$$[A\bar{y} = \int y_c\,dA] \qquad \bar{y}\int_0^b (a - x)\,dy = \int_0^b y(a - x)\,dy$$

where $y_c = y$ for the horizontal strip. The evaluation of these integrals will check the previous results for $\bar{x}$ and $\bar{y}$.



Helpful Hint

(1) Note that $x_c = x$ for the vertical element.

SAMPLE PROBLEM 5/5

Hemispherical volume. Locate the centroid of the volume of a hemisphere of radius r with respect to its base.

Solution I. With the axes chosen as shown in the figure, $\bar{x} = \bar{z} = 0$ by symmetry. The most convenient element is a circular slice of thickness dy parallel to the x-z plane. Since the hemisphere intersects the y-z plane in the circle $y^2 + z^2 = r^2$, the radius of the circular slice is $z = +\sqrt{r^2 - y^2}$. The volume of the elemental slice becomes

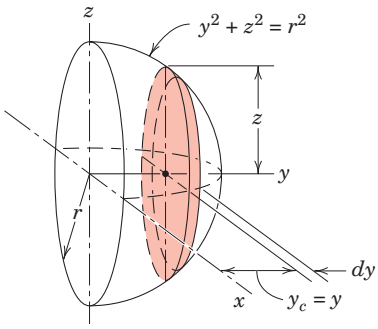
$$dV = \pi(r^2 - y^2)\,dy \tag{1}$$

The second of Eqs. 5/6*a* requires

$$[V\bar{y} = \int y_c\,dV] \qquad \bar{y}\int_0^r \pi(r^2 - y^2)\,dy = \int_0^r y\pi(r^2 - y^2)\,dy$$

where $y_c = y$. Integrating gives

$$\tfrac{2}{3}\pi r^3\bar{y} = \tfrac{1}{4}\pi r^4 \qquad \bar{y} = \tfrac{3}{8}r \qquad \textit{Ans.}$$



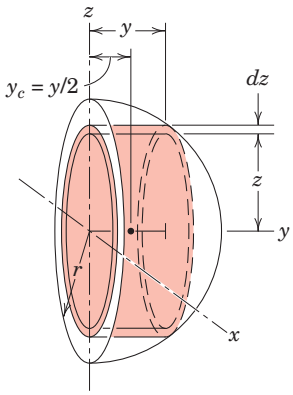
Solution I

Solution II. Alternatively we may use for our differential element a cylindrical shell of length y, radius z, and thickness dz, as shown in the lower figure. By expanding the radius of the shell from zero to r, we cover the entire volume. By symmetry the centroid of the elemental shell lies at its center, so that $y_c = y/2$. The volume of the element is $dV = (2\pi z\,dz)(y)$. Expressing y in terms of z from the equation of the circle gives $y = +\sqrt{r^2 - z^2}$. Using the value of $\frac{2}{3}\pi r^3$ computed in *Solution I* for the volume of the hemisphere and substituting in the second of Eqs. 5/6*a* give us

$$[V\bar{y} = \int y_c\,dV] \qquad (\tfrac{2}{3}\pi r^3)\bar{y} = \int_0^r \frac{\sqrt{r^2 - z^2}}{2}(2\pi z\sqrt{r^2 - z^2})\,dz$$

$$= \int_0^r \pi(r^2 z - z^3)\,dz = \frac{\pi r^4}{4}$$

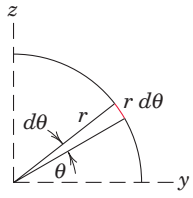
$$\bar{y} = \tfrac{3}{8}r \qquad \textit{Ans.}$$



Solution II

Solutions I and *II* are of comparable use since each involves an element of simple shape and requires integration with respect to one variable only.

Solution III. As an alternative, we could use the angle θ as our variable with limits of 0 and $\pi/2$. The radius of either element would become $r \sin\theta$, whereas the thickness of the slice in *Solution I* would be $dy = (r\,d\theta)\sin\theta$ and that of the shell in *Solution II* would be $dz = (r\,d\theta)\cos\theta$. The length of the shell would be $y = r\cos\theta$.



Solution III

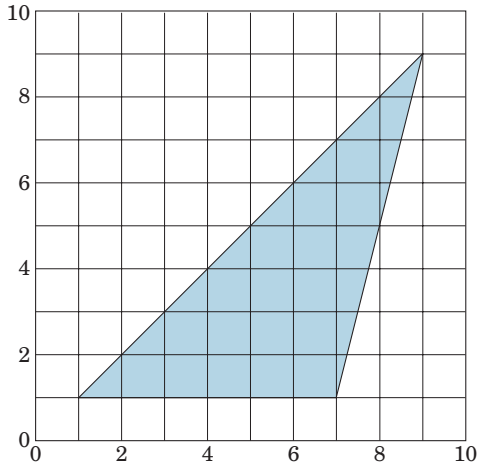
Helpful Hint

1. Can you identify the higher-order element of volume which is omitted from the expression for dV?

PROBLEMS

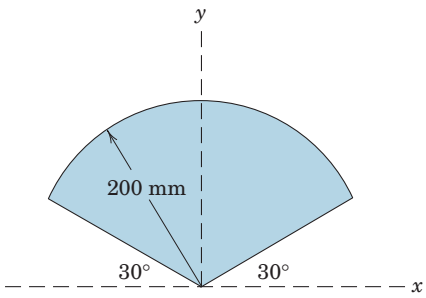
Introductory Problems

5/1 With your pencil, make a dot on the position of your best visual estimate of the centroid of the triangular area. Check the position of your estimate by referring to the results of Sample Problem 5/2 and to Table D/3.



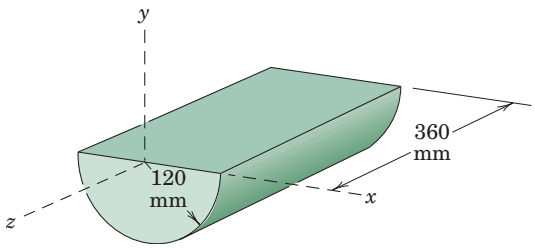
Problem 5/1

5/2 With your pencil, make a dot on the position of your best visual estimate of the centroid of the area of the circular sector. Check your estimate by using the results of Sample Problem 5/3.



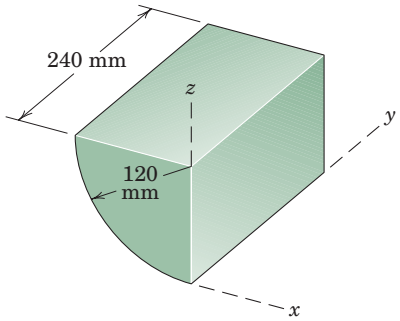
Problem 5/2

5/3 Specify the x-, y-, and z-coordinates of the mass center of the homogeneous semicylinder.



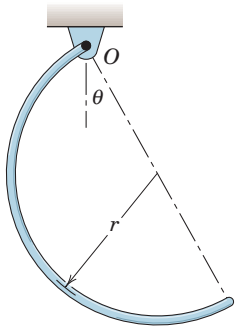
Problem 5/3

5/4 Specify the x-, y-, and z-coordinates of the mass center of the quadrant of the homogeneous solid cylinder.



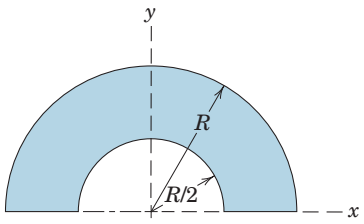
Problem 5/4

5/5 A uniform semicircular rod of radius r is supported in a bearing at its upper end and is free to swing in the vertical plane. Calculate the angle θ made by the diameter with the vertical for the equilibrium position.



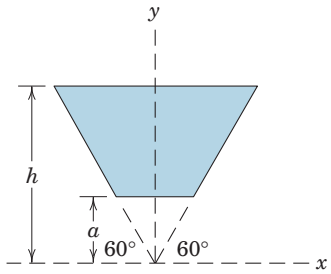
Problem 5/5

5/6 Determine the y-coordinate of the centroid of the area by direct integration.



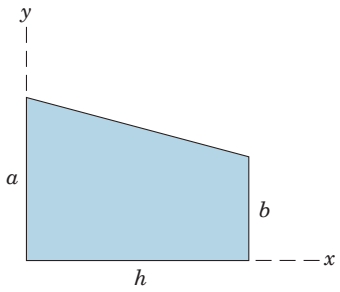
Problem 5/6

5/7 Determine the y-coordinate of the centroid of the shaded area. Check your result for the special case $a = 0$.



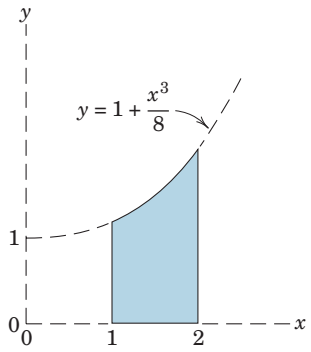
Problem 5/7

5/8 Determine the x- and y-coordinates of the centroid of the trapezoidal area.



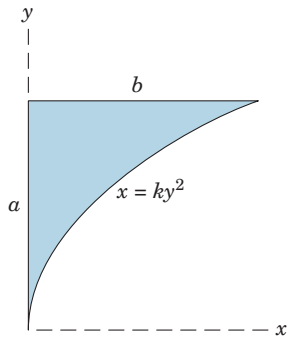
Problem 5/8

5/9 Determine the x- and y-coordinates of the centroid of the shaded area.



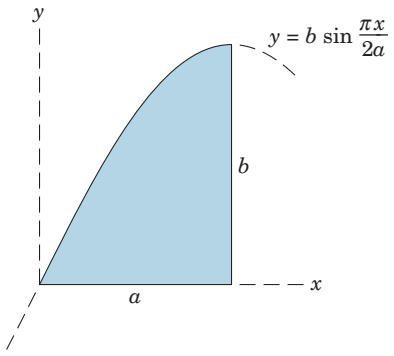
Problem 5/9

5/10 Determine the coordinates of the centroid of the shaded area.



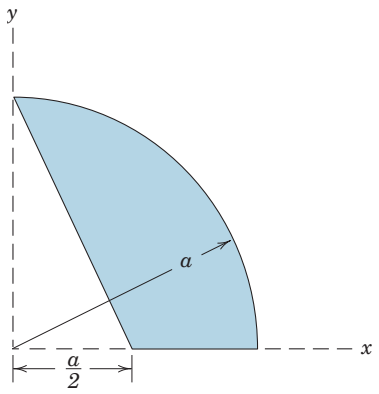
Problem 5/10

5/11 Determine the coordinates of the centroid of the shaded area.



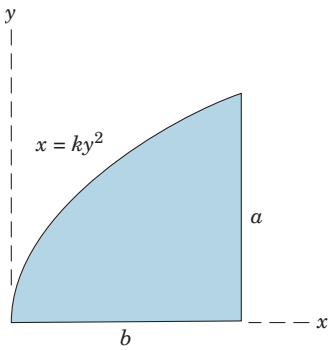
Problem 5/11

5/12 Determine the x- and y-coordinates of the centroid of the shaded area.



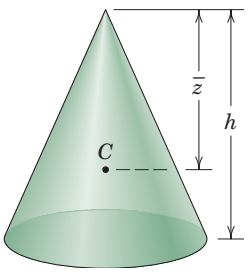
Problem 5/12

5/13 Determine the coordinates of the centroid of the shaded area.



Problem 5/13

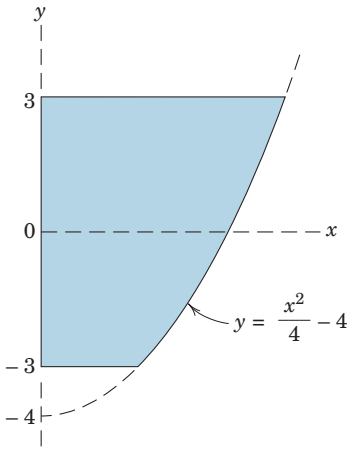
5/14 Find the distance $\bar{z}$ from the vertex of the right-circular cone to the centroid of its volume.



Problem 5/14

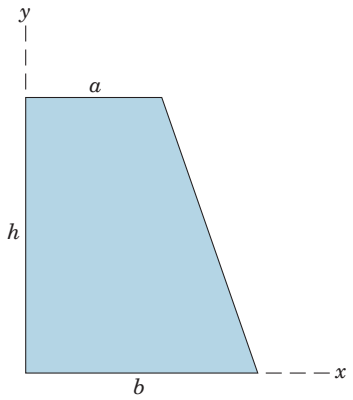
Representative Problems

5/15 Locate the centroid of the shaded area shown.



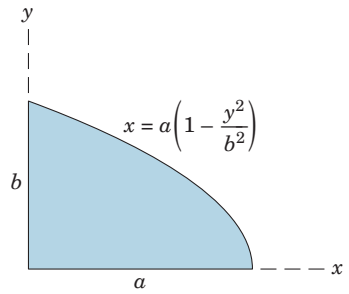
Problem 5/15

5/16 Determine the x- and y-coordinates of the centroid of the trapezoidal area.



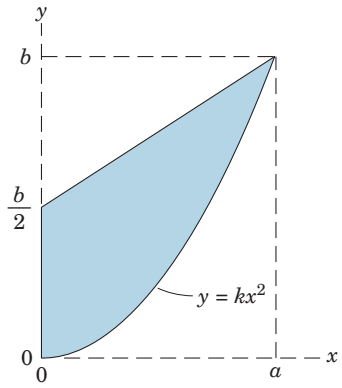
Problem 5/16

5/17 Locate the centroid of the shaded area.



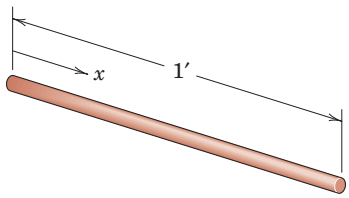
Problem 5/17

5/18 Determine the coordinates of the centroid of the shaded area.



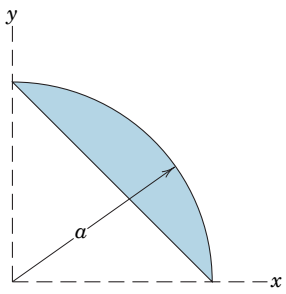
Problem 5/18

5/19 The mass per unit length of the slender rod varies with position according to $\rho = \rho_0(1 - x/2)$, where x is in feet. Determine the location of the center of mass of the rod.



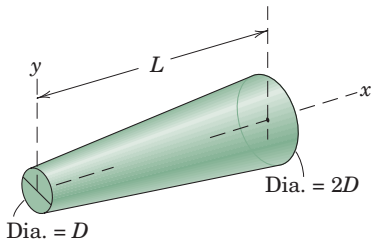
Problem 5/19

5/20 Calculate the coordinates of the centroid of the segment of the circular area.



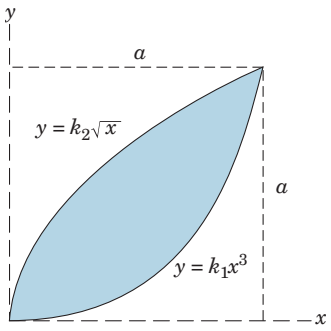
Problem 5/20

5/21 Determine the x-coordinate of the mass center of the tapered steel rod of length L where the diameter at the large end is twice the diameter at the small end.



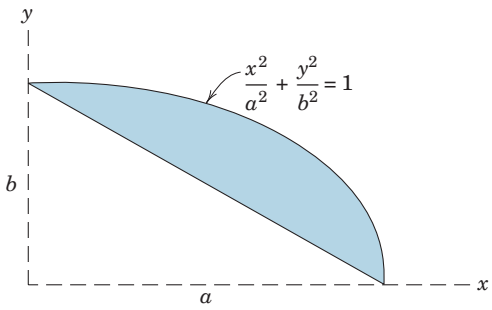
Problem 5/21

5/22 Determine the x- and y-coordinates of the centroid of the shaded area.



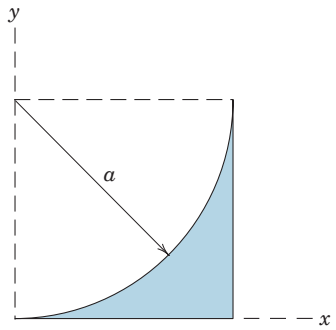
Problem 5/22

5/23 Determine the x- and y-coordinates of the centroid of the shaded area shown.



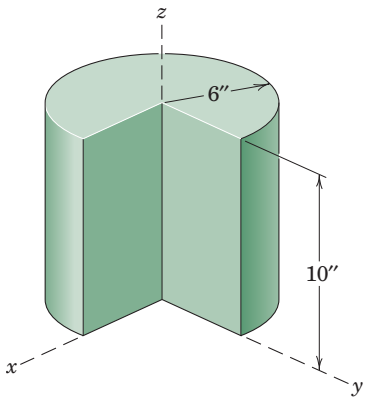
Problem 5/23

5/24 Locate the centroid of the area shown in the figure by direct integration. (*Caution:* Observe carefully the proper sign of the radical involved.)



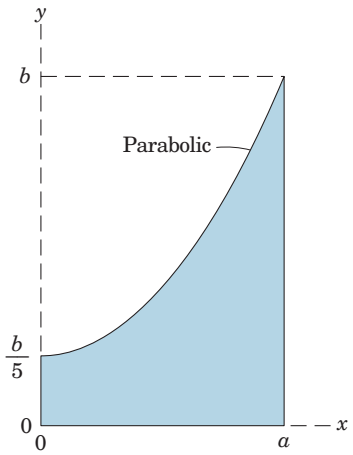
Problem 5/24

5/25 Use the results of Sample Problem 5/3 to compute the coordinates of the mass center of the portion of the solid homogeneous cylinder shown.



Problem 5/25

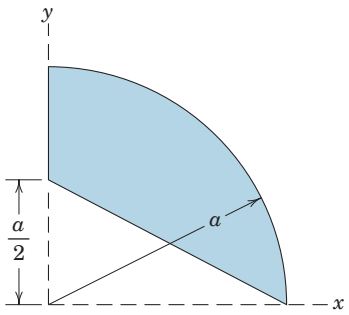
5/26 Determine the y-coordinate of the centroid of the shaded area shown.



Problem 5/26

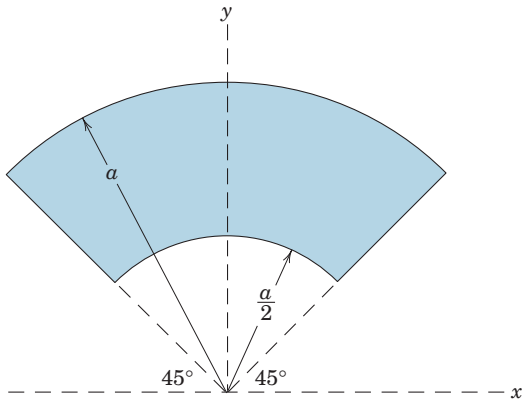
5/27 If the shaded area of Prob. 5/26 is rotated 360° about the y-axis, determine the y-coordinate of the centroid of the resulting volume.

5/28 Determine the x- and y-coordinates of the centroid of the shaded area.



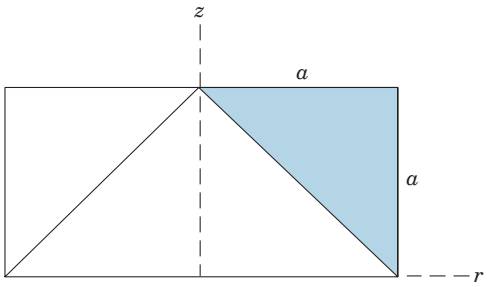
Problem 5/28

5/29 Determine the y-coordinate of the centroid of the shaded area.



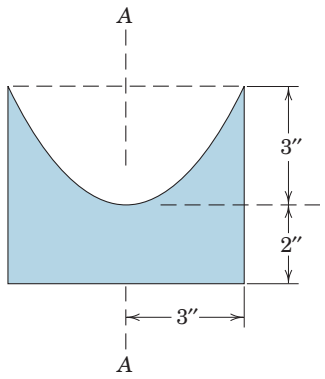
Problem 5/29

5/30 Determine the z-coordinate of the centroid of the volume generated by revolving the shaded area around the z-axis through 360°.



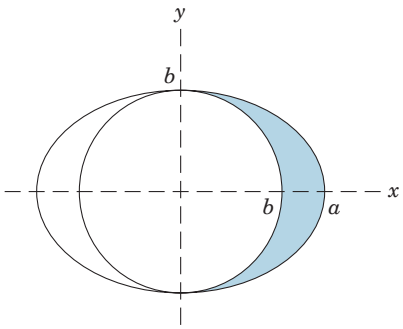
Problem 5/30

5/31 The figure represents a flat piece of sheet metal symmetrical about axis A-A and having a parabolic upper boundary. Choose your own coordinates and calculate the distance $\bar{h}$ from the base to the center of gravity of the piece.



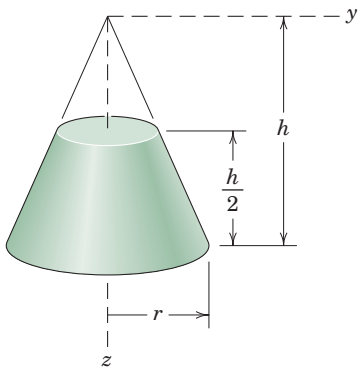
Problem 5/31

5/32 Locate the centroid of the shaded area between the ellipse and the circle.



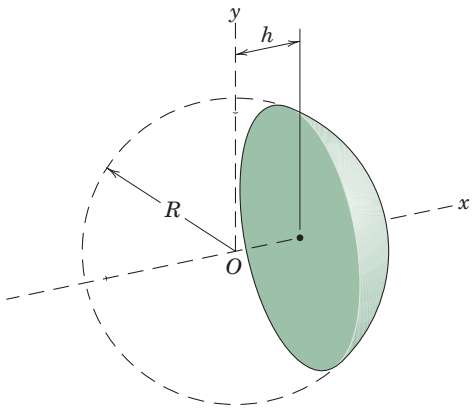
Problem 5/32

5/33 Calculate the distance $\bar{h}$ measured from the base to the centroid of the volume of the frustum of the right-circular cone.



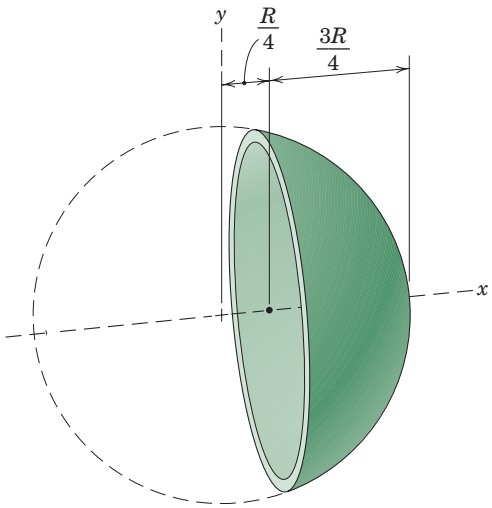
Problem 5/33

5/34 Determine the x-coordinate of the centroid of the solid spherical segment. Evaluate your expression for $h = R/4$ and $h = 0$.



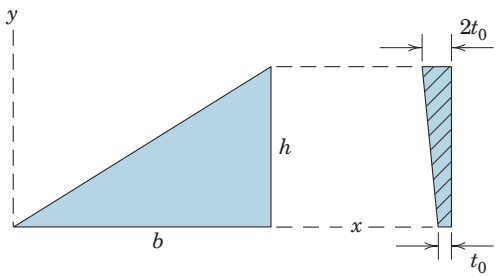
Problem 5/34

5/35 Determine the x-coordinate of the mass center of the portion of the spherical shell of uniform but small thickness.



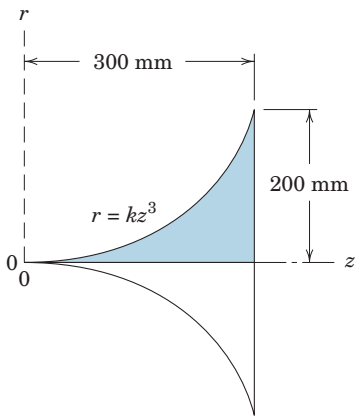
Problem 5/35

5/36 The thickness of the triangular plate varies linearly with y from a value t_0 along its base $y = 0$ to $2t_0$ at $y = h$. Determine the y-coordinate of the center of mass of the plate.



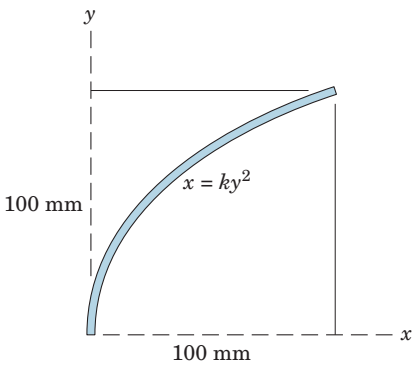
Problem 5/36

5/37 Locate the mass center of the homogeneous solid body whose volume is determined by revolving the shaded area through 360° about the z-axis.



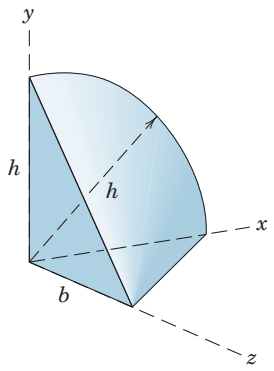
Problem 5/37

5/38 The homogeneous slender rod has a uniform cross section and is bent into the shape shown. Calculate the y-coordinate of the mass center of the rod. (*Reminder:* A differential arc length is $dL = \sqrt{(dx)^2 + (dy)^2} = \sqrt{1 + (dx/dy)^2}\,dy$.)



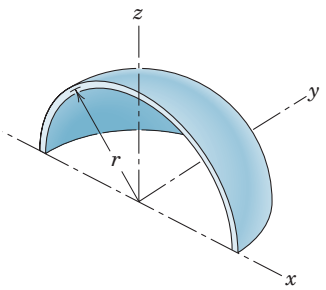
Problem 5/38

5/39 Determine the y-coordinate of the centroid of the volume generated by revolving the shaded area of the right triangle about the z-axis through 90° as shown in the figure.



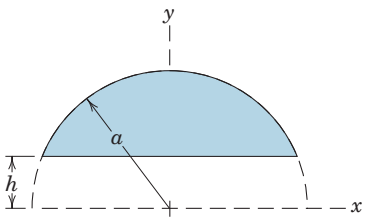
Problem 5/39

▶5/40 Determine the z-coordinate of the mass center of the homogeneous quarter-spherical shell, which has a radius r.



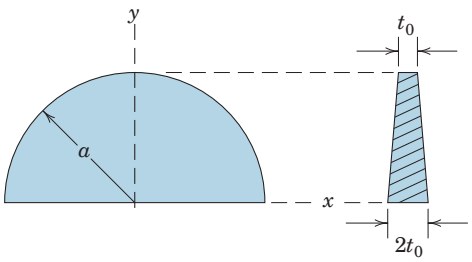
Problem 5/40

▶5/41 Determine the y-coordinate of the centroid of the plane area shown. Set $h = 0$ in your result and compare with the result $\bar{y} = \frac{4a}{3\pi}$ for a full semicircular area (see Sample Problem 5/3 and Table D/3). Also evaluate your result for the conditions $h = \frac{a}{4}$ and $h = \frac{a}{2}$.



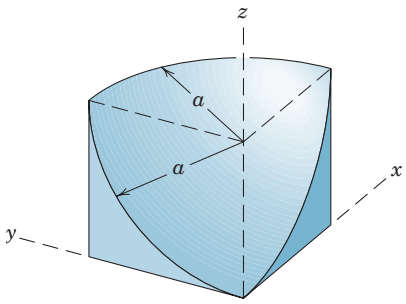
Problem 5/41

▶5/42 The thickness of the semicircular plate varies linearly with y from a value $2t_0$ along its base $y = 0$ to t_0 at $y = a$. Determine the y-coordinate of the mass center of the plate.



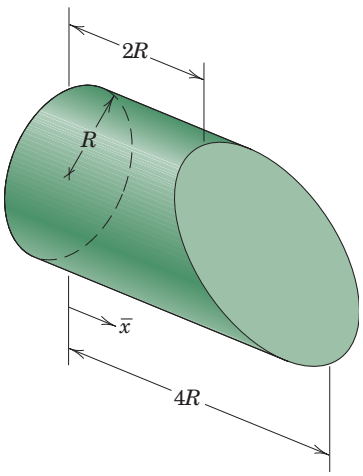
Problem 5/42

▶5/43 Determine the coordinates of the centroid of the volume obtained by revolving the shaded area about the z-axis through the 90° angle.



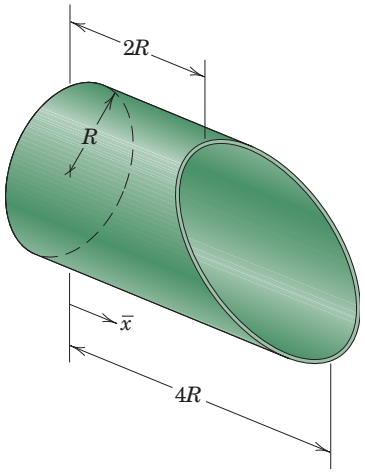
Problem 5/43

▶5/44 Determine the x-coordinate of the mass center of the solid homogeneous body shown.



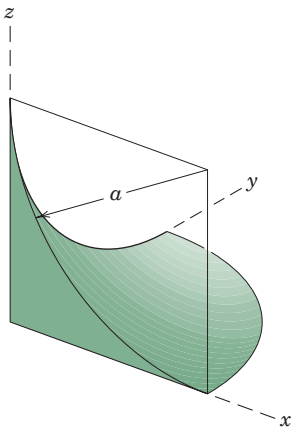
Problem 5/44

▶5/45 Determine the x-coordinate of the mass center of the cylindrical shell of small uniform thickness.



Problem 5/45

▶5/46 Determine the x- and y-coordinates of the centroid of the volume generated by rotating the shaded area about the z-axis through 90°.



Problem 5/46

5/4 Composite Bodies and Figures; Approximations

When a body or figure can be conveniently divided into several parts whose mass centers are easily determined, we use the principle of moments and treat each part as a finite element of the whole. Such a body is illustrated schematically in Fig. 5/13. Its parts have masses m_1, m_2, m_3 with the respective mass-center coordinates $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$ in the x-direction. The moment principle gives

$$(m_1 + m_2 + m_3)\bar{X} = m_1\bar{x}_1 + m_2\bar{x}_2 + m_3\bar{x}_3$$

where $\bar{X}$ is the x-coordinate of the center of mass of the whole. Similar relations hold for the other two coordinate directions.

We generalize, then, for a body of any number of parts and express the sums in condensed form to obtain the mass-center coordinates

$$\bar{X} = \frac{\Sigma m\bar{x}}{\Sigma m} \qquad \bar{Y} = \frac{\Sigma m\bar{y}}{\Sigma m} \qquad \bar{Z} = \frac{\Sigma m\bar{z}}{\Sigma m} \tag{5/7}$$

Analogous relations hold for composite lines, areas, and volumes, where the m's are replaced by L's, A's, and V's, respectively. Note that if a hole or cavity is considered one of the component parts of a composite body or figure, the corresponding mass represented by the cavity or hole is treated as a negative quantity.

An Approximation Method

In practice the boundaries of an area or volume might not be expressible in terms of simple geometrical shapes or as shapes which can be represented mathematically. For such cases we must resort to a method of approximation. As an example, consider the problem of locating the

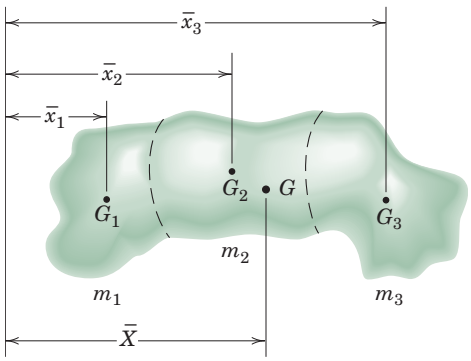


Figure 5/13

centroid C of the irregular area shown in Fig. 5/14. The area is divided into strips of width Δx and variable height h. The area A of each strip, such as the one shown in red, is $h\ \Delta x$ and is multiplied by the coordinates x_c and y_c of its *centroid* to obtain the moments of the element of area. The sum of the moments for all strips divided by the total area of the strips will give the corresponding centroidal coordinate. A systematic tabulation of the results will permit an orderly evaluation of the total area ΣA, the sums $\Sigma A x_c$ and $\Sigma A y_c$, and the centroidal coordinates

$$\bar{x} = \frac{\Sigma A x_c}{\Sigma A} \qquad \bar{y} = \frac{\Sigma A y_c}{\Sigma A}$$

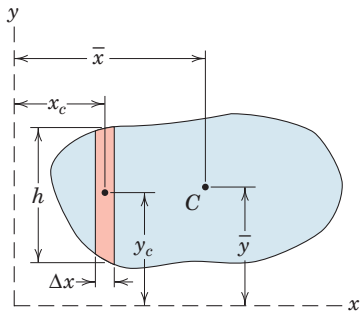


Figure 5/14

We can increase the accuracy of the approximation by decreasing the widths of the strips. In all cases the average height of the strip should be estimated in approximating the areas. Although it is usually advantageous to use elements of constant width, it is not necessary. In fact, we may use elements of any size and shape which approximate the given area to satisfactory accuracy.

Irregular Volumes

To locate the centroid of an irregular volume, we may reduce the problem to one of locating the centroid of an area. Consider the volume shown in Fig. 5/15, where the magnitudes A of the cross-sectional areas normal to the x-direction are plotted against x as shown. A vertical strip of area under the curve is $A\ \Delta x$, which equals the corresponding element of volume ΔV. Thus, the area under the plotted curve represents the volume of the body, and the x-coordinate of the centroid of the area under the curve is given by

$$\bar{x} = \frac{\Sigma (A\ \Delta x) x_c}{\Sigma A\ \Delta x} \qquad \text{which equals} \qquad \bar{x} = \frac{\Sigma V x_c}{\Sigma V}$$

for the centroid of the actual volume.

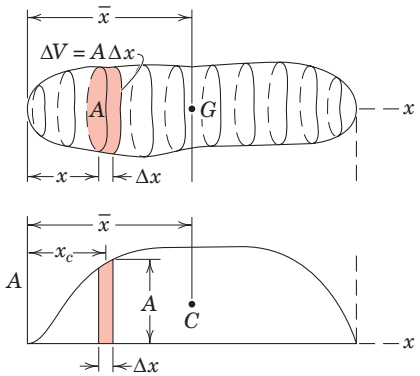
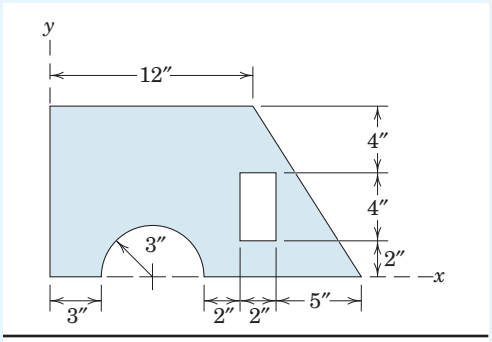


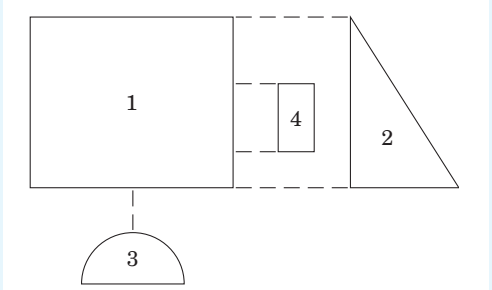
Figure 5/15

SAMPLE PROBLEM 5/6

Locate the centroid of the shaded area.



Solution. The composite area is divided into the four elementary shapes shown in the lower figure. The centroid locations of all these shapes may be obtained from Table D/3. Note that the areas of the "holes" (parts 3 and 4) are taken as negative in the following table:



PART	A in.^{2}	$\bar{x}$ in.	$\bar{y}$ in.	$\bar{x}A$ in.^{3}	$\bar{y}A$ in.^{3}
1	120	6	5	720	600
2	30	14	10/3	420	100
3	−14.14	6	1.273	−84.8	−18
4	−8	12	4	−96	−32
TOTALS	127.9			959	650

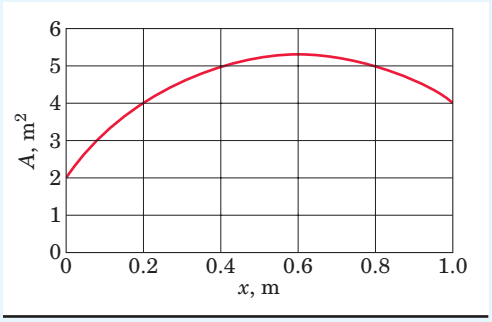
The area counterparts to Eqs. 5/7 are now applied and yield

$$\left[\bar{X} = \frac{\Sigma A\bar{x}}{\Sigma A}\right] \qquad \bar{X} = \frac{959}{127.9} = 7.50 \text{ in.} \qquad \textit{Ans.}$$

$$\left[\bar{Y} = \frac{\Sigma A\bar{y}}{\Sigma A}\right] \qquad \bar{Y} = \frac{650}{127.9} = 5.08 \text{ in.} \qquad \textit{Ans.}$$

SAMPLE PROBLEM 5/7

Approximate the x-coordinate of the volume centroid of a body whose length is 1 m and whose cross-sectional area varies with x as shown in the figure.



Solution. The body is divided into five sections. For each section, the average area, volume, and centroid location are determined and entered in the following table:

INTERVAL	A_{av} m^2	Volume V m^3	$\bar{x}$ m	$V\bar{x}$ m^4
0–0.2	3	0.6	0.1	0.060
0.2–0.4	4.5	0.90	0.3	0.270
0.4–0.6	5.2	1.04	0.5	0.520
0.6–0.8	5.2	1.04	0.7	0.728
0.8–1.0	4.5	0.90	0.9	0.810
TOTALS		4.48		2.388

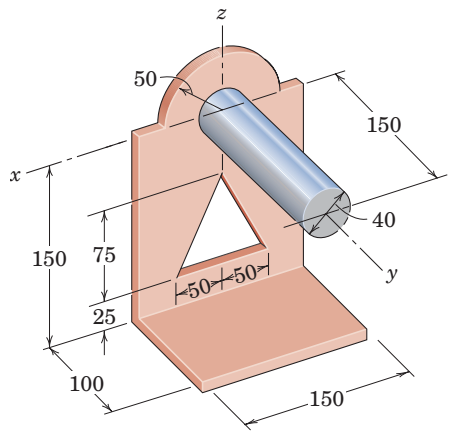
(1) $$\left[\bar{X} = \frac{\Sigma V\bar{x}}{\Sigma V}\right] \qquad \bar{X} = \frac{2.388}{4.48} = 0.533 \text{ m} \qquad \textit{Ans.}$$

Helpful Hint

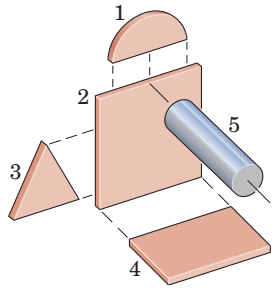
(1) Note that the shape of the body as a function of y and z does not affect $\bar{X}$.

SAMPLE PROBLEM 5/8

Locate the center of mass of the bracket-and-shaft combination. The vertical face is made from sheet metal which has a mass of 25 kg/m^2. The material of the horizontal base has a mass of 40 kg/m^2, and the steel shaft has a density of 7.83 Mg/m^3.



Dimensions in millimeters



Solution. The composite body may be considered to be composed of the five elements shown in the lower portion of the illustration. The triangular part will be taken as a negative mass. For the reference axes indicated it is clear by symmetry that the x-coordinate of the center of mass is zero.

The mass m of each part is easily calculated and should need no further explanation. For Part 1 we have from Sample Problem 5/3

$$\bar{z} = \frac{4r}{3\pi} = \frac{4(50)}{3\pi} = 21.2 \text{ mm}$$

For Part 3 we see from Sample Problem 5/2 that the centroid of the triangular mass is one-third of its altitude above its base. Measurement from the coordinate axes becomes

$$\bar{z} = -[150 - 25 - \tfrac{1}{3}(75)] = -100 \text{ mm}$$

The y- and z-coordinates to the mass centers of the remaining parts should be evident by inspection. The terms involved in applying Eqs. 5/7 are best handled in the form of a table as follows:

PART	m kg	$\bar{y}$ mm	$\bar{z}$ mm	$m\bar{y}$ kg·m	$m\bar{z}$ kg·mm
1	0.098	0	21.2	0	2.08
2	0.562	0	−75.0	0	−42.19
3	−0.094	0	−100.0	0	9.38
4	0.600	50.0	−150.0	30.0	−90.00
5	1.476	75.0	0	110.7	0
TOTALS	2.642			140.7	−120.73

Equations 5/7 are now applied and the results are

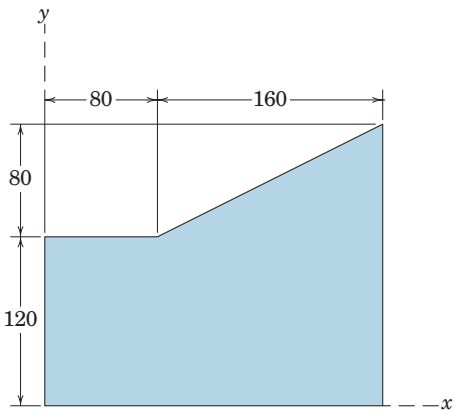
$$\left[\bar{Y} = \frac{\Sigma m\bar{y}}{\Sigma m}\right] \qquad \bar{Y} = \frac{140.7}{2.642} = 53.3 \text{ mm} \qquad \textit{Ans.}$$

$$\left[\bar{Z} = \frac{\Sigma m\bar{z}}{\Sigma m}\right] \qquad \bar{Z} = \frac{-120.73}{2.642} = -45.7 \text{ mm} \qquad \textit{Ans.}$$

PROBLEMS

Introductory Problems

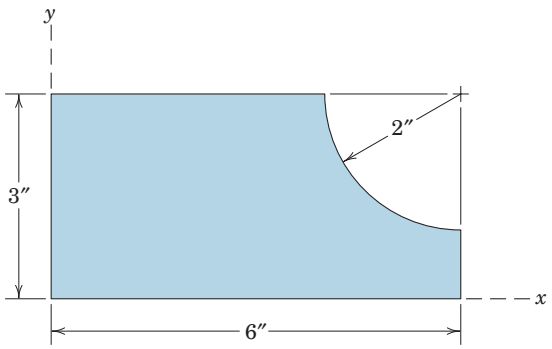
5/47 Determine the x- and y-coordinates of the centroid of the shaded area.



Dimensions in millimeters

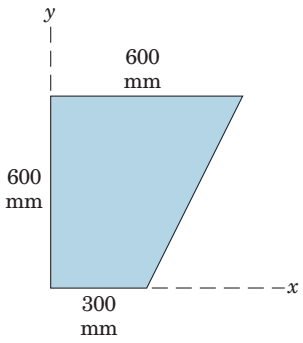
Problem 5/47

5/48 Determine the coordinates of the centroid of the shaded area.



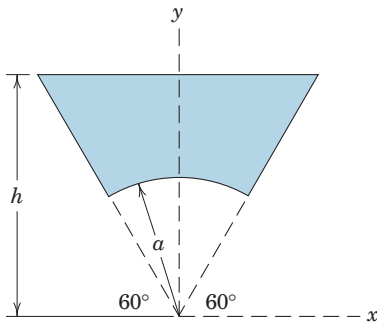
Problem 5/48

5/49 Determine the coordinates of the centroid of the trapezoidal area shown.



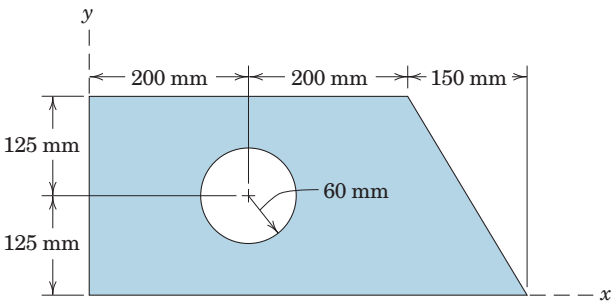
Problem 5/49

5/50 Determine the y-coordinate of the centroid of the shaded area.



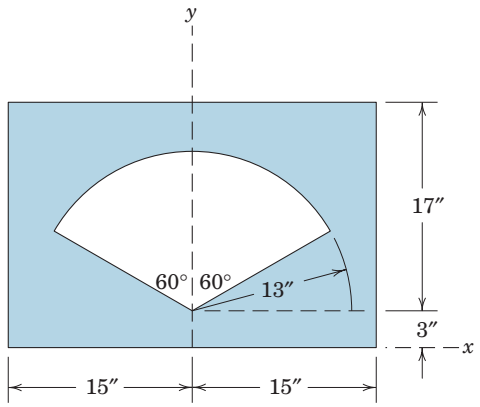
Problem 5/50

5/51 Determine the coordinates of the centroid of the shaded area.



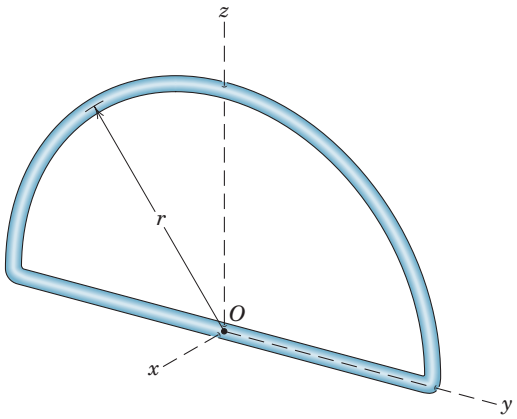
Problem 5/51

5/52 Calculate the y-coordinate of the centroid of the shaded area.



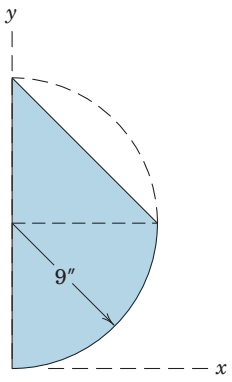
Problem 5/52

5/53 Determine the x-, y-, and z-coordinates of the mass center of the body constructed of uniform slender rod.



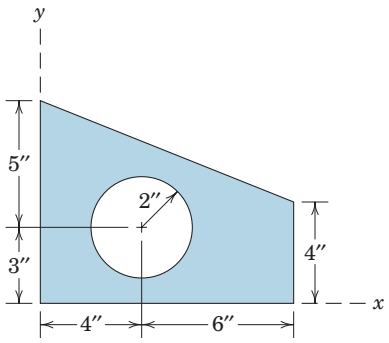
Problem 5/53

5/54 Calculate the x- and y-coordinates of the centroid of the shaded area.



Problem 5/54

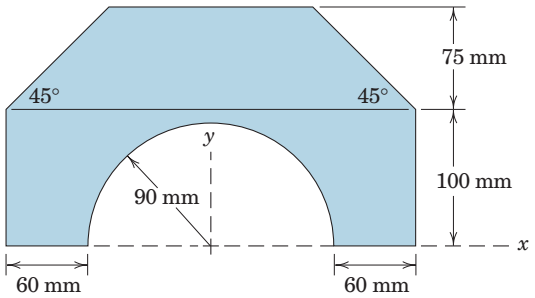
5/55 Determine the x- and y-coordinates of the centroid of the shaded area.



Problem 5/55

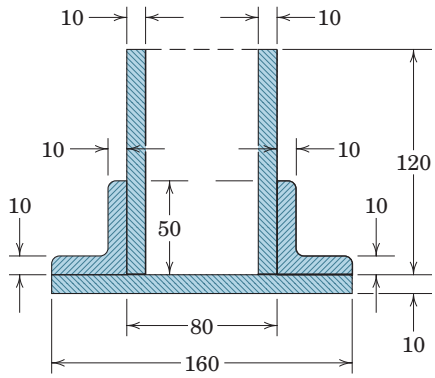
Representative Problems

5/56 Determine the y-coordinate of the centroid of the shaded area.



Problem 5/56

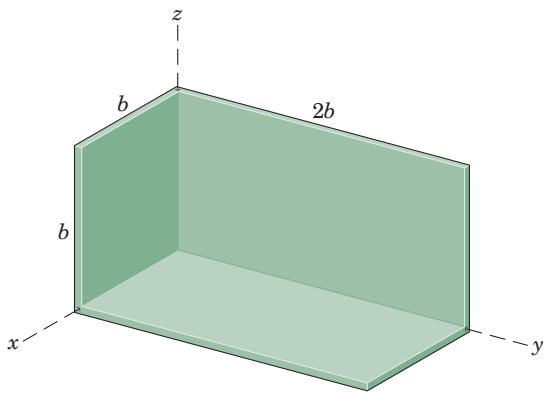
5/57 Determine the distance $\overline{H}$ from the bottom of the base plate to the centroid of the built-up structural section shown.



Dimensions in millimeters

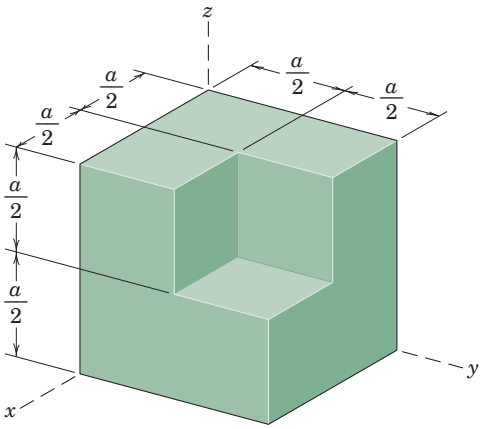
Problem 5/57

5/58 Determine the coordinates of the mass center of the body which is constructed of three pieces of uniform thin plate welded together.



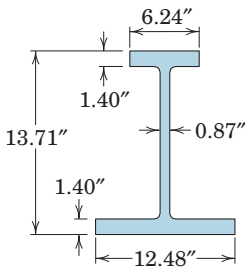
Problem 5/58

5/59 Determine the coordinates of the mass center of the homogeneous body.



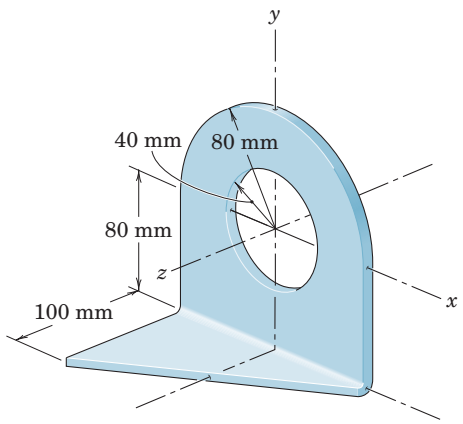
Problem 5/59

5/60 Determine the height above the base of the centroid of the cross-sectional area of the beam. Neglect the fillets.



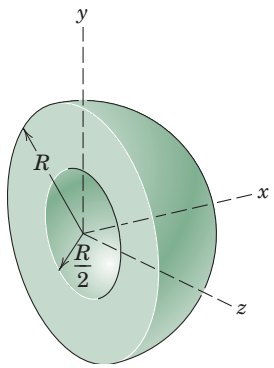
Problem 5/60

5/61 Determine the coordinates of the center of mass of the bracket, which is made from a plate of uniform thickness.



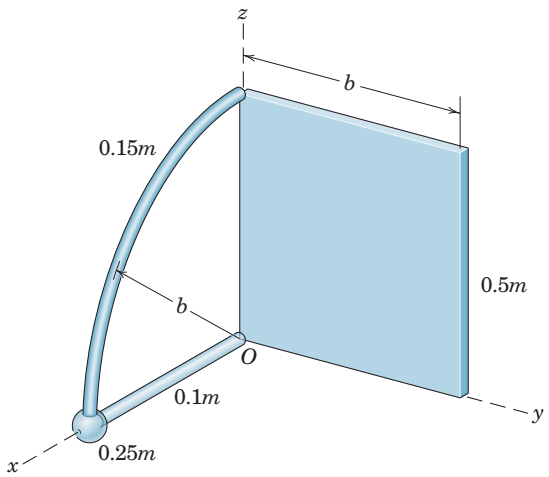
Problem 5/61

5/62 Determine the x-coordinate of the mass center of the homogeneous hemisphere with the smaller hemispherical portion removed.



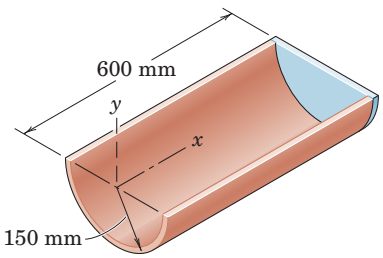
Problem 5/62

5/63 Determine the coordinates of the mass center of the body constructed of a uniform square plate, a uniform straight rod, a uniform quarter-circular rod, and a particle (negligible dimensions). The mass of each component is given in the figure.



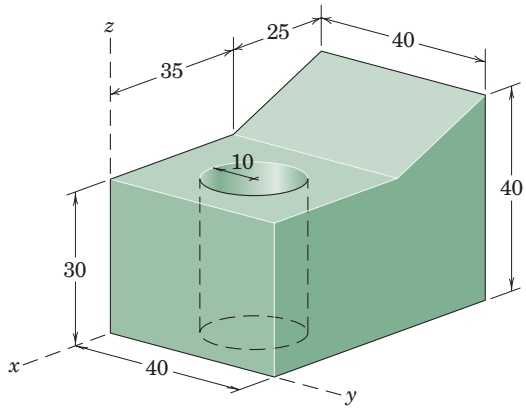
Problem 5/63

5/64 Determine the position of the mass center of the cylindrical shell with a closed semicircular end. The shell is made from sheet metal with a mass of 24 kg/m^2, and the end is made from metal plate with a mass of 36 kg/m^2.



Problem 5/64

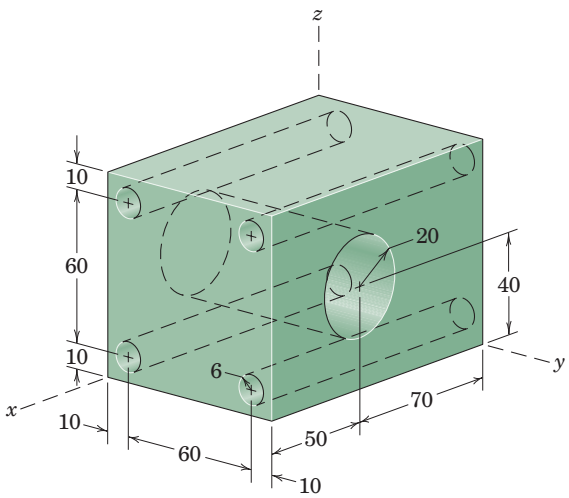
5/65 Determine the coordinates of the mass center of the homogeneous body shown. The hole is centered in the upper horizontal surface.



Dimensions in millimeters

Problem 5/65

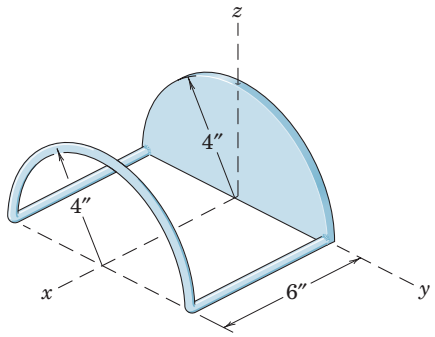
5/66 Determine the coordinates of the centroid of the body with five holes drilled completely through it. Each of the four small holes has a radius of 6 mm.



Dimensions in millimeters

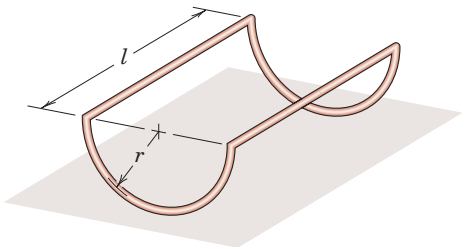
Problem 5/66

5/67 The welded assembly is made of a uniform rod weighing 0.370 lb per foot of length and the semicircular plate weighing 8 lb per square foot. Calculate the coordinates of the center of gravity of the assembly.



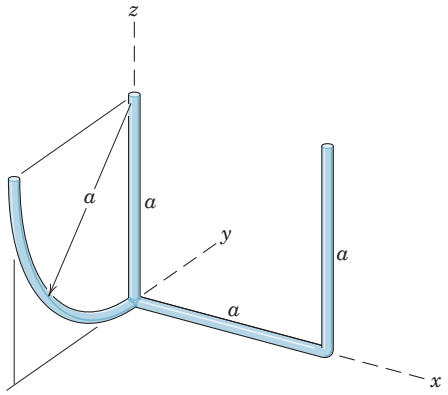
Problem 5/67

5/68 The assembly shown is formed of uniform rod. For what value of the length l will the mass center be located at a height of $\frac{3r}{4}$ above the supporting surface?



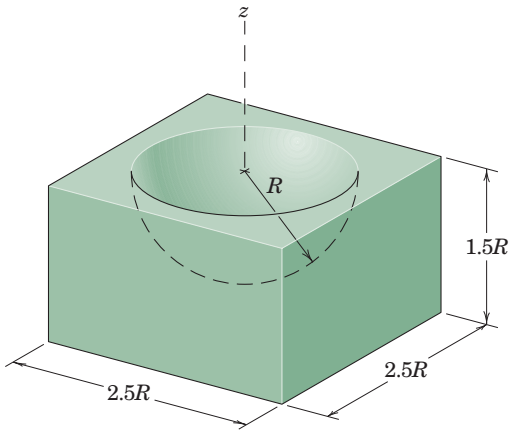
Problem 5/68

5/69 Determine the coordinates of the mass center of the welded assembly of uniform slender rods made from the same bar stock.



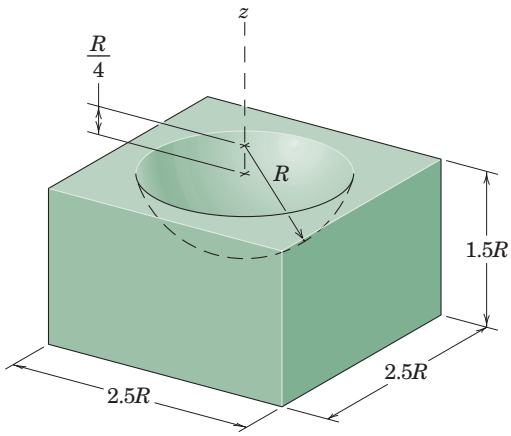
Problem 5/69

5/70 Determine the z-coordinate of the centroid of the rectangular solid with the hemispherical hole. The center of the hemisphere is centered on the upper face of the solid, and z is measured upward from the lower face.



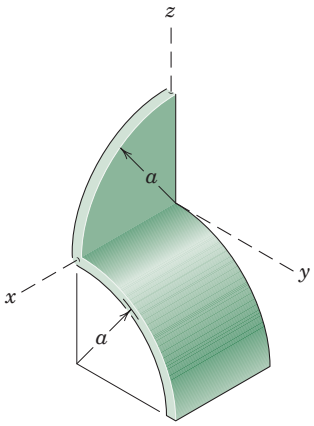
Problem 5/70

5/71 Determine the z-coordinate of the centroid of the rectangular solid with the spherical segment (less than a hemisphere) removed. The center of the hemisphere of which the segment is a portion is located a distance $R/4$ above the center of the upper face of the solid, and z is measured upward from the lower face. Refer to the results of Prob. 5/34.



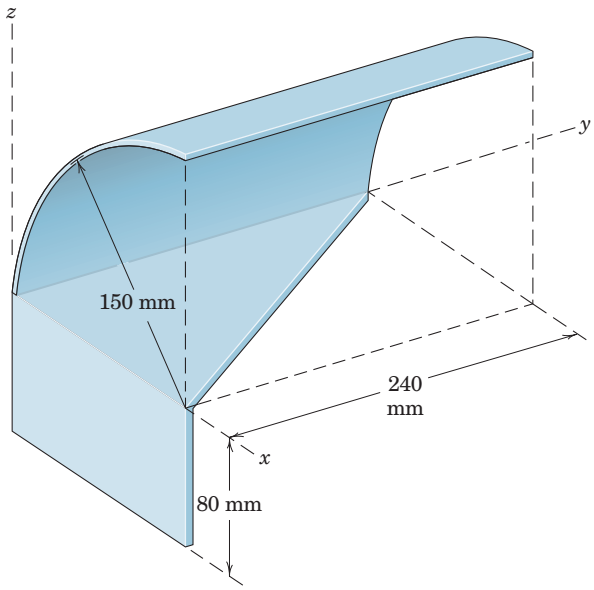
Problem 5/71

5/72 Determine the x-, y-, and z-coordinates of the mass center of the sheet-metal shape shown.



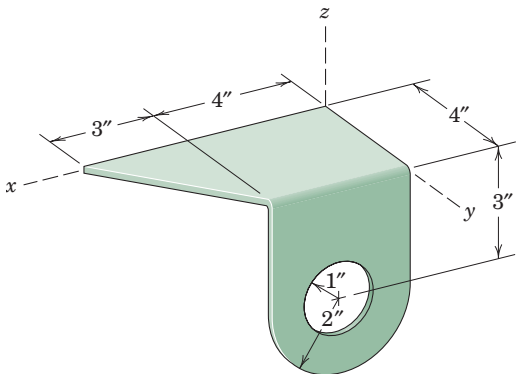
Problem 5/72

5/73 Calculate the x-, y-, and z-coordinates of the mass center of the fixture formed from aluminum plate of uniform thickness.



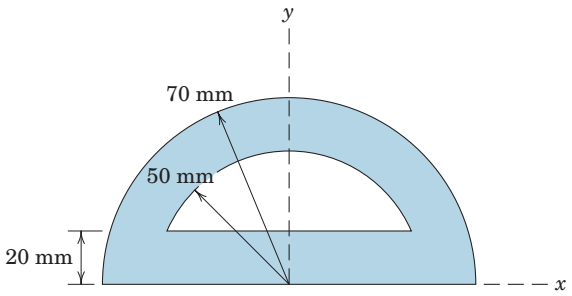
Problem 5/73

5/74 Determine the coordinates of the mass center of the bracket, which is constructed from sheet metal of uniform thickness.



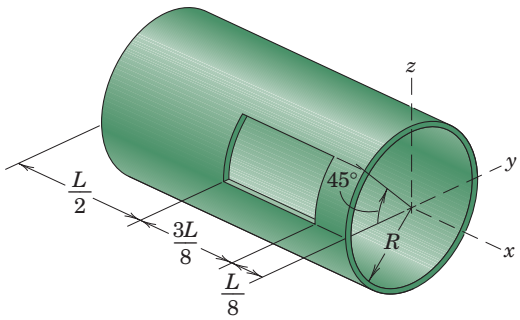
Problem 5/74

5/75 Determine the y-coordinate of the centroid of the shaded area. (*Hint:* Refer to the solution to Prob. 5/41.)



Problem 5/75

▶5/76 An opening is formed in the thin cylindrical shell. Determine the x-, y-, and z-coordinates of the mass center of the homogeneous body.



Problem 5/76

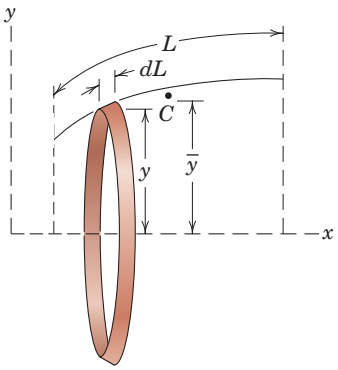


Figure 5/16

5/5 Theorems of Pappus*

A very simple method exists for calculating the surface area generated by revolving a plane curve about a nonintersecting axis in the plane of the curve. In Fig. 5/16 the line segment of length L in the x-y plane generates a surface when revolved about the x-axis. An element of this surface is the ring generated by dL. The area of this ring is its circumference times its slant height or $dA = 2\pi y\, dL$. The total area is then

$$A = 2\pi \int y\, dL$$

Because $\bar{y}L = \int y\, dL$, the area becomes

$$A = 2\pi \bar{y} L \tag{5/8}$$

where $\bar{y}$ is the y-coordinate of the centroid C for the line of length L. Thus, the generated area is the same as the lateral area of a right-circular cylinder of length L and radius $\bar{y}$.

In the case of a volume generated by revolving an area about a nonintersecting line in its plane, an equally simple relation exists for finding the volume. An element of the volume generated by revolving the area A about the x-axis, Fig. 5/17, is the elemental ring of cross section dA and radius y. The volume of the element is its circumference times dA or $dV = 2\pi y\, dA$, and the total volume is

$$V = 2\pi \int y\, dA$$

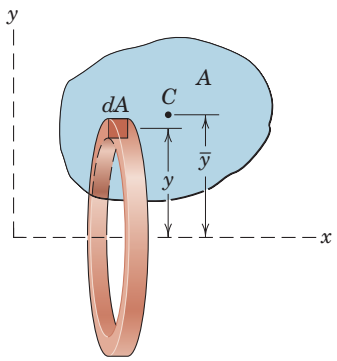


Figure 5/17

*Attributed to Pappus of Alexandria, a Greek geometer who lived in the third century A.D. The theorems often bear the name of Guldinus (Paul Guldin, 1577–1643), who claimed original authorship, although the works of Pappus were apparently known to him.

Because $\bar{y}A = \int y\,dA$, the volume becomes

$$V = 2\pi\bar{y}A \tag{5/9}$$

where $\bar{y}$ is the y-coordinate of the centroid C of the revolved area A. Thus, we obtain the generated volume by multiplying the generating area by the circumference of the circular path described by its centroid.

The two theorems of Pappus, expressed by Eqs. 5/8 and 5/9, are useful for determining areas and volumes of revolution. They are also used to find the centroids of plane curves and plane areas when we know the corresponding areas and volumes created by revolving these figures about a nonintersecting axis. Dividing the area or volume by 2π times the corresponding line segment length or plane area gives the distance from the centroid to the axis.

If a line or an area is revolved through an angle θ less than 2π, we can determine the generated surface or volume by replacing 2π by θ in Eqs. 5/8 and 5/9. Thus, the more general relations are

$$A = \theta\bar{y}L \tag{5/8a}$$

and

$$V = \theta\bar{y}A \tag{5/9a}$$

where θ is expressed in radians.

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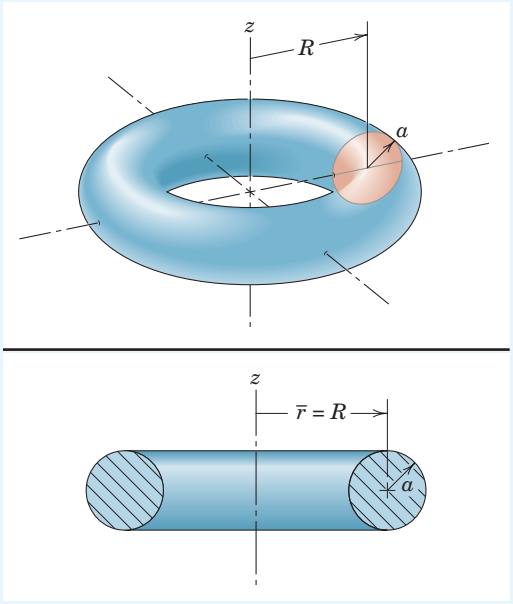
Another view of the Millau Viaduct in southern France, which appears on the cover. Note that the full height of the pylons is obscured here.

SAMPLE PROBLEM 5/9

Determine the volume V and surface area A of the complete torus of circular cross section.



Solution. The torus can be generated by revolving the circular area of radius a through 360° about the z-axis. With the use of Eq. 5/9a, we have

(1) $$V = \theta \bar{r} A = 2\pi(R)(\pi a^2) = 2\pi^2 R a^2 \qquad \textit{Ans.}$$

Similarly, using Eq. 5/8a gives

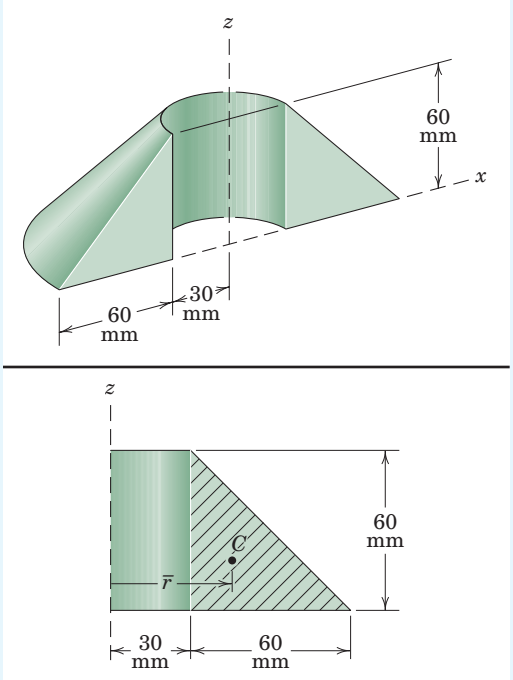
$$A = \theta \bar{r} L = 2\pi(R)(2\pi a) = 4\pi^2 R a \qquad \textit{Ans.}$$

Helpful Hint

(1) We note that the angle θ of revolution is 2π for the complete ring. This common but special-case result is given by Eq. 5/9.

SAMPLE PROBLEM 5/10

Calculate the volume V of the solid generated by revolving the 60-mm right-triangular area through 180° about the z-axis. If this body were constructed of steel, what would be its mass m?



Solution. With the angle of revolution $\theta = 180°$, Eq. 5/9a gives

(1) $$V = \theta \bar{r} A = \pi[30 + \tfrac{1}{3}(60)][\tfrac{1}{2}(60)(60)] = 2.83(10^5)\ \text{mm}^3 \qquad \textit{Ans.}$$

The mass of the body is then

$$m = \rho V = \left[7830\ \frac{\text{kg}}{\text{m}^3}\right][2.83(10^5)\ \text{mm}^3]\left[\frac{1\ \text{m}}{1000\ \text{mm}}\right]^3$$

$$= 2.21\ \text{kg} \qquad \textit{Ans.}$$

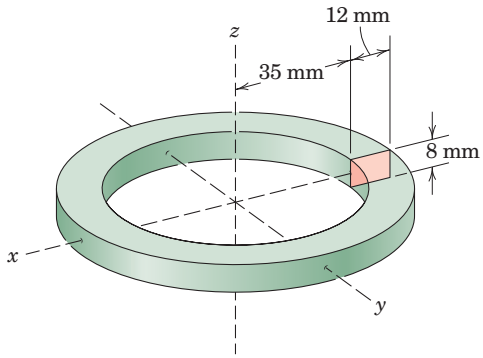
Helpful Hint

(1) Note that θ must be in radians.

PROBLEMS

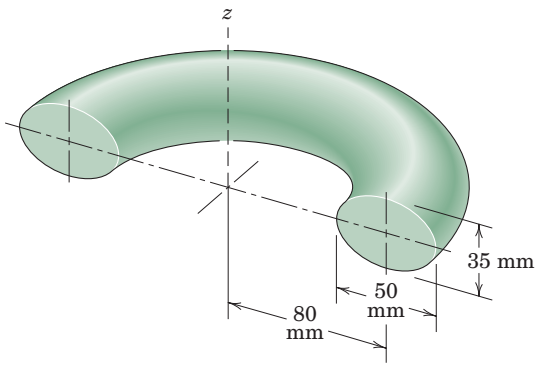
Introductory Problems

5/77 Using the methods of this article, determine the surface area A and volume V of the body formed by revolving the rectangular area through 360° about the z-axis.



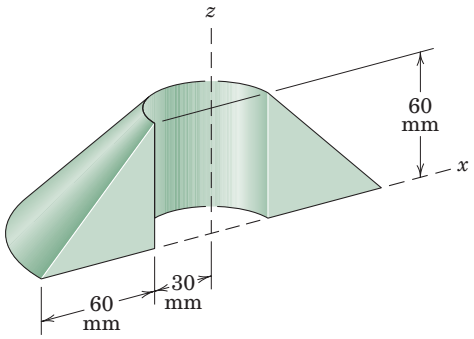
Problem 5/77

5/78 Determine the volume V generated by revolving the elliptical area through 180° about the z-axis.



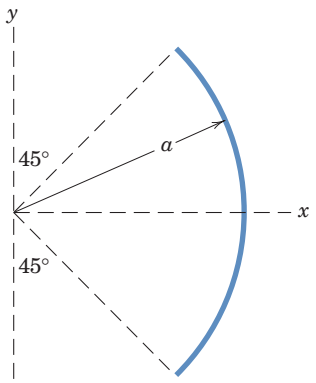
Problem 5/78

5/79 Calculate the volume V of the solid generated by revolving the 60-mm right-triangular area about the z-axis through 180°.



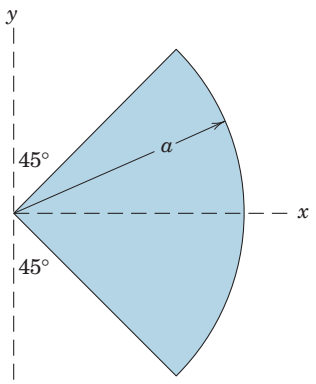
Problem 5/79

5/80 The circular arc is rotated through 360° about the y-axis. Determine the outer surface area S of the resulting body, which is a portion of a spherical shell.



Problem 5/80

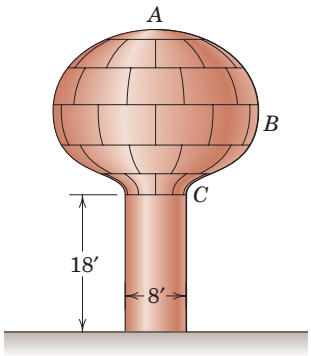
5/81 The quarter-circular area is rotated through 360° about the y-axis. Determine the volume of the resulting body, which is a portion of a sphere.



Problem 5/81

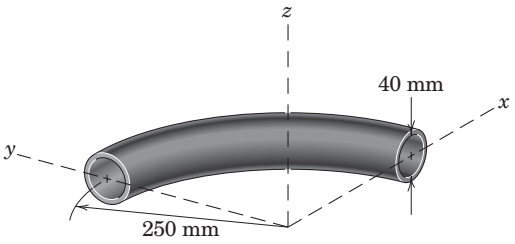
Representative Problems

5/82 The water storage tank is a shell of revolution and is to be sprayed with two coats of paint which has a coverage of 500 ft^2 per gallon. The engineer (who remembers mechanics) consults a scale drawing of the tank and determines that the curved line *ABC* has a length of 34 ft and that its centroid is 8.2 ft from the centerline of the tank. How many gallons of paint will be used for the tank including the vertical cylindrical column?



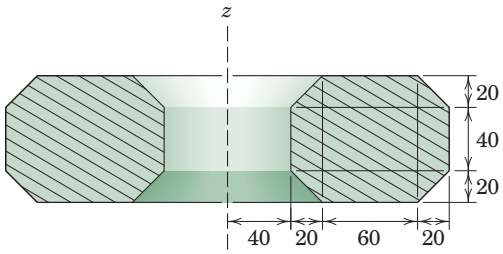
Problem 5/82

5/83 Determine the mass of the quarter-circular length of radiator hose. The inside diameter is 40 mm, the wall thickness is 4 mm, and the density of the rubber material is 1050 kg/m^3.



Problem 5/83

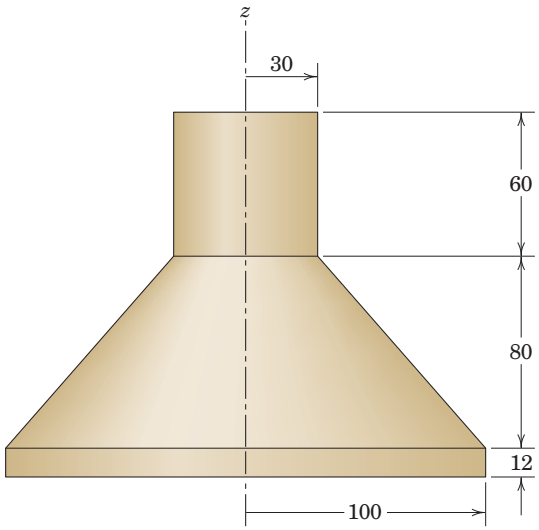
5/84 The body shown in cross section is a complete circular ring formed by revolving the octagonal area about the *z*-axis. The entire surface is to be covered with a special coating. Determine this surface area.



Dimensions in millimeters

Problem 5/84

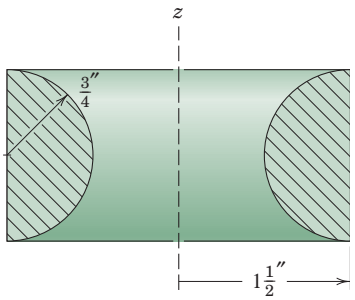
5/85 The lampshade shown is constructed of 0.6-mm-thick steel and is symmetric about the *z*-axis. Both the upper and lower ends are open. Determine the mass of the lampshade. Take the radii to be to midthickness.



Dimensions in millimeters

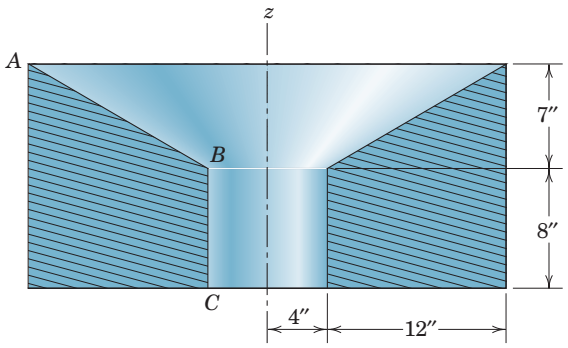
Problem 5/85

5/86 Calculate the volume V of the spacer in the form of the complete ring of the semicircular section shown. Also compute the total surface area A of the spacer.



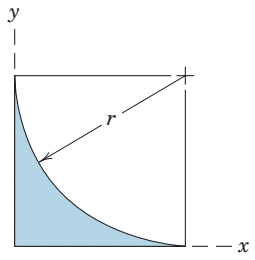
Problem 5/86

5/87 Determine the inner surface area (generated by revolving ABC completely about the z-axis) of the body shown in cross section.



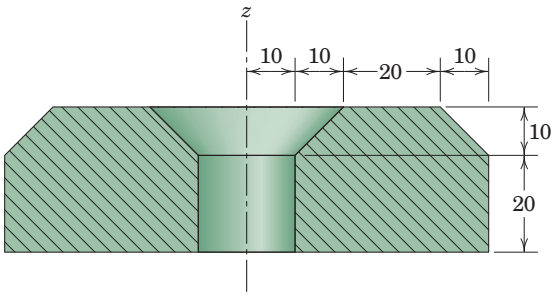
Problem 5/87

5/88 Determine the total surface area of the body generated by revolving the shaded area through 360° about the x-axis.



Problem 5/88

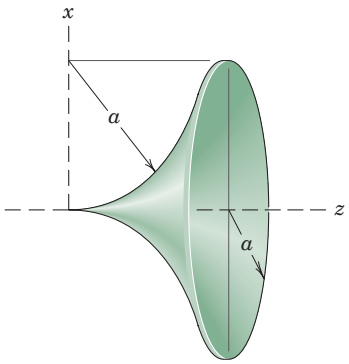
5/89 Determine the mass of the titanium casting formed by completely revolving the cross section shown about the z-axis of symmetry.



Dimensions in millimeters

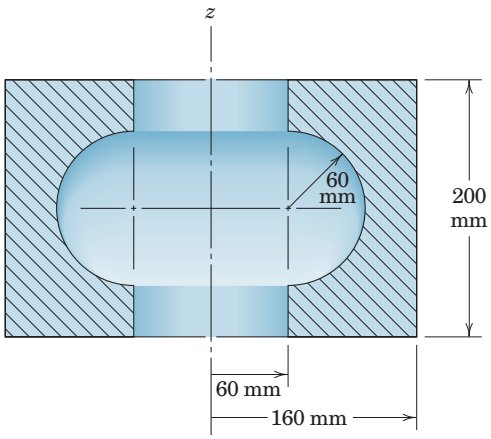
Problem 5/89

5/90 Determine the surface area of one side of the bell-shaped shell of uniform but negligible thickness.



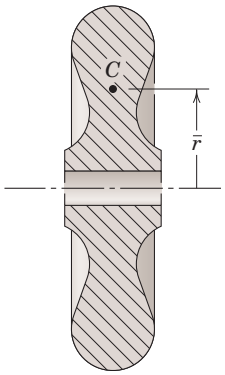
Problem 5/90

5/91 A steel die, shown in section, has the form of a solid generated by revolving the shaded area around the z-axis. Calculate the mass m of the die.



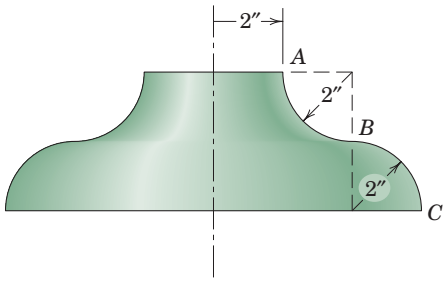
Problem 5/91

5/92 A hand-operated control wheel made of aluminum has the proportions shown in the cross-sectional view. The area of the total section shown is 15 200 mm^2, and the wheel has a mass of 10 kg. Calculate the distance $\bar{r}$ to the centroid of the half-section. The aluminum has a density of 2.69 Mg/m^3.



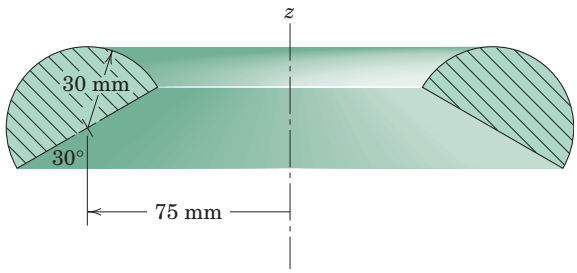
Problem 5/92

5/93 The two circular arcs *AB* and *BC* are revolved about the vertical axis to obtain the surface of revolution shown. Compute the area *A* of the outside of this surface.



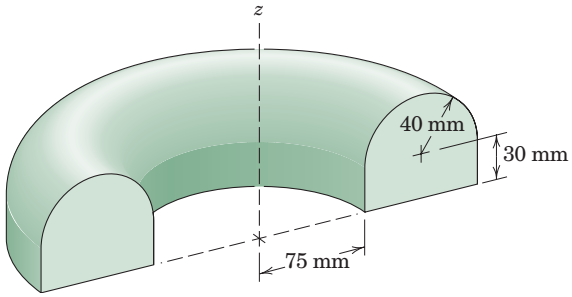
Problem 5/93

5/94 Determine the volume generated by rotating the semicircular area through 180°.



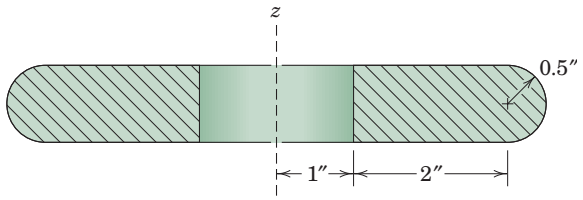
Problem 5/94

5/95 Determine the volume *V* and total surface area *A* of the solid generated by revolving the area shown through 180° about the *z*-axis.



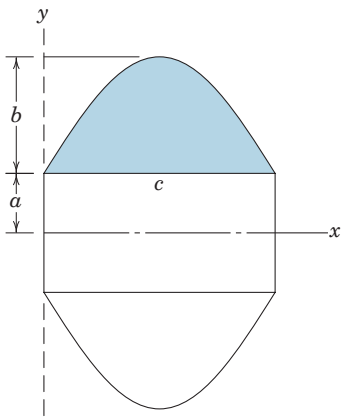
Problem 5/95

5/96 Calculate the volume *V* of the large neoprene washer in the form of the complete ring of section shown. Also compute the overall surface area *A*.



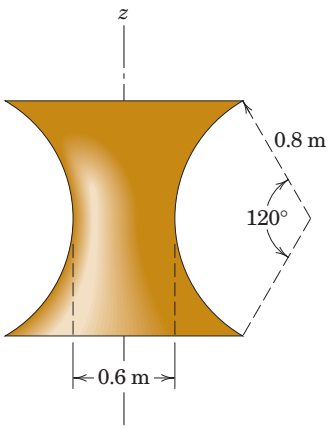
Problem 5/96

5/97 The shaded area is bounded by one half-cycle of a sine wave and the axis of the sine wave. Determine the volume generated by completely revolving the area about the x-axis.



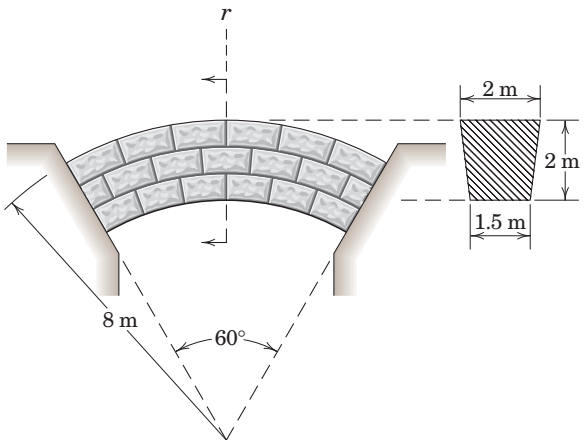
Problem 5/97

5/98 A surface is generated by revolving the circular arc of 0.8-m radius and subtended angle of 120° completely about the z-axis. The diameter of the neck is 0.6 m. Determine the outside area A generated.



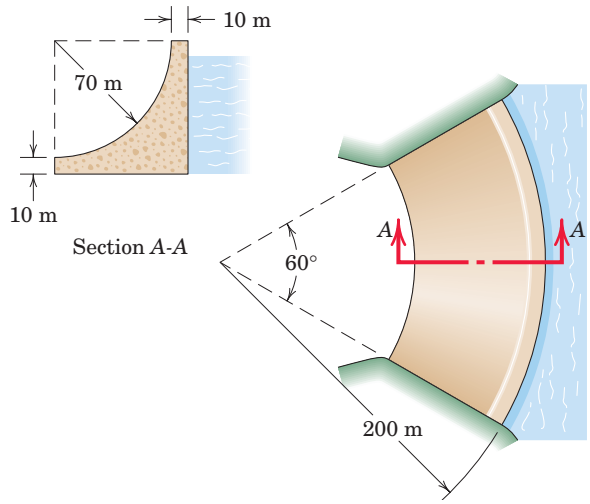
Problem 5/98

5/99 In order to provide sufficient support for the stone masonry arch designed as shown, it is necessary to know its total weight W. Use the results of Prob. 5/16 and determine W. The density of stone masonry is 2.40 Mg/m ^{3}.



Problem 5/99

5/100 Calculate the mass m of concrete required to construct the arched dam shown. Concrete has a density of 2.40 Mg/m^3.



Problem 5/100

SECTION B SPECIAL TOPICS

5/6 Beams—External Effects

Beams are structural members which offer resistance to bending due to applied loads. Most beams are long prismatic bars, and the loads are usually applied normal to the axes of the bars.

Beams are undoubtedly the most important of all structural members, so it is important to understand the basic theory underlying their design. To analyze the load-carrying capacities of a beam we must first establish the equilibrium requirements of the beam as a whole and any portion of it considered separately. Second, we must establish the relations between the resulting forces and the accompanying internal resistance of the beam to support these forces. The first part of this analysis requires the application of the principles of statics. The second part involves the strength characteristics of the material and is usually treated in studies of the mechanics of solids or the mechanics of materials.

This article is concerned with the *external* loading and reactions acting on a beam. In Art. 5/7 we calculate the distribution along the beam of the *internal* force and moment.

Types of Beams

Beams supported so that their external support reactions can be calculated by the methods of statics alone are called *statically determinate beams*. A beam which has more supports than needed to provide equilibrium is *statically indeterminate*. To determine the support reactions for such a beam we must consider its load-deformation properties in addition to the equations of static equilibrium. Figure 5/18 shows examples

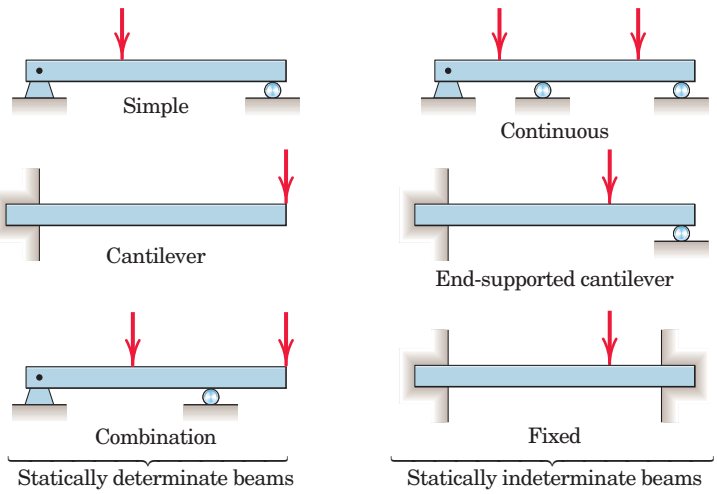


Figure 5/18

of both types of beams. In this article we will analyze statically determinate beams only.

Beams may also be identified by the type of external loading they support. The beams in Fig. 5/18 are supporting concentrated loads, whereas the beam in Fig. 5/19 is supporting a distributed load. The intensity w of a distributed load may be expressed as force per unit length of beam. The intensity may be constant or variable, continuous or discontinuous. The intensity of the loading in Fig. 5/19 is constant from C to D and variable from A to C and from D to B. The intensity is discontinuous at D, where it changes magnitude abruptly. Although the intensity itself is not discontinuous at C, the rate of change of intensity dw/dx is discontinuous.

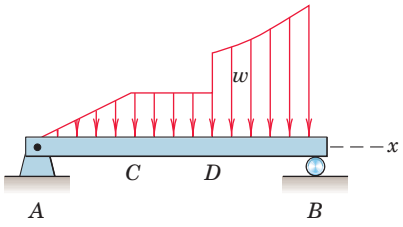


Figure 5/19

Distributed Loads

Loading intensities which are constant or which vary linearly are easily handled. Figure 5/20 illustrates the three most common cases and the resultants of the distributed loads in each case.

In cases a and b of Fig. 5/20, we see that the resultant load R is represented by the area formed by the intensity w (force per unit length of beam) and the length L over which the force is distributed. The resultant passes through the centroid of this area.

In part c of Fig. 5/20, the trapezoidal area is broken into a rectangular and a triangular area, and the corresponding resultants R_1 and R_2 of these subareas are determined separately. Note that a single resultant could be determined by using the composite technique for finding centroids, which was discussed in Art. 5/4. Usually, however, the determination of a single resultant is unnecessary.

For a more general load distribution, Fig. 5/21, we must start with a differential increment of force $dR = w\,dx$. The total load R is then the sum of the differential forces, or

$$R = \int w\,dx$$

As before, the resultant R is located at the centroid of the area under consideration. The x-coordinate of this centroid is found by the principle of moments $R\bar{x} = \int xw\,dx$, or

$$\bar{x} = \frac{\int xw\,dx}{R}$$

For the distribution of Fig. 5/21, the vertical coordinate of the centroid need not be found.

Once the distributed loads have been reduced to their equivalent concentrated loads, the external reactions acting on the beam may be found by a straightforward static analysis as developed in Chapter 3.

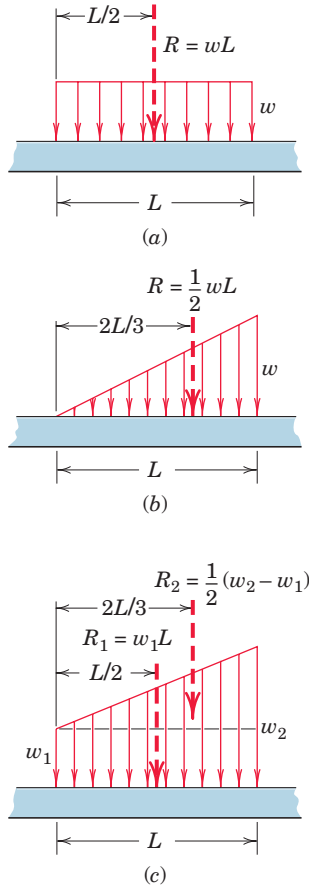


Figure 5/20

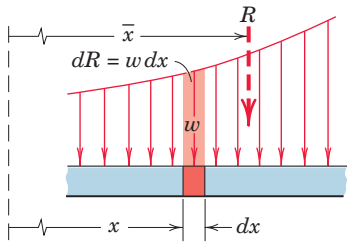
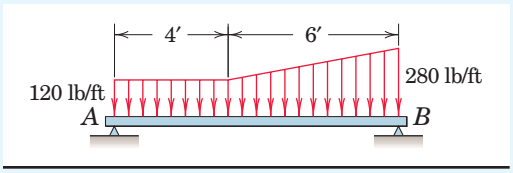


Figure 5/21

SAMPLE PROBLEM 5/11

Determine the equivalent concentrated load(s) and external reactions for the simply supported beam which is subjected to the distributed load shown.



Solution. The area associated with the load distribution is divided into the rectangular and triangular areas shown. The concentrated-load values are determined by computing the areas, and these loads are located at the centroids of the respective areas. (1)

Once the concentrated loads are determined, they are placed on the free-body diagram of the beam along with the external reactions at A and B. Using principles of equilibrium, we have

$$[\Sigma M_A = 0] \qquad 1200(5) + 480(8) - R_B(10) = 0$$

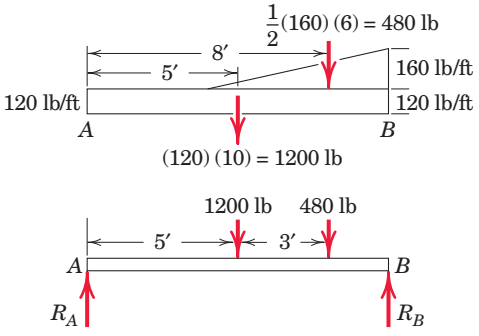
$$R_B = 984 \text{ lb} \qquad \textit{Ans.}$$

$$[\Sigma M_B = 0] \qquad R_A(10) - 1200(5) - 480(2) = 0$$

$$R_A = 696 \text{ lb} \qquad \textit{Ans.}$$

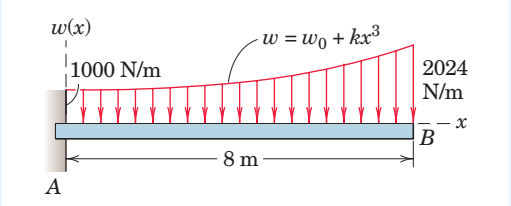
Helpful Hint

(1) Note that it is usually unnecessary to reduce a given distributed load to a *single* concentrated load.



SAMPLE PROBLEM 5/12

Determine the reaction at the support A of the loaded cantilever beam.



Solution. The constants in the load distribution are found to be $w_0 = 1000$ N/m and $k = 2$ N/m^4. The load R is then (1)

$$R = \int w\,dx = \int_0^8 (1000 + 2x^3)\,dx = \left(1000x + \frac{x^4}{2}\right)\Big|_0^8 = 10\,050 \text{ N}$$

(2) The x-coordinate of the centroid of the area is found by

$$\bar{x} = \frac{\int xw\,dx}{R} = \frac{1}{10\,050}\int_0^8 x(1000 + 2x^3)\,dx$$

$$= \frac{1}{10\,050}\left(500x^2 + \tfrac{2}{5}x^5\right)\Big|_0^8 = 4.49 \text{ m}$$

From the free-body diagram of the beam, we have

$$[\Sigma M_A = 0] \qquad M_A - (10\,050)(4.49) = 0$$

$$M_A = 45\,100 \text{ N}\cdot\text{m} \qquad \textit{Ans.}$$

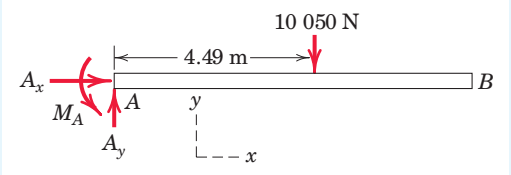
$$[\Sigma F_y = 0] \qquad A_y = 10\,050 \text{ N} \qquad \textit{Ans.}$$

Note that $A_x = 0$ by inspection.

Helpful Hints

(1) Use caution with the units of the constants w_0 and k.

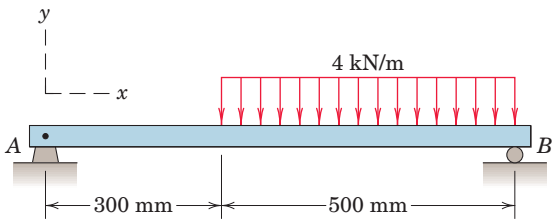
(2) The student should recognize that the calculation of R and its location $\bar{x}$ is simply an application of centroids as treated in Art. 5/3.



PROBLEMS

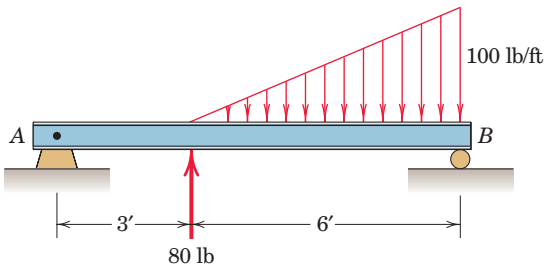
Introductory Problems

5/101 Determine the reactions at A and B for the beam subjected to the uniform load distribution.



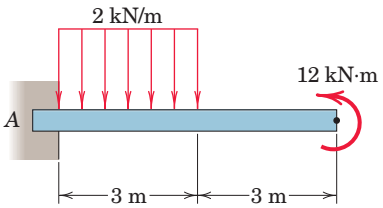
Problem 5/101

5/102 Calculate the support reactions at A and B for the loaded beam.



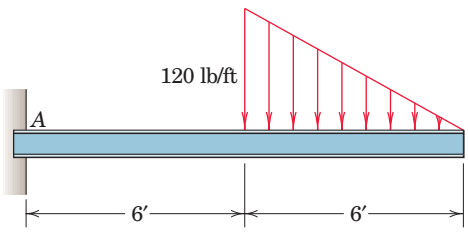
Problem 5/102

5/103 Find the reaction at A due to the uniform loading and the applied couple.



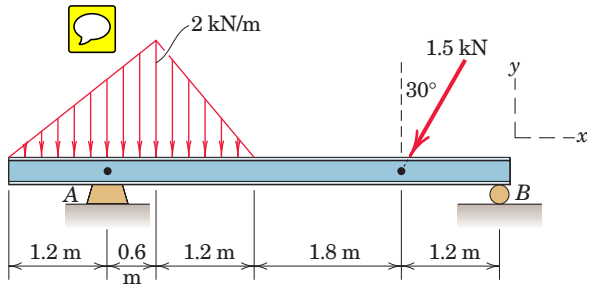
Problem 5/103

5/104 Calculate the supporting force R_A and moment M_A acting on the loaded cantilever beam at A.



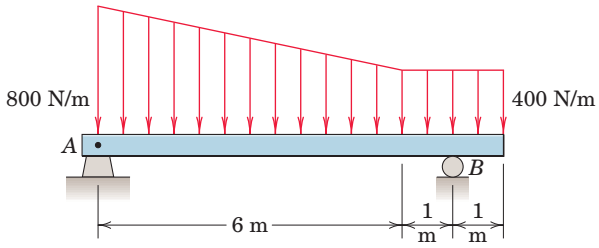
Problem 5/104

5/105 Determine the reactions at A and B for the beam subjected to a combination of distributed and point loads.



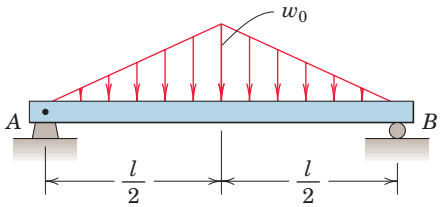
Problem 5/105

5/106 Determine the reactions at the supports of the beam which is loaded as shown.



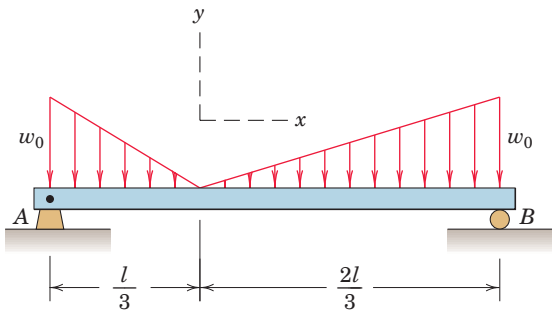
Problem 5/106

5/107 Determine the reactions at the supports A and B for the beam loaded as shown.



Problem 5/107

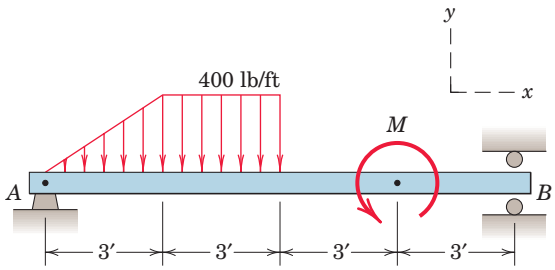
5/108 Calculate the support reactions at A and B for the loaded beam.



Problem 5/108

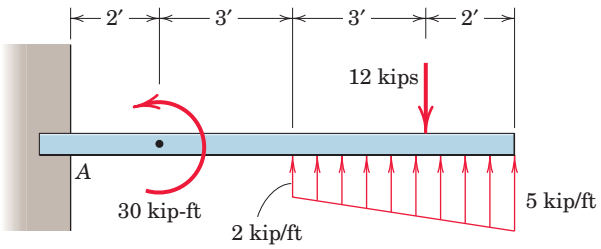
Representative Problems

5/109 The beam is subjected to the distributed load and the couple shown. If M is slowly increased starting from zero, at what value M_0 will contact at B change from the lower surface to the upper surface?



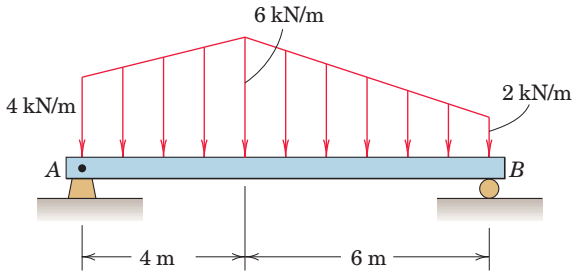
Problem 5/109

5/110 Determine the force and moment reactions at A for the beam which is subjected to the load combination shown.



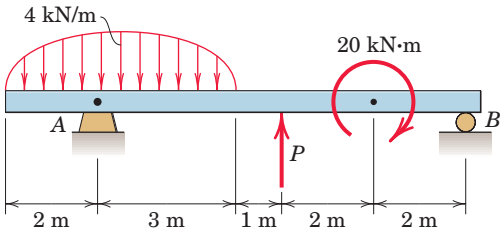
Problem 5/110

5/111 Calculate the support reactions at A and B for the beam subjected to the two linearly varying load distributions.



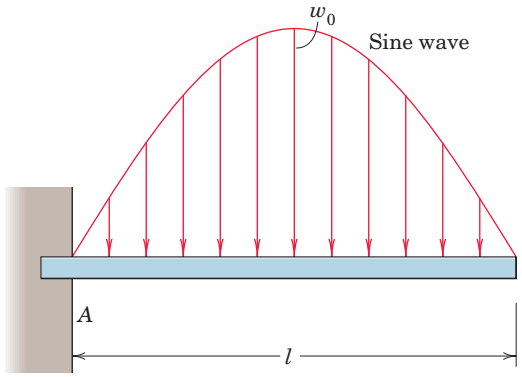
Problem 5/111

5/112 The beam is subjected to an elliptical load distribution and the point loads shown. For what value of the force P will the reaction at B go to zero?



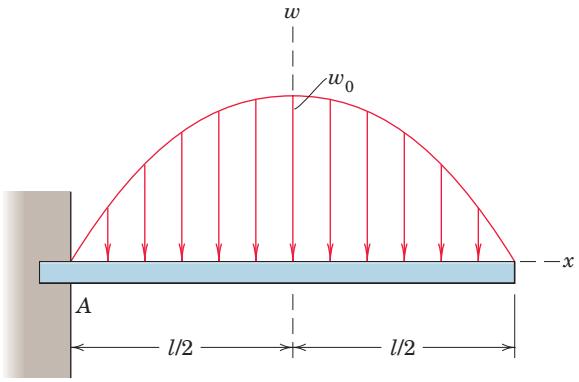
Problem 5/112

5/113 Determine the force and moment reactions at the support A of the built-in beam which is subjected to the sine-wave load distribution.



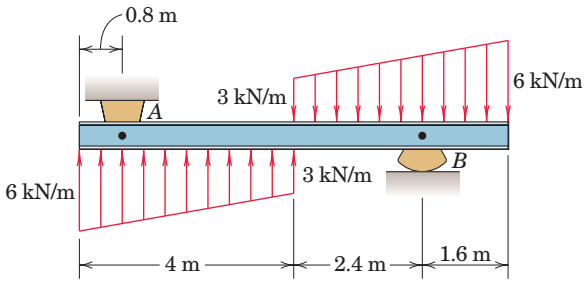
Problem 5/113

5/114 The cantilever beam is subjected to a parabolic distribution of load symmetrical about the middle of the beam. Determine the supporting force R_A and moment M_A acting on the beam at A.



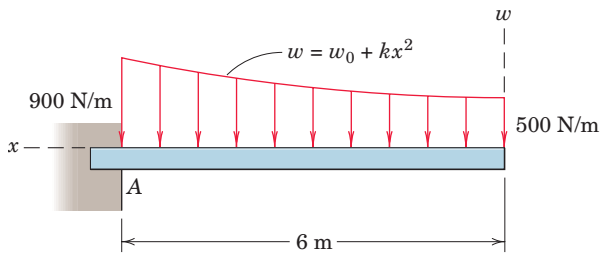
Problem 5/114

5/115 Calculate the support reactions at A and B for the beam subjected to the two linearly distributed loads.



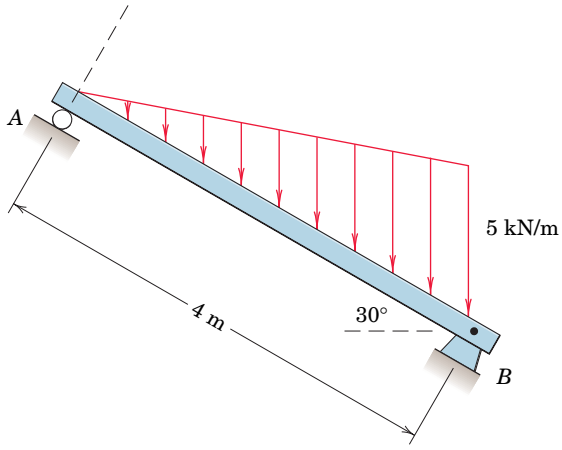
Problem 5/115

5/116 A cantilever beam supports the variable load shown. Calculate the supporting force R_A and moment M_A at A.



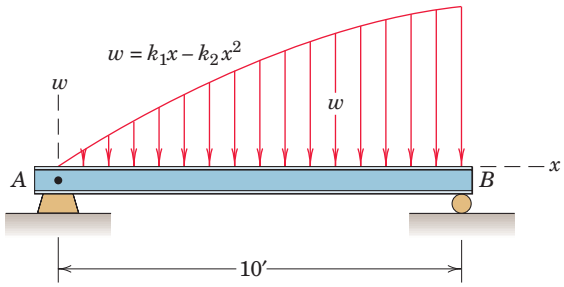
Problem 5/116

5/117 Determine the reactions at points A and B of the inclined beam subjected to the vertical load distribution shown. The value of the load distribution at the right end of the beam is 5 kN per *horizontal* meter.



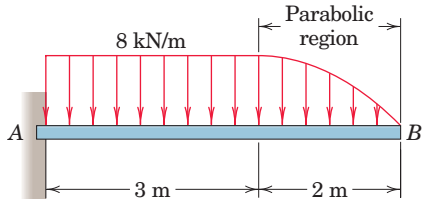
Problem 5/117

5/118 The load per foot of beam length varies as shown. For $x = 10$ ft, the unit load is $w = 300$ lb/ft. At $x = 0$, the load is increasing at the rate of 50 lb/ft per foot. Calculate the support reactions at A and B.



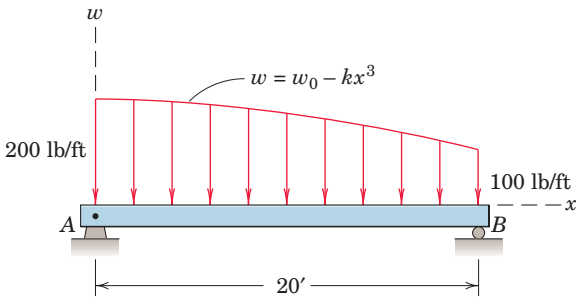
Problem 5/118

5/119 Determine the reactions at the support for the beam which is subjected to the combination of uniform and parabolic loading distributions.



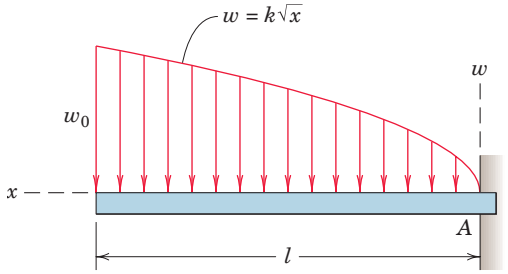
Problem 5/119

5/120 A beam is subjected to the variable loading shown. Calculate the support reactions at A and B.



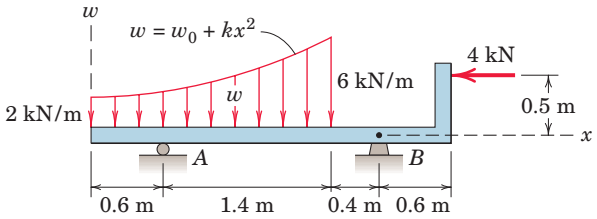
Problem 5/120

5/121 Determine the force and moment reactions at the support A of the cantilever beam subjected to the load distribution shown.



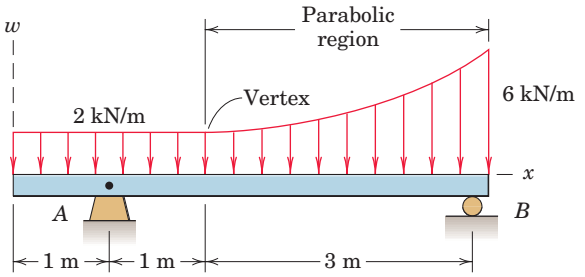
Problem 5/121

5/122 Determine the reactions at A and B for the beam subjected to the distributed and concentrated loads.



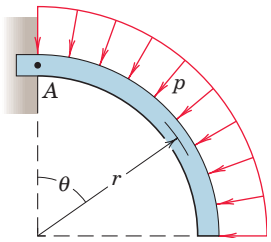
Problem 5/122

5/123 Determine the reactions at the supports of the beam which is acted on by the combination of uniform and parabolic loading distributions.



Problem 5/123

▶**5/124** The quarter-circular cantilever beam is subjected to a uniform pressure on its upper surface as shown. The pressure is expressed in terms of the force p per unit length of circumferential arc. Determine the reactions on the beam at its support A in terms of the compression C_A, shear V_A, and bending moment M_A.



Problem 5/124

5/7 Beams—Internal Effects

The previous article treated the reduction of a distributed force to one or more equivalent concentrated forces and the subsequent determination of the external reactions acting on the beam. In this article we introduce internal beam effects and apply principles of statics to calculate the internal shear force and bending moment as functions of location along the beam.

Shear, Bending, and Torsion

In addition to supporting tension or compression, a beam can resist shear, bending, and torsion. These three effects are illustrated in Fig. 5/22. The force V is called the *shear force*, the couple M is called the *bending moment*, and the couple T is called a *torsional moment*. These effects represent the vector components of the resultant of the forces acting on a transverse section of the beam as shown in the lower part of the figure.

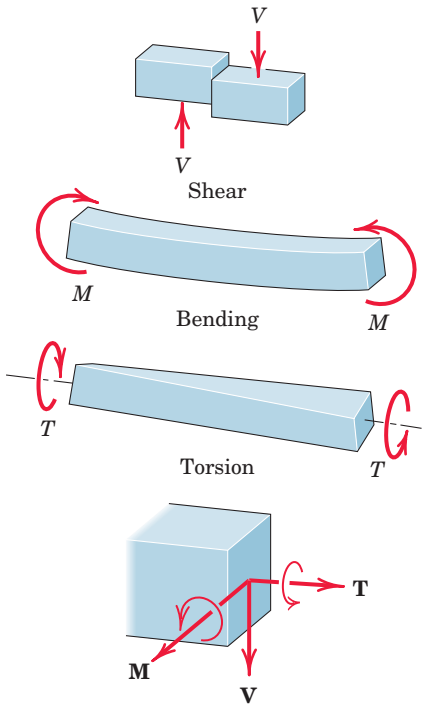


Figure 5/22

Consider the shear force V and bending moment M caused by forces applied to the beam in a single plane. The conventions for positive values of shear V and bending moment M shown in Fig. 5/23 are the ones generally used. From the principle of action and reaction we can see that the directions of V and M are reversed on the two sections. It is frequently impossible to tell without calculation whether the shear and moment at a particular section are positive or negative. For this reason it is advisable to represent V and M in their positive directions on the free-body diagrams and let the algebraic signs of the calculated values indicate the proper directions.

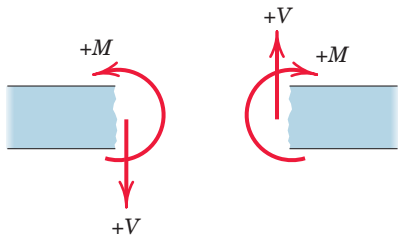


Figure 5/23

As an aid to the physical interpretation of the bending couple M, consider the beam shown in Fig. 5/24 bent by the two equal and opposite positive moments applied at the ends. The cross section of the beam is treated as an H-section with a very narrow center web and heavy top and bottom flanges. For this beam we may neglect the load carried by the small web compared with that carried by the two flanges. The upper flange of the beam clearly is shortened and is under compression, whereas the lower flange is lengthened and is under tension. The resultant of the two forces, one tensile and the other compressive, acting on any section is a couple and has the value of the bending moment on the section. If a beam having some other cross-sectional shape were loaded

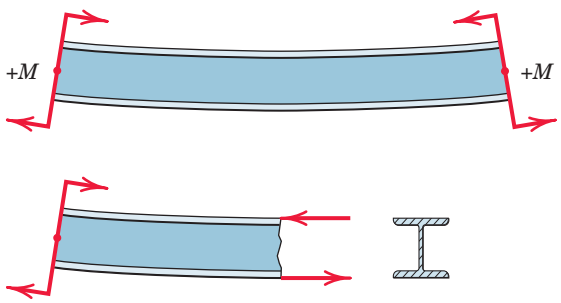


Figure 5/24

in the same way, the distribution of force over the cross section would be different, but the resultant would be the same couple.

Shear-Force and Bending-Moment Diagrams

The variation of shear force V and bending moment M over the length of a beam provides information necessary for the design analysis of the beam. In particular, the maximum magnitude of the bending moment is usually the primary consideration in the design or selection of a beam, and its value and position should be determined. The variations in shear and moment are best shown graphically, and the expressions for V and M when plotted against distance along the beam give the *shear-force* and *bending-moment diagrams* for the beam.

The first step in the determination of the shear and moment relations is to establish the values of all external reactions on the beam by applying the equations of equilibrium to a free-body diagram of the beam as a whole. Next, we isolate a portion of the beam, either to the right or to the left of an arbitrary transverse section, with a free-body diagram, and apply the equations of equilibrium to this isolated portion of the beam. These equations will yield expressions for the shear force V and bending moment M acting at the cut section on the part of the beam isolated. The part of the beam which involves the smaller number of forces, either to the right or to the left of the arbitrary section, usually yields the simpler solution.

We should avoid using a transverse section which coincides with the location of a concentrated load or couple, as such a position represents a point of discontinuity in the variation of shear or bending moment. Finally, it is important to note that the calculations for V and M on each section chosen should be consistent with the positive convention illustrated in Fig. 5/23.

General Loading, Shear, and Moment Relationships

For any beam with distributed loads we can establish certain general relationships which will aid greatly in the determination of the shear and moment distributions along the beam. Figure 5/25 represents a portion of a loaded beam, where an element dx of the beam is isolated. The loading w represents the force per unit length of beam. At the location x the shear V and moment M acting on the element are drawn in their positive directions. On the opposite side of the element where the

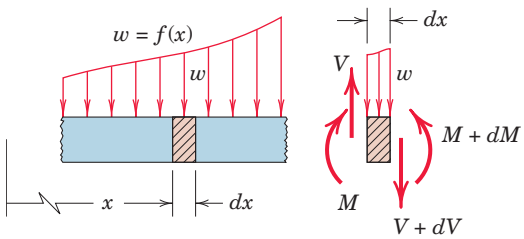


Figure 5/25

coordinate is $x + dx$, these quantities are also shown in their positive directions. They must, however, be labeled $V + dV$ and $M + dM$, since V and M change with x. The applied loading w may be considered constant over the length of the element, since this length is a differential quantity and the effect of any change in w disappears in the limit compared with the effect of w itself.

Equilibrium of the element requires that the sum of the vertical forces be zero. Thus, we have

$$V - w\,dx - (V + dV) = 0$$

or

$$w = -\frac{dV}{dx} \tag{5/10}$$

We see from Eq. 5/10 that the slope of the shear diagram must everywhere be equal to the negative of the value of the applied loading. Equation 5/10 holds on either side of a concentrated load but not at the concentrated load because of the discontinuity produced by the abrupt change in shear.

We may now express the shear force V in terms of the loading w by integrating Eq. 5/10. Thus,

$$\int_{V_0}^{V} dV = -\int_{x_0}^{x} w\,dx$$

or

$$V = V_0 + (\text{the negative of the area under the loading curve from } x_0 \text{ to } x)$$

In this expression V_0 is the shear force at x_0 and V is the shear force at x. Summing the area under the loading curve is usually a simple way to construct the shear-force diagram.

Equilibrium of the element in Fig. 5/25 also requires that the moment sum be zero. Summing moments about the left side of the element gives

$$M + w\,dx\,\frac{dx}{2} + (V + dV)\,dx - (M + dM) = 0$$

The two M's cancel, and the terms $w(dx)^2/2$ and $dV\,dx$ may be dropped, since they are differentials of higher order than those which remain. This leaves

$$V = \frac{dM}{dx} \tag{5/11}$$

© Jake Wyman/Photonica/Getty Images

Because of its economical use of material in achieving bending stiffness, the I-beam is a very common structural element.

which expresses the fact that the shear everywhere is equal to the slope of the moment curve. Equation 5/11 holds on either side of a concentrated couple but not at the concentrated couple because of the discontinuity caused by the abrupt change in moment.

We may now express the moment M in terms of the shear V by integrating Eq. 5/11. Thus,

$$\int_{M_0}^{M} dM = \int_{x_0}^{x} V\,dx$$

or

$$M = M_0 + (\text{area under the shear diagram from } x_0 \text{ to } x)$$

In this expression M_0 is the bending moment at x_0 and M is the bending moment at x. For beams where there is no externally applied moment M_0 at $x_0 = 0$, the total moment at any section equals the area under the shear diagram up to that section. Summing the area under the shear diagram is usually the simplest way to construct the moment diagram.

When V passes through zero and is a continuous function of x with $dV/dx \neq 0$, the bending moment M will be a maximum or a minimum, since $dM/dx = 0$ at such a point. Critical values of M also occur when V crosses the zero axis discontinuously, which occurs for beams under concentrated loads.

We observe from Eqs. 5/10 and 5/11 that the degree of V in x is one higher than that of w. Also M is of one higher degree in x than is V. Consequently, M is two degrees higher in x than w. Thus for a beam loaded by $w = kx$, which is of the first degree in x, the shear V is of the second degree in x and the bending moment M is of the third degree in x.

Equations 5/10 and 5/11 may be combined to yield

$$\frac{d^2M}{dx^2} = -w \qquad \textbf{(5/12)}$$

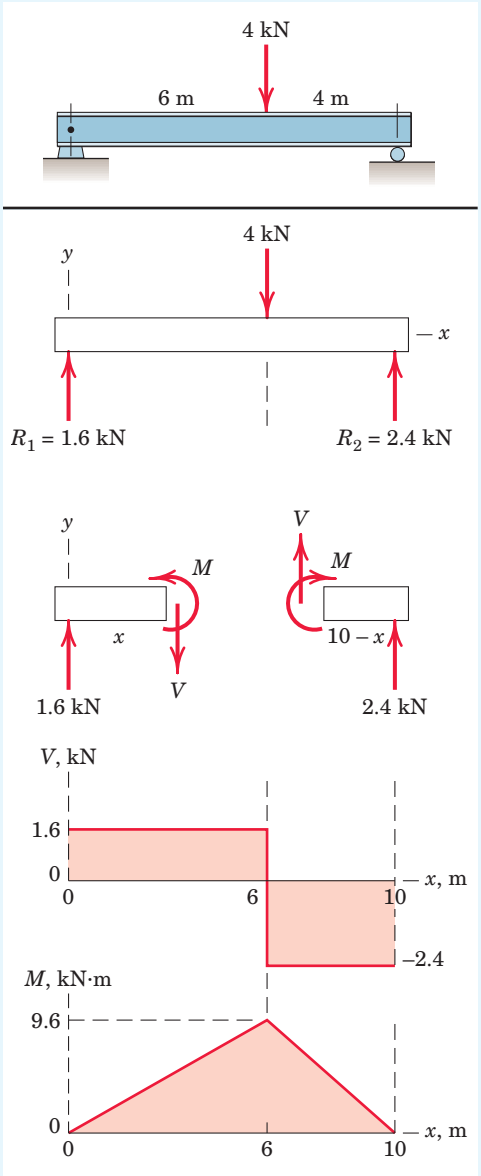
Thus, if w is a known function of x, the moment M can be obtained by two integrations, provided that the limits of integration are properly evaluated each time. This method is usable only if w is a continuous function of x.*

When bending in a beam occurs in more than a single plane, we may perform a separate analysis in each plane and combine the results vectorially.

*When w is a discontinuous function of x, it is possible to introduce a special set of expressions called *singularity functions* which permit writing analytical expressions for shear V and moment M over an interval which includes discontinuities. These functions are not discussed in this book.

SAMPLE PROBLEM 5/13

Determine the shear and moment distributions produced in the simple beam by the 4-kN concentrated load.



Solution. From the free-body diagram of the entire beam we find the support reactions, which are

$$R_1 = 1.6 \text{ kN} \qquad R_2 = 2.4 \text{ kN}$$

A section of the beam of length x is next isolated with its free-body diagram on which we show the shear V and the bending moment M in their positive directions. Equilibrium gives

$$[\Sigma F_y = 0] \qquad 1.6 - V = 0 \qquad V = 1.6 \text{ kN}$$

$$[\Sigma M_{R_1} = 0] \qquad M - 1.6x = 0 \qquad M = 1.6x$$

(1) These values of V and M apply to all sections of the beam to the left of the 4-kN load.

A section of the beam to the right of the 4-kN load is next isolated with its free-body diagram on which V and M are shown in their positive directions. Equilibrium requires

$$[\Sigma F_y = 0] \qquad V + 2.4 = 0 \qquad V = -2.4 \text{ kN}$$

$$[\Sigma M_{R_2} = 0] \qquad -(2.4)(10 - x) + M = 0 \qquad M = 2.4(10 - x)$$

These results apply only to sections of the beam to the right of the 4-kN load.

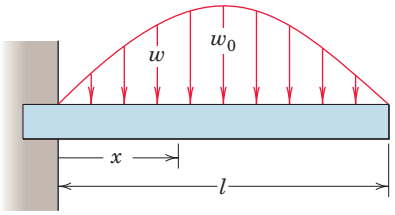
The values of V and M are plotted as shown. The maximum bending moment occurs where the shear changes direction. As we move in the positive x-direction starting with $x = 0$, we see that the moment M is merely the accumulated area under the shear diagram.

Helpful Hint

(1) We must be careful not to take our section at a concentrated load (such as $x = 6$ m) since the shear and moment relations involve discontinuities at such positions.

SAMPLE PROBLEM 5/14

The cantilever beam is subjected to the load intensity (force per unit length) which varies as $w = w_0 \sin(\pi x/l)$. Determine the shear force V and bending moment M as functions of the ratio x/l.



Solution. The free-body diagram of the entire beam is drawn first so that the shear force V_0 and bending moment M_0 which act at the supported end at $x = 0$ can be computed. By convention V_0 and M_0 are shown in their positive mathematical senses. A summation of vertical forces for equilibrium gives

$$[\Sigma F_y = 0] \qquad V_0 - \int_0^l w\,dx = 0 \qquad V_0 = \int_0^l w_0 \sin\frac{\pi x}{l}\,dx = \frac{2w_0 l}{\pi}$$

1 A summation of moments about the left end at $x = 0$ for equilibrium gives

$$[\Sigma M = 0] \qquad -M_0 - \int_0^l x(w\,dx) = 0 \qquad M_0 = -\int_0^l w_0 x \sin\frac{\pi x}{l}\,dx$$

$$M_0 = \frac{-w_0 l^2}{\pi^2}\left[\sin\frac{\pi x}{l} - \frac{\pi x}{l}\cos\frac{\pi x}{l}\right]_0^l = -\frac{w_0 l^2}{\pi}$$

From a free-body diagram of an arbitrary section of length x, integration of Eq. 5/10 permits us to find the shear force internal to the beam. Thus,

2 $[dV = -w\,dx]$

$$\int_{V_0}^{V} dV = -\int_0^x w_0 \sin\frac{\pi x}{l}\,dx$$

$$V - V_0 = \left[\frac{w_0 l}{\pi}\cos\frac{\pi x}{l}\right]_0^x \qquad V - \frac{2w_0 l}{\pi} = \frac{w_0 l}{\pi}\left(\cos\frac{\pi x}{l} - 1\right)$$

or in dimensionless form

$$\frac{V}{w_0 l} = \frac{1}{\pi}\left(1 + \cos\frac{\pi x}{l}\right) \qquad \textit{Ans.}$$

The bending moment is obtained by integration of Eq. 5/11, which gives

$$[dM = V\,dx] \qquad \int_{M_0}^{M} dM = \int_0^x \frac{w_0 l}{\pi}\left(1 + \cos\frac{\pi x}{l}\right)dx$$

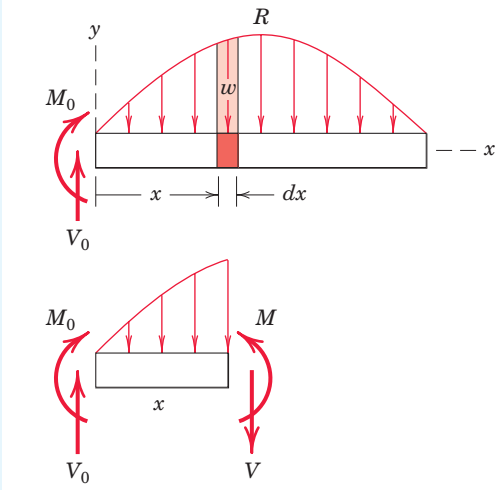
$$M - M_0 = \frac{w_0 l}{\pi}\left[x + \frac{l}{\pi}\sin\frac{\pi x}{l}\right]_0^x$$

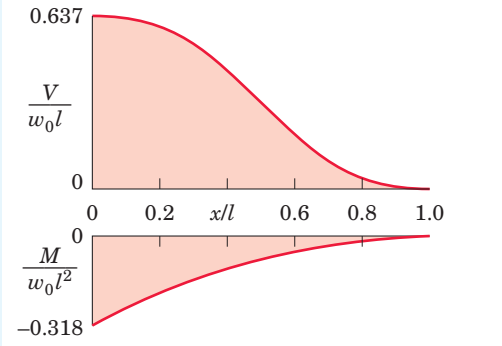
$$M = -\frac{w_0 l^2}{\pi} + \frac{w_0 l}{\pi}\left[x + \frac{l}{\pi}\sin\frac{\pi x}{l} - 0\right]$$

or in dimensionless form

$$\frac{M}{w_0 l^2} = \frac{1}{\pi}\left(\frac{x}{l} - 1 + \frac{1}{\pi}\sin\frac{\pi x}{l}\right) \qquad \textit{Ans.}$$

The variations of $V/w_0 l$ and $M/w_0 l^2$ with x/l are shown in the bottom figures. The negative values of $M/w_0 l^2$ indicate that physically the bending moment is in the direction opposite to that shown.





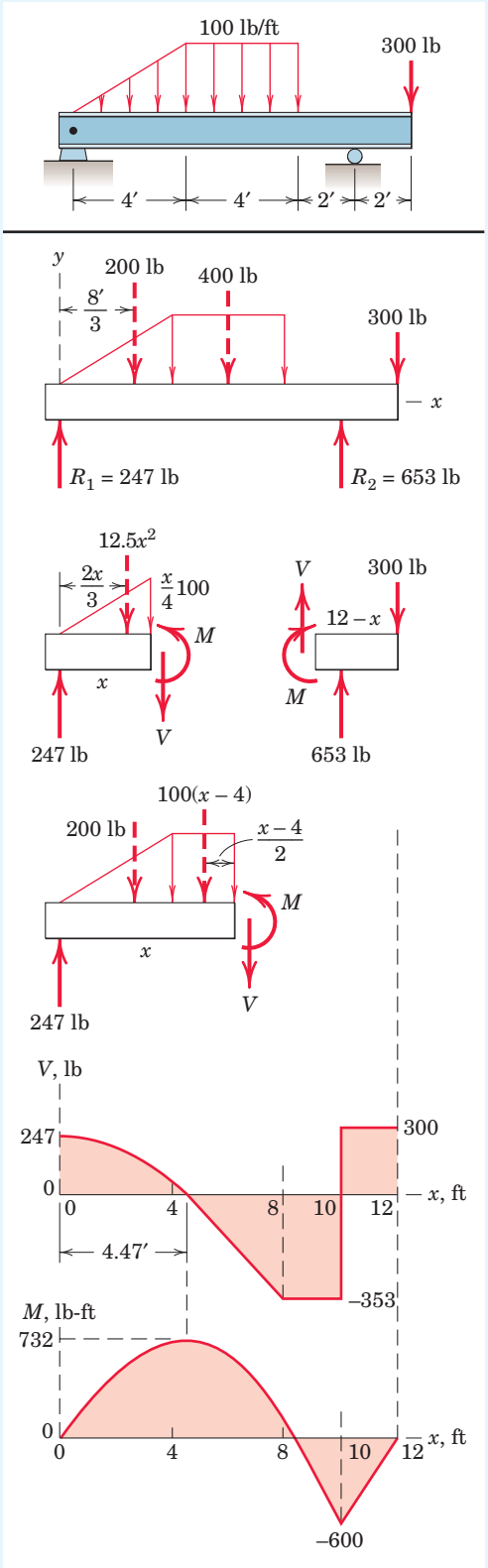
Helpful Hints

1 In this case of symmetry it is clear that the resultant $R = V_0 = 2w_0 l/\pi$ of the load distribution acts at midspan, so that the moment requirement is simply $M_0 = -Rl/2 = -w_0 l^2/\pi$. The minus sign tells us that physically the bending moment at $x = 0$ is opposite to that represented on the free-body diagram.

2 The free-body diagram serves to remind us that the integration limits for V as well as for x must be accounted for. We see that the expression for V is positive, so that the shear force is as represented on the free-body diagram.

SAMPLE PROBLEM 5/15

Draw the shear-force and bending-moment diagrams for the loaded beam and determine the maximum moment M and its location x from the left end.



Solution. The support reactions are most easily obtained by considering the resultants of the distributed loads as shown on the free-body diagram of the beam as a whole. The first interval of the beam is analyzed from the free-body diagram of the section for $0 < x < 4$ ft. A summation of vertical forces and a moment summation about the cut section yield

$$[\Sigma F_y = 0] \qquad V = 247 - 12.5x^2$$

$$[\Sigma M = 0] \qquad M + (12.5x^2)\frac{x}{3} - 247x = 0 \qquad M = 247x - 4.17x^3$$

These values of V and M hold for $0 < x < 4$ ft and are plotted for that interval in the shear and moment diagrams shown.

From the free-body diagram of the section for which $4 < x < 8$ ft, equilibrium in the vertical direction and a moment sum about the cut section give

$$[\Sigma F_y = 0] \qquad V + 100(x - 4) + 200 - 247 = 0 \qquad V = 447 - 100x$$

$$[\Sigma M = 0] \qquad M + 100(x - 4)\frac{x - 4}{2} + 200[x - \tfrac{2}{3}(4)] - 247x = 0$$

$$M = -267 + 447x - 50x^2$$

These values of V and M are plotted on the shear and moment diagrams for the interval $4 < x < 8$ ft.

The analysis of the remainder of the beam is continued from the free-body diagram of the portion of the beam to the right of a section in the next interval. It should be noted that V and M are represented in their positive directions. A vertical-force summation and a moment summation about the section yield

$$V = -353 \text{ lb} \qquad \text{and} \qquad M = 2930 - 353x$$

These values of V and M are plotted on the shear and moment diagrams for the interval $8 < x < 10$ ft.

The last interval may be analyzed by inspection. The shear is constant at +300 lb, and the moment follows a straight-line relation beginning with zero at the right end of the beam.

The maximum moment occurs at $x = 4.47$ ft, where the shear curve crosses the zero axis, and the magnitude of M is obtained for this value of x by substitution into the expression for M for the second interval. The maximum moment is

$$M = 732 \text{ lb-ft} \qquad \textit{Ans.}$$

As before, note that the change in moment M up to any section equals the area under the shear diagram up to that section. For instance, for $x < 4$ ft,

$$[\Delta M = \int V\,dx] \qquad M - 0 = \int_0^x (247 - 12.5x^2)\,dx$$

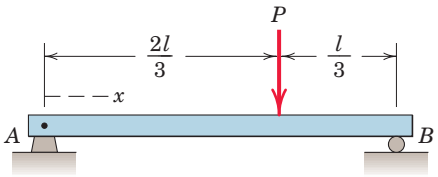
and, as above,

$$M = 247x - 4.17x^3$$

PROBLEMS

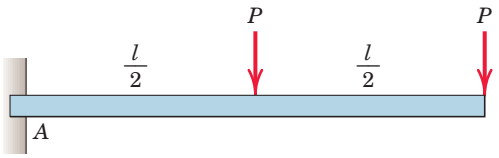
Introductory Problems

5/125 Determine the shear-force and bending-moment distributions produced in the beam by the concentrated load. What are the values of the shear and moment when $x = l/2$?



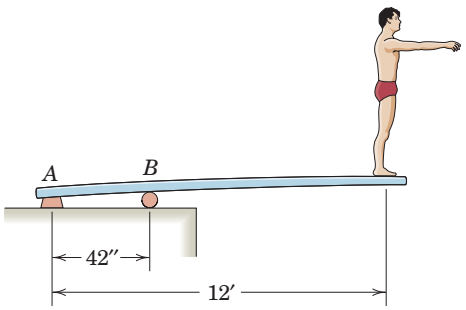
Problem 5/125

5/126 Draw the shear and moment diagrams for the loaded cantilever beam. State the value of the bending moment at midbeam.



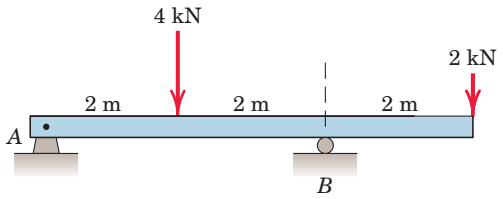
Problem 5/126

5/127 Draw the shear and moment diagrams for the diving board, which supports the 175-lb man poised to dive. Specify the bending moment with the maximum magnitude.



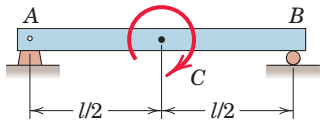
Problem 5/127

5/128 Draw the shear and moment diagrams for the loaded beam and determine the distance d to the right of A where the moment is zero.



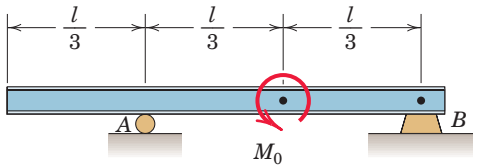
Problem 5/128

5/129 Draw the shear and moment diagrams for the beam loaded at its center by the couple C. State the value of the shear force at midbeam.



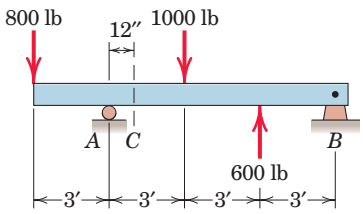
Problem 5/129

5/130 Draw the shear and moment diagrams for the loaded beam. What are the values of the shear and moment at midbeam?



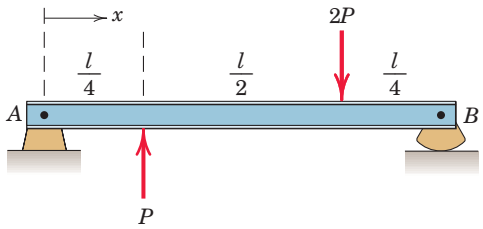
Problem 5/130

5/131 Draw the shear and moment diagrams for the beam shown and find the bending moment M at section C.



Problem 5/131

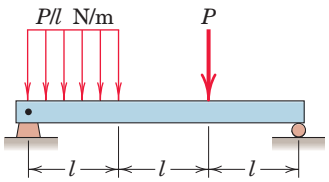
5/132 Draw the shear and moment diagrams for the beam subjected to the two point loads. Determine the maximum bending moment M_{max} and its location.



Problem 5/132

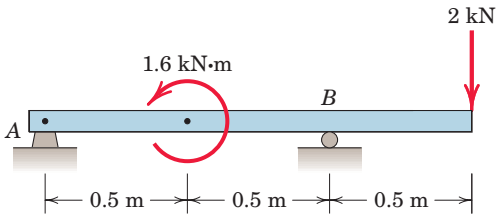
Representative Problems

5/133 Draw the shear and moment diagrams for the loaded beam and find the maximum magnitude M of the bending moment.



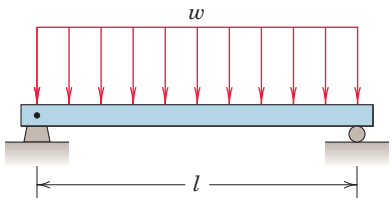
Problem 5/133

5/134 Construct the shear and moment diagrams for the beam loaded by the 2-kN force and the 1.6-kN·m couple. State the value of the bending moment at point B.



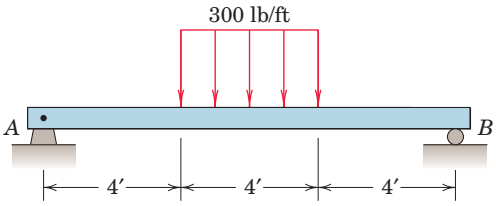
Problem 5/134

5/135 Draw the shear and moment diagrams for the uniformly loaded beam and find the maximum bending moment M_{max}.



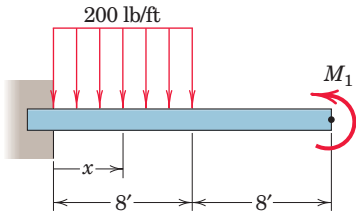
Problem 5/135

5/136 Draw the shear and moment diagrams for the loaded beam and determine the maximum value M_{max} of the moment.



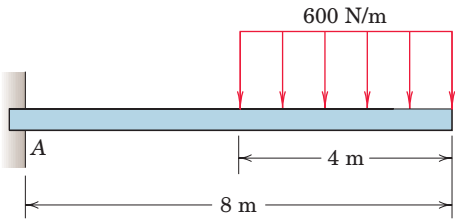
Problem 5/136

5/137 Draw the shear and moment diagrams for the loaded cantilever beam where the end couple M_1 is adjusted so as to produce zero moment at the fixed end of the beam. Find the bending moment M at $x = 4$ ft.



Problem 5/137

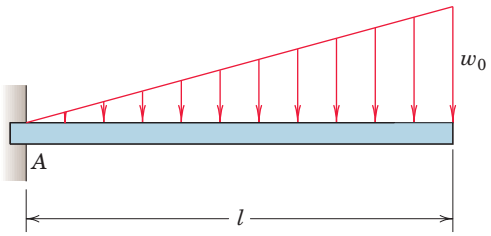
5/138 Determine the shear and moment diagrams for the loaded cantilever beam. Specify the shear V and moment M at the middle section of the beam.



Problem 5/138

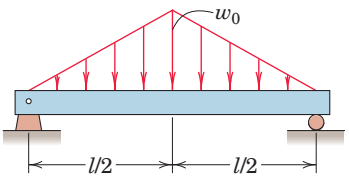
5/139 The shear force in pounds in a certain beam is given by $V = 2200x - 40x^3$ where x is the distance in feet measured along the beam. Determine the corresponding variation with x of the normal loading w in pounds per foot of length. Also determine the bending moment M at $x = 3$ ft if the bending moment at $x = 1$ ft is -1600 lb-ft.

5/140 Draw the shear and moment diagrams for the linearly loaded cantilever beam and specify the bending moment M_A at the support A.



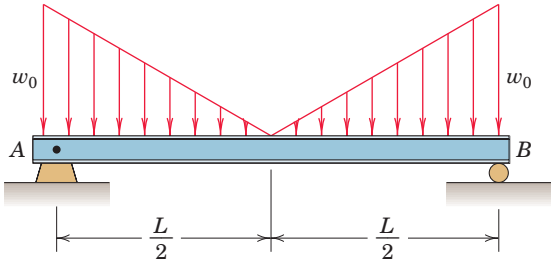
Problem 5/140

5/141 Draw the shear and moment diagrams for the linearly loaded simple beam shown. Determine the maximum magnitude of the bending moment M.



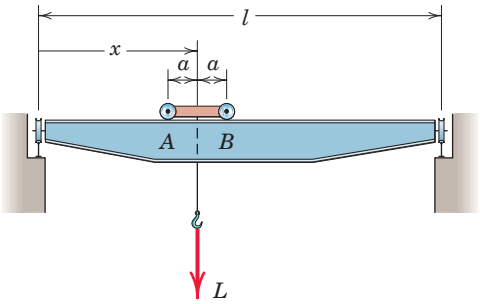
Problem 5/141

5/142 Draw the shear and moment diagrams for the beam loaded as shown. Specify the maximum moment $M_{\max}$.



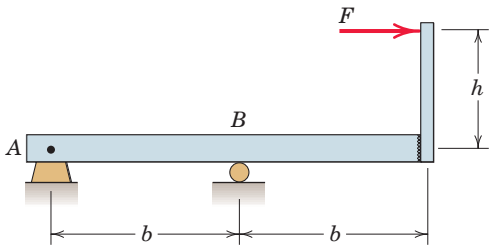
Problem 5/142

5/143 Determine the maximum bending moment M and the corresponding value of x in the crane beam and indicate the section where this moment acts.



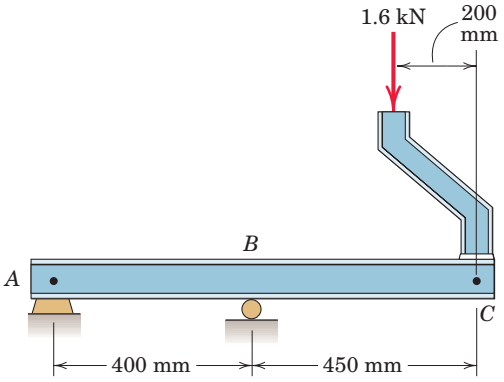
Problem 5/143

5/144 Draw the shear and moment diagrams for the beam loaded by the force F applied to the strut welded to the beam as shown. Specify the bending moment at point B.



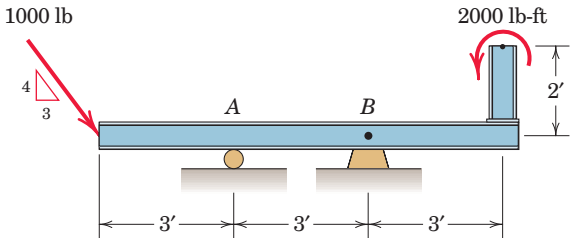
Problem 5/144

5/145 The angle strut is welded to the end C of the I-beam and supports the 1.6-kN vertical force. Determine the bending moment at B and the distance x to the left of C at which the bending moment is zero. Also construct the moment diagram for the beam.



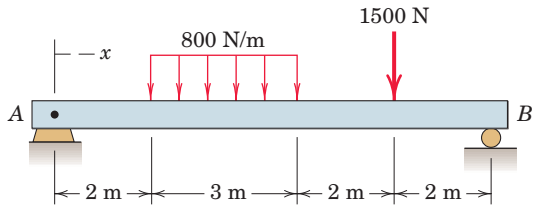
Problem 5/145

5/146 The I-beam supports the 1000-lb force and the 2000-lb-ft couple applied to the 2-ft strut welded to the end of the beam. Calculate the shear V and moment M at the section midway between A and B.



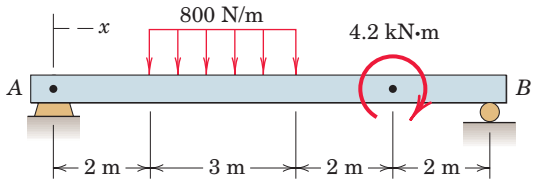
Problem 5/146

5/147 Plot the shear and moment diagrams for the beam loaded with both the distributed and point loads. What are the values of the shear and moment at $x = 6$ m? Determine the maximum bending moment M_{max}.



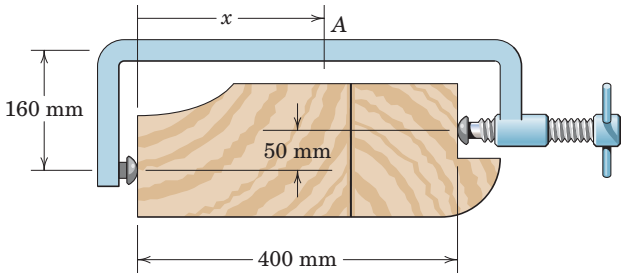
Problem 5/147

5/148 Repeat Prob. 5/147, where the 1500-N load has been replaced by the 4.2-kN·m couple.



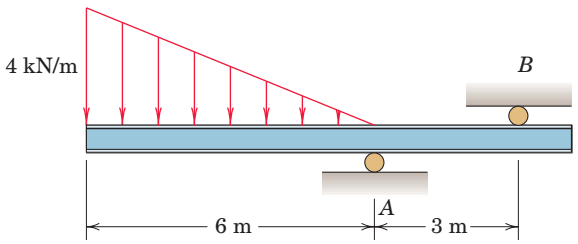
Problem 5/148

5/149 The adjusting screw of the special-purpose offset clamp supports a compression of 500 N. Calculate the shear force V, the tension T, and the bending moment M at section A of the clamp bar for $x = 250$ mm. Which of these three quantities changes with x?



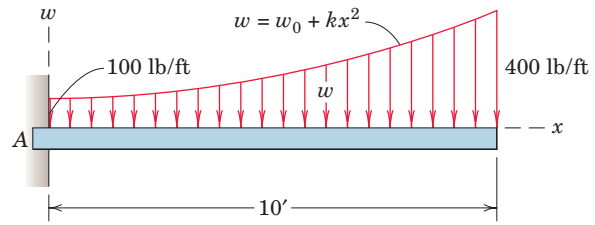
Problem 5/149

5/150 Draw the shear and moment diagrams for the loaded beam and specify the shear V and moment M at a section 3 m to the left of the support at A.



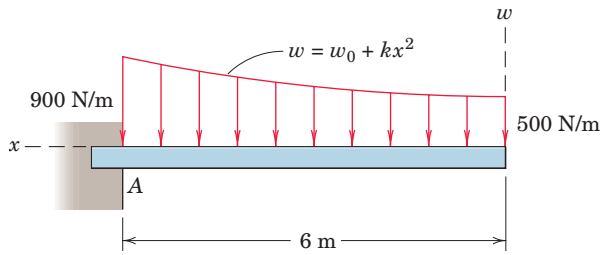
Problem 5/150

5/151 Derive expressions for the shear force V and bending moment M as functions of x in the cantilever beam loaded as shown.



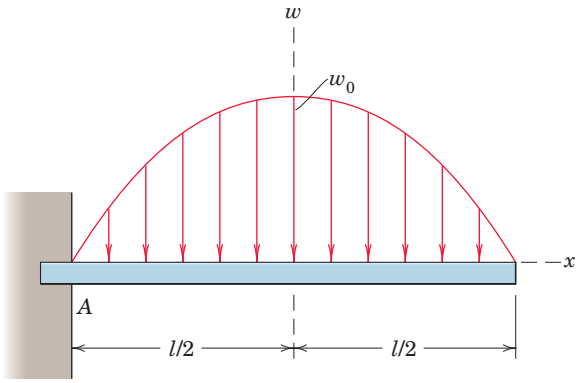
Problem 5/151

5/152 Draw the shear and moment diagrams for the beam of Prob. 5/116 repeated here and specify the shear V and moment M at the midlength of the beam.



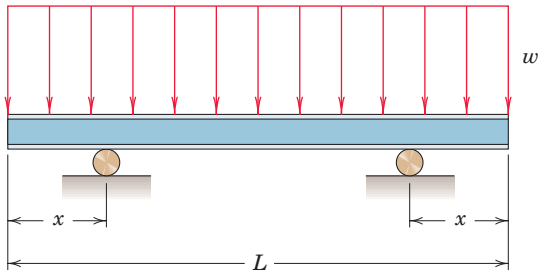
Problem 5/152

5/153 Derive expressions for the shear V and moment M in terms of x for the cantilever beam of Prob. 5/114 shown again here. The loading is parabolic.



Problem 5/153

▶5/154 The beam supports a uniform unit load w. Determine the location x of the two supports so as to minimize the maximum bending moment M_{max} in the beam. Specify M_{max}.



Problem 5/154

5/8 Flexible Cables

One important type of structural member is the flexible cable which is used in suspension bridges, transmission lines, messenger cables for supporting heavy trolley or telephone lines, and many other applications. To design these structures we must know the relations involving the tension, span, sag, and length of the cables. We determine these quantities by examining the cable as a body in equilibrium. In the analysis of flexible cables we assume that any resistance offered to bending is negligible. This assumption means that the force in the cable is always in the direction of the cable.

Flexible cables may support a series of distinct concentrated loads, as shown in Fig. 5/26*a*, or they may support loads continuously distributed over the length of the cable, as indicated by the variable-intensity loading w in 5/26*b*. In some instances the weight of the cable is negligible compared with the loads it supports. In other cases the weight of the cable may be an appreciable load or the sole load and cannot be neglected. Regardless of which of these conditions is present, the equilibrium requirements of the cable may be formulated in the same manner.

General Relationships

If the intensity of the variable and continuous load applied to the cable of Fig. 5/26*b* is expressed as w units of force per unit of horizontal length x, then the resultant R of the vertical loading is

$$R = \int dR = \int w\,dx$$

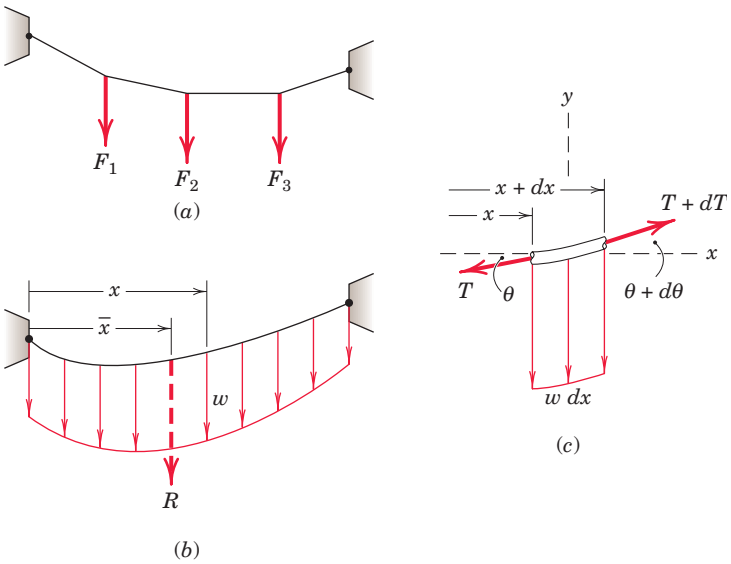


Figure 5/26

where the integration is taken over the desired interval. We find the position of R from the moment principle, so that

$$R\bar{x} = \int x\,dR \qquad \bar{x} = \frac{\int x\,dR}{R}$$

The elemental load $dR = w\,dx$ is represented by an elemental strip of vertical length w and width dx of the shaded area of the loading diagram, and R is represented by the total area. It follows from the foregoing expressions that R passes through the *centroid* of the shaded area.

The equilibrium condition of the cable is satisfied if each infinitesimal element of the cable is in equilibrium. The free-body diagram of a differential element is shown in Fig. 5/26*c*. At the general position x the tension in the cable is T, and the cable makes an angle θ with the horizontal x-direction. At the section $x + dx$ the tension is $T + dT$, and the angle is $\theta + d\theta$. Note that the changes in both T and θ are taken to be positive with a positive change in x. The vertical load $w\,dx$ completes the free-body diagram. The equilibrium of vertical and horizontal forces requires, respectively, that

$$(T + dT)\sin(\theta + d\theta) = T\sin\theta + w\,dx$$
$$(T + dT)\cos(\theta + d\theta) = T\cos\theta$$

The trigonometric expansion for the sine and cosine of the sum of two angles and the substitutions $\sin d\theta = d\theta$ and $\cos d\theta = 1$, which hold in the limit as $d\theta$ approaches zero, yield

$$(T + dT)(\sin\theta + \cos\theta\,d\theta) = T\sin\theta + w\,dx$$
$$(T + dT)(\cos\theta - \sin\theta\,d\theta) = T\cos\theta$$

Dropping the second-order terms and simplifying give us

$$T\cos\theta\,d\theta + dT\sin\theta = w\,dx$$
$$-T\sin\theta\,d\theta + dT\cos\theta = 0$$

which we write as

$$d(T\sin\theta) = w\,dx \qquad \text{and} \qquad d(T\cos\theta) = 0$$

The second relation expresses the fact that the horizontal component of T remains unchanged, which is clear from the free-body diagram. If we introduce the symbol $T_0 = T\cos\theta$ for this constant horizontal force, we may then substitute $T = T_0/\cos\theta$ into the first of the two equations just derived and obtain $d(T_0\tan\theta) = w\,dx$. Because $\tan\theta = dy/dx$, the equilibrium equation may be written in the form

$$\frac{d^2y}{dx^2} = \frac{w}{T_0} \qquad \textbf{(5/13)}$$

Equation 5/13 is the differential equation for the flexible cable. The solution to the equation is that functional relation $y = f(x)$ which satisfies the equation and also satisfies the conditions at the fixed ends of the cable, called *boundary conditions*. This relationship defines the shape of the cable, and we will use it to solve two important and limiting cases of cable loading.

Parabolic Cable

When the intensity of vertical loading w is constant, the condition closely approximates that of a suspension bridge where the uniform weight of the roadway may be expressed by the constant w. The mass of the cable itself is not distributed uniformly with the horizontal but is relatively small, and thus we neglect its weight. For this limiting case we will prove that the cable hangs in a *parabolic arc*.

We start with a cable suspended from two points A and B which are not on the same horizontal line, Fig. 5/27*a*. We place the coordinate origin at the lowest point of the cable, where the tension is horizontal and is T_0. Integration of Eq. 5/13 once with respect to x gives

$$\frac{dy}{dx} = \frac{wx}{T_0} + C$$

where C is a constant of integration. For the coordinate axes chosen, $dy/dx = 0$ when $x = 0$, so that $C = 0$. Thus,

$$\frac{dy}{dx} = \frac{wx}{T_0}$$

which defines the slope of the curve as a function of x. One further integration yields

$$\int_0^y dy = \int_0^x \frac{wx}{T_0}\,dx \qquad \text{or} \qquad y = \frac{wx^2}{2T_0} \qquad \textbf{(5/14)}$$

Alternatively, you should be able to obtain the identical results with the indefinite integral together with the evaluation of the constant of

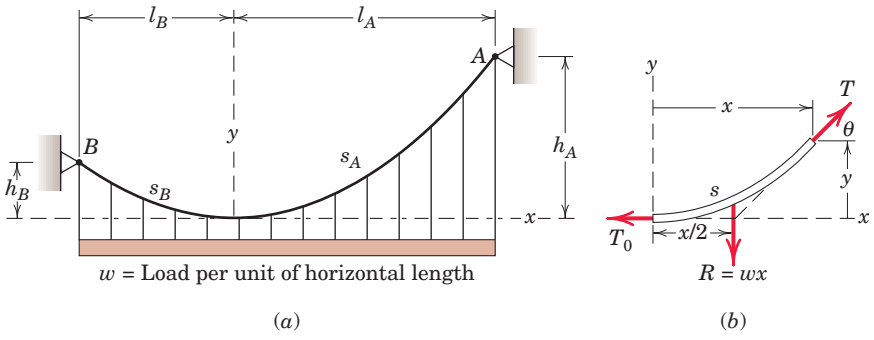


Figure 5/27

integration. Equation 5/14 gives the shape of the cable, which we see is a vertical parabola. The constant horizontal component of cable tension becomes the cable tension at the origin.

Inserting the corresponding values $x = l_A$ and $y = h_A$ in Eq. 5/14 gives

$$T_0 = \frac{wl_A^2}{2h_A} \qquad \text{so that} \qquad y = h_A(x/l_A)^2$$

The tension T is found from a free-body diagram of a finite portion of the cable, shown in Fig. 5/27*b*. From the Pythagorean theorem

$$T = \sqrt{T_0^2 + w^2x^2}$$

Elimination of T_0 gives

$$T = w\sqrt{x^2 + (l_A^2/2h_A)^2} \tag{5/15}$$

The maximum tension occurs where $x = l_A$ and is

$$T_{\max} = wl_A\sqrt{1 + (l_A/2h_A)^2} \tag{5/15a}$$

We obtain the length s_A of the cable from the origin to point A by integrating the expression for a differential length $ds = \sqrt{(dx)^2 + (dy)^2}$. Thus,

$$\int_0^{s_A} ds = \int_0^{l_A} \sqrt{1 + (dy/dx)^2}\, dx = \int_0^{l_A} \sqrt{1 + (wx/T_0)^2}\, dx$$

Although we can integrate this expression in closed form, for computational purposes it is more convenient to express the radical as a convergent series and then integrate it term by term. For this purpose we use the binomial expansion

$$(1 + x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots$$

which converges for $x^2 < 1$. Replacing x in the series by $(wx/T_0)^2$ and setting $n = \frac{1}{2}$ give the expression

$$\begin{aligned} s_A &= \int_0^{l_A} \left(1 + \frac{w^2x^2}{2T_0^2} - \frac{w^4x^4}{8T_0^4} + \cdots\right) dx \\ &= l_A\left[1 + \frac{2}{3}\left(\frac{h_A}{l_A}\right)^2 - \frac{2}{5}\left(\frac{h_A}{l_A}\right)^4 + \cdots\right] \end{aligned} \tag{5/16}$$

This series is convergent for values of $h_A/l_A < \frac{1}{2}$, which holds for most practical cases.

The relationships which apply to the cable section from the origin to point B can be easily obtained by replacing h_A, l_A, and s_A by h_B, l_B, and s_B, respectively.

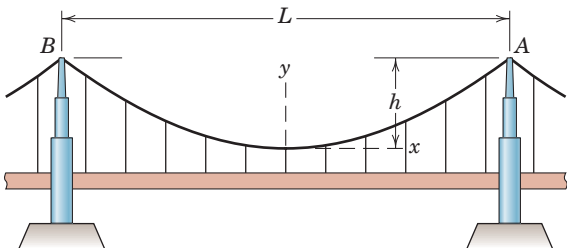


Figure 5/28

For a suspension bridge where the supporting towers are on the same horizontal line, Fig. 5/28, the total span is $L = 2l_A$, the sag is $h = h_A$, and the total length of the cable is $S = 2s_A$. With these substitutions, the maximum tension and the total length become

$$T_{\max} = \frac{wL}{2}\sqrt{1 + (L/4h)^2} \qquad \textbf{(5/15}\boldsymbol{b}\textbf{)}$$

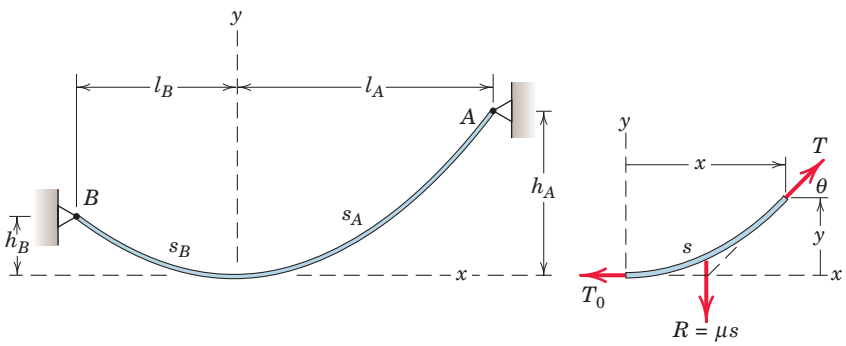
$$S = L\left[1 + \frac{8}{3}\left(\frac{h}{L}\right)^2 - \frac{32}{5}\left(\frac{h}{L}\right)^4 + \cdots\right] \qquad \textbf{(5/16}\boldsymbol{a}\textbf{)}$$

This series converges for all values of $h/L < \frac{1}{4}$. In most cases h is much smaller than $L/4$, so that the three terms of Eq. 5/16a give a sufficiently accurate approximation.

Catenary Cable

Consider now a uniform cable, Fig. 5/29a, suspended from two points A and B and hanging under the action of its own weight only. We will show in this limiting case that the cable assumes a curved shape known as a *catenary*.

The free-body diagram of a finite portion of the cable of length s measured from the origin is shown in part b of the figure. This free-body diagram differs from the one in Fig. 5/27b in that the total vertical force supported is equal to the weight of the cable section of length s rather



(*a*) (*b*)

Figure 5/29

than the load distributed uniformly with respect to the horizontal. If the cable has a weight μ per unit of its length, the resultant R of the load is $R = \mu s$, and the incremental vertical load $w\,dx$ of Fig. 5/26*c* is replaced by $\mu\,ds$. With this replacement the differential relation, Eq. 5/13, for the cable becomes

$$\frac{d^2y}{dx^2} = \frac{\mu}{T_0}\frac{ds}{dx} \tag{5/17}$$

Because $s = f(x, y)$, we must change this equation to one containing only the two variables.

We may substitute the identity $(ds)^2 = (dx)^2 + (dy)^2$ to obtain

$$\frac{d^2y}{dx^2} = \frac{\mu}{T_0}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \tag{5/18}$$

Equation 5/18 is the differential equation of the curve (catenary) formed by the cable. This equation is easier to solve if we substitute $p = dy/dx$ to obtain

$$\frac{dp}{\sqrt{1 + p^2}} = \frac{\mu}{T_0}\,dx$$

Integrating this equation gives us

$$\ln\,(p + \sqrt{1 + p^2}) = \frac{\mu}{T_0}x + C$$

The constant C is zero because $dy/dx = p = 0$ when $x = 0$. Substituting $p = dy/dx$, changing to exponential form, and clearing the equation of the radical give

$$\frac{dy}{dx} = \frac{e^{\mu x/T_0} - e^{-\mu x/T_0}}{2} = \sinh\frac{\mu x}{T_0}$$

where the hyperbolic function* is introduced for convenience. The slope may be integrated to obtain

$$y = \frac{T_0}{\mu}\cosh\frac{\mu x}{T_0} + K$$

The integration constant K is evaluated from the boundary condition $x = 0$ when $y = 0$. This substitution requires that $K = -T_0/\mu$, and hence,

$$y = \frac{T_0}{\mu}\left(\cosh\frac{\mu x}{T_0} - 1\right) \tag{5/19}$$

*See Arts. C/8 and C/10, Appendix C, for the definition and integral of hyperbolic functions.

Equation 5/19 is the equation of the curve (catenary) formed by the cable hanging under the action of its weight only.

From the free-body diagram in Fig. 5/29*b* we see that $dy/dx = \tan\theta = \mu s/T_0$. Thus, from the previous expression for the slope,

$$s = \frac{T_0}{\mu}\sinh\frac{\mu x}{T_0} \qquad \textbf{(5/20)}$$

We obtain the tension T in the cable from the equilibrium triangle of the forces in Fig. 5/29*b*. Thus,

$$T^2 = \mu^2 s^2 + {T_0}^2$$

which, when combined with Eq. 5/20, becomes

$${T^2} = {T_0}^2\left(1 + \sinh^2\frac{\mu x}{T_0}\right) = {T_0}^2\cosh^2\frac{\mu x}{T_0}$$

or

$$T = T_0\cosh\frac{\mu x}{T_0} \qquad \textbf{(5/21)}$$

We may also express the tension in terms of y with the aid of Eq. 5/19, which, when substituted into Eq. 5/21, gives

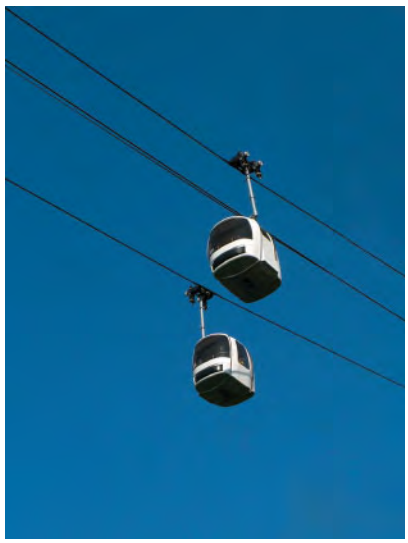
$$T = T_0 + \mu y \qquad \textbf{(5/22)}$$

Equation 5/22 shows that the change in cable tension from that at the lowest position depends only on μy.

Most problems dealing with the catenary involve solutions of Eqs. 5/19 through 5/22, which can be handled by a graphical approximation or solved by computer. The procedure for a graphical or computer solution is illustrated in Sample Problem 5/17 following this article.

The solution of catenary problems where the sag-to-span ratio is small may be approximated by the relations developed for the parabolic cable. A small sag-to-span ratio means a tight cable, and the uniform distribution of weight along the cable is not very different from the same load intensity distributed uniformly along the horizontal.

Many problems dealing with both the catenary and parabolic cables involve suspension points which are not on the same level. In such cases we may apply the relations just developed to the part of the cable on each side of the lowest point.



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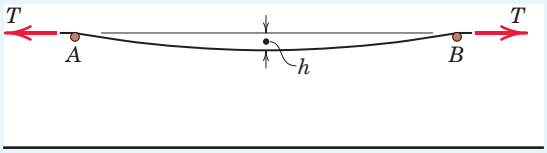
In addition to the distributed weight of the cable, these tramway cars exert a concentrated load on the suspending cable.

SAMPLE PROBLEM 5/16

A 100-ft length of surveyor's tape weighs 0.6 lb. When the tape is stretched between two points on the same level by a tension of 10 lb at each end, calculate the sag h in the middle.



Solution. The weight per unit length is $\mu = 0.6/100 = 0.006$ lb/ft. The total length is $2s = 100$ or $s = 50$ ft.

$[T^2 = \mu^2 s^2 + T_0^2]$ $\quad 10^2 = (0.006)^2(50)^2 + T_0^2$

(1) $\quad T_0 = 9.995$ lb

$[T = T_0 + \mu y]$ $\quad 10 = 9.995 + 0.006h$

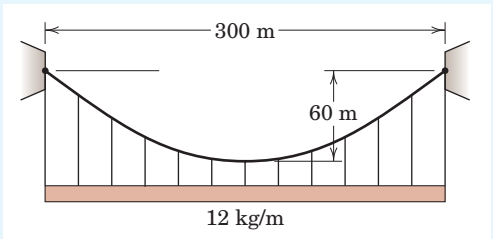
$h = 0.750$ ft or 9.00 in. *Ans.*

Helpful Hint

(1) An extra significant figure is displayed here for clarity.

SAMPLE PROBLEM 5/17

The light cable supports a mass of 12 kg per meter of horizontal length and is suspended between the two points on the same level 300 m apart. If the sag is 60 m, find the tension at midlength, the maximum tension, and the total length of the cable.



Solution. With a uniform horizontal distribution of load, the solution of part (b) of Art. 5/8 applies, and we have a parabolic shape for the cable. For $h = 60$ m, $L = 300$ m, and $w = 12(9.81)(10^{-3})$ kN/m the relation following Eq. 5/14 with $l_A = L/2$ gives for the midlength tension

$$\left[T_0 = \frac{wL^2}{8h}\right] \qquad T_0 = \frac{0.1177(300)^2}{8(60)} = 22.1 \text{ kN} \qquad \textit{Ans.}$$

The maximum tension occurs at the supports and is given by Eq. 5/15b. Thus,

(1) $$\left[T_{\max} = \frac{wL}{2}\sqrt{1 + \left(\frac{L}{4h}\right)^2}\right]$$

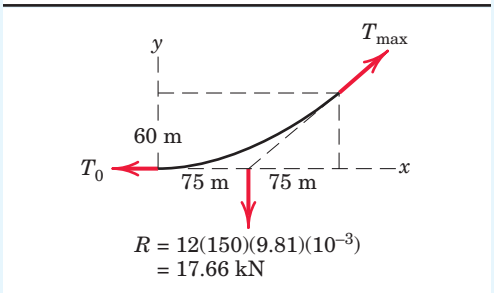
$$T_{\max} = \frac{12(9.81)(10^{-3})(300)}{2}\sqrt{1 + \left(\frac{300}{4(60)}\right)^2} = 28.3 \text{ kN} \qquad \textit{Ans.}$$

The sag-to-span ratio is $60/300 = 1/5 < 1/4$. Therefore, the series expression developed in Eq. 5/16a is convergent, and we may write for the total length

$$S = 300\left[1 + \frac{8}{3}\left(\frac{1}{5}\right)^2 - \frac{32}{5}\left(\frac{1}{5}\right)^4 + \cdots\right]$$

$$= 300[1 + 0.1067 - 0.01024 + \cdots]$$

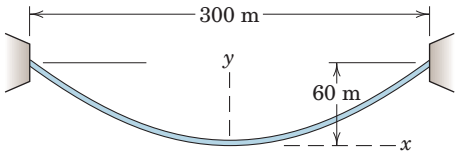
$$= 329 \text{ m} \qquad \textit{Ans.}$$



Helpful Hint

(1) *Suggestion:* Check the value of $T_{\max}$ directly from the free-body diagram of the right-hand half of the cable, from which a force polygon may be drawn.

SAMPLE PROBLEM 5/18



Replace the cable of Sample Problem 5/17, which is loaded uniformly along the horizontal, by a cable which has a mass of 12 kg per meter of its own length and supports its own weight only. The cable is suspended between two points on the same level 300 m apart and has a sag of 60 m. Find the tension at midlength, the maximum tension, and the total length of the cable.

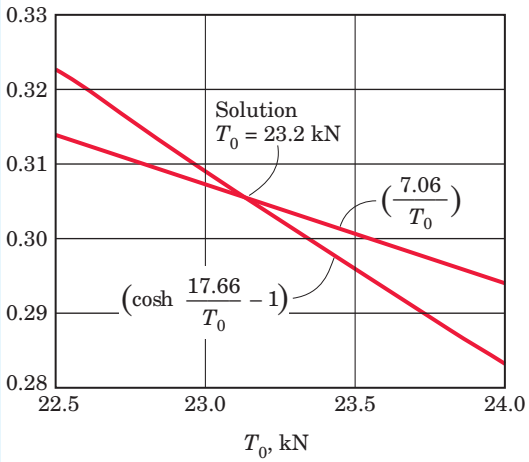
Solution. With a load distributed uniformly along the length of the cable, the solution of part (*c*) of Art. 5/8 applies, and we have a catenary shape of the cable. Equations 5/20 and 5/21 for the cable length and tension both involve the minimum tension T_0 at midlength, which must be found from Eq. 5/19. Thus, for $x = 150$ m, $y = 60$ m, and $\mu = 12(9.81)(10^{-3}) = 0.1177$ kN/m, we have

$$60 = \frac{T_0}{0.1177}\left[\cosh\frac{(0.1177)(150)}{T_0} - 1\right]$$

or

$$\frac{7.06}{T_0} = \cosh\frac{17.66}{T_0} - 1$$

This equation can be solved graphically. We compute the expression on each side of the equals sign and plot it as a function of T_0. The intersection of the two curves establishes the equality and determines the correct value of T_0. This plot is shown in the figure accompanying this problem and yields the solution



$$T_0 = 23.2 \text{ kN}$$

Alternatively, we may write the equation as

$$f(T_0) = \cosh\frac{17.66}{T_0} - \frac{7.06}{T_0} - 1 = 0$$

and set up a computer program to calculate the value(s) of T_0 which renders $f(T_0) = 0$. See Art. C/11 of Appendix C for an explanation of one applicable numerical method.

The maximum tension occurs for maximum y and from Eq. 5/22 is

$$T_{\max} = 23.2 + (0.1177)(60) = 30.2 \text{ kN} \qquad \textit{Ans.}$$

(1) From Eq. 5/20 the total length of the cable becomes

$$2s = 2\frac{23.2}{0.1177}\sinh\frac{(0.1177)(150)}{23.2} = 330 \text{ m} \qquad \textit{Ans.}$$

Helpful Hint

(1) Note that the solution of Sample Problem 5/17 for the parabolic cable gives a very close approximation to the values for the catenary even though we have a fairly large sag. The approximation is even better for smaller sag-to-span ratios.

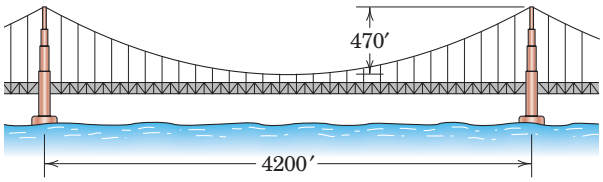
PROBLEMS

(The problems marked with an asterisk (*) involve transcendental equations which may be solved with a computer or by graphical methods.)

Introductory Problems

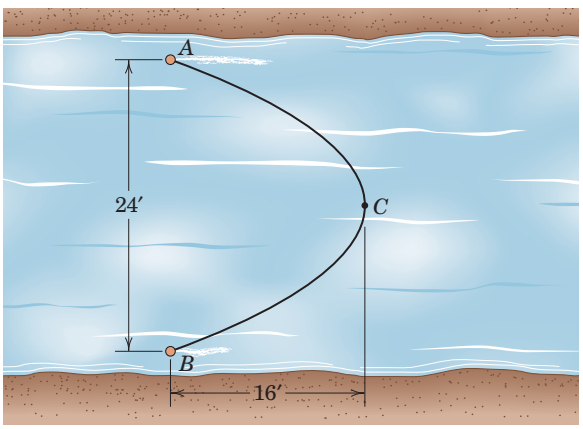
5/155 A mason stretches a string between two points 50 ft apart on the same level with a tension of 10 lb at each end. If the string weighs 0.1 lb, determine the sag h at the middle of the string.

5/156 The Golden Gate Bridge in San Francisco has a main span of 4200 ft, a sag of 470 ft, and a total static loading of 21,300 lb per lineal foot of horizontal measurement. The weight of both of the main cables is included in this figure and is assumed to be uniformly distributed along the horizontal. The angle made by the cable with the horizontal at the top of the tower is the same on each side of each tower. Calculate the midspan tension T_0 in each of the main cables and the compressive force C exerted by each cable on the top of each tower.



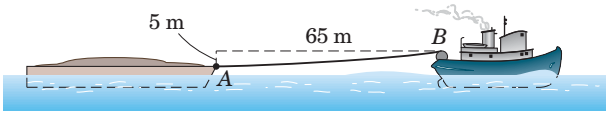
Problem 5/156

5/157 The left-to-right current in the stream causes a uniform drag of 4 lb per cross-stream foot on the floating cable which is attached to posts A and B. Determine the minimum and maximum tensions in the cable and the location of each.



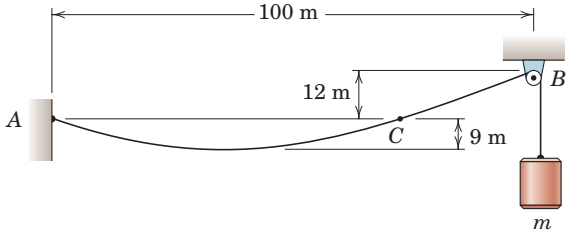
Problem 5/157

***5/158** A power tug is towing a barge with a cable which has a mass of 14 kg per meter of its length. It is observed that the tangent to the cable at point A is horizontal. Determine the tensions at A and B.



Problem 5/158

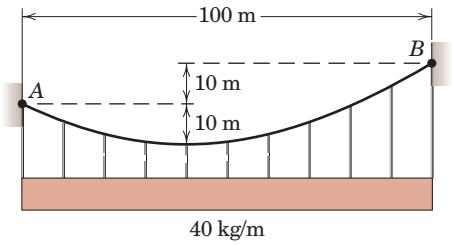
5/159 A cable weighing 25 newtons per meter of length is suspended from point A and passes over the small pulley at B. Calculate the mass m of the attached cylinder which will produce the sag of 9 m. Also determine the horizontal distance from A to C. Because of the small sag-to-span ratio, use the approximation of a parabolic cable.



Problem 5/159

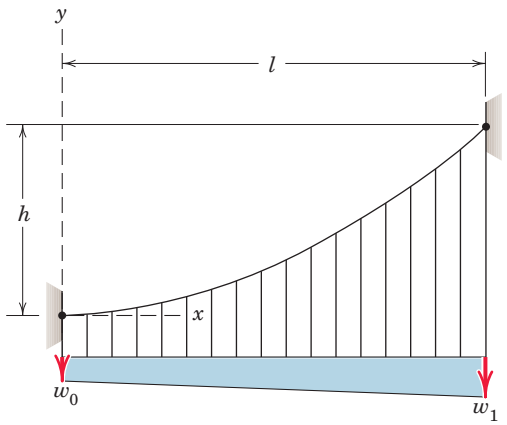
***5/160** Repeat Prob. 5/159, but do not use the approximation of a parabolic cable. Compare your results with the printed answers for Prob. 5/159.

5/161 A cable supports a load of 40 kg /m uniformly distributed along the horizontal and is suspended from two fixed points A and B located as shown. Calculate the cable tensions at A and B and the minimum tension T_0.



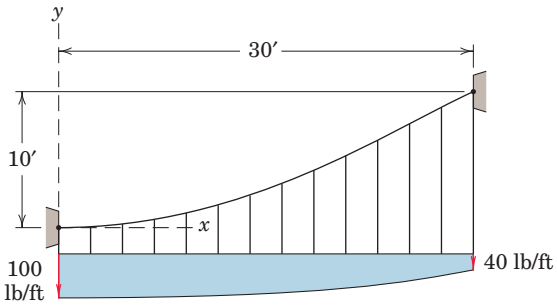
Problem 5/161

5/162 The light cable has zero slope at its lower point of attachment and supports the unit load which varies linearly with x from w_0 to w_1 as shown. Derive the expression for the tension T_0 in the cable at the origin.



Problem 5/162

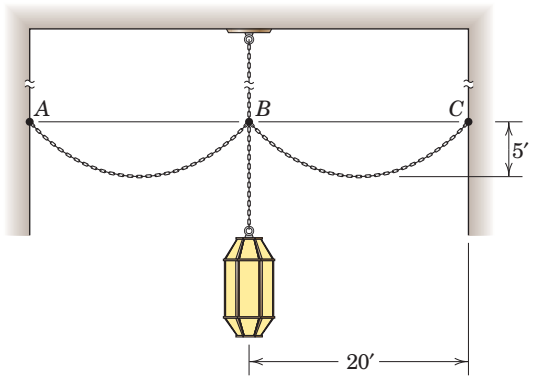
5/163 The cable of negligible mass is suspended from the two fixed points shown, and the tension at the lower support is sufficient to maintain zero slope of the cable at that point. The cable supports a unit load w which decreases from 100 lb/ft at $x = 0$ to 40 lb/ft at $x = 30$ ft with the decrease proportional to x^2. Determine the equation of the curve assumed by the cable.



Problem 5/163

5/164 Expand Eq. 5/19 in a power series for cosh $(\mu x/T_0)$ and show that the equation for the parabola, Eq. 5/14, is obtained by taking only the first two terms in the series. (See Art. C/8, Appendix C, for the series expansion of the hyperbolic function.)

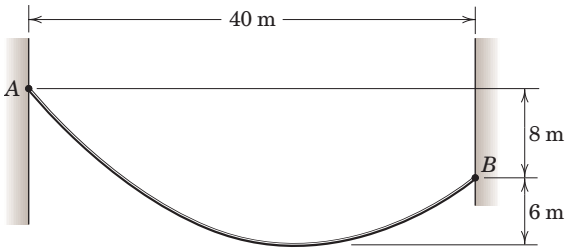
***5/165** A light fixture is suspended from the ceiling of an outside portico. Four chains, two of which are shown, prevent excessive motion of the fixture during windy conditions. If the chains weigh 15 lb per foot of length, determine the chain tension at C and the length L of chain BC.



Problem 5/165

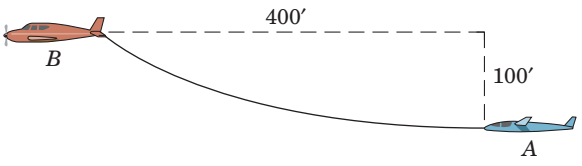
Representative Problems

***5/166** Find the total length L of cable which will have the configuration shown when suspended from points A and B.



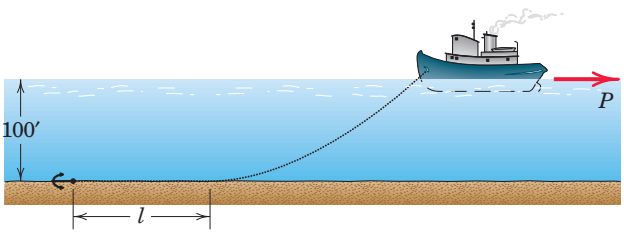
Problem 5/166

***5/167** The glider A is being towed in level flight and is 400 ft behind and 100 ft below the tow plane B. The tangent to the cable at the glider is horizontal. The cable weighs 0.5 lb per foot of length. Calculate the horizontal tension T_0 in the cable at the glider. Neglect air resistance and compare your result with that obtained by approximating the cable shape by a parabola.



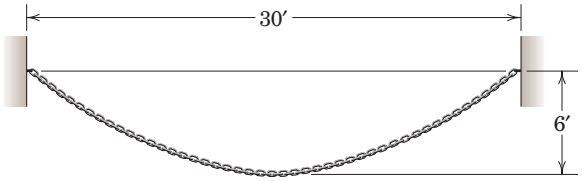
Problem 5/167

5/168 In setting its anchor in 100 ft of water, a small power boat reverses its propeller, which gives a reverse thrust $P = 800$ lb. A total of 400 ft of anchor chain from anchor to bow has been released. The chain weighs 1.63 lb/ft, and the upward force due to water buoyancy is 0.21 lb/ft. Calculate the length l of chain in contact with the bottom.



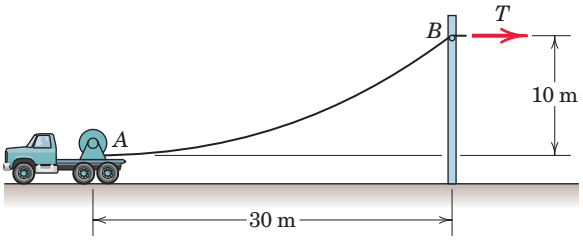
Problem 5/168

***5/169** Find the total length L of chain which will have a sag of 6 ft when suspended from two points on the same horizontal line 30 ft apart.



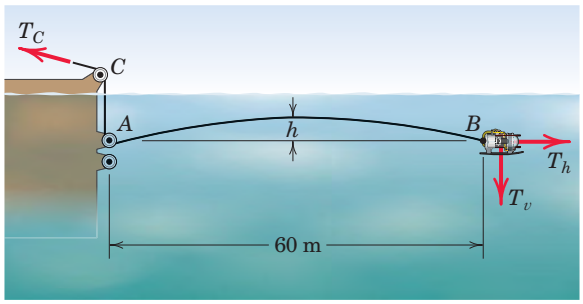
Problem 5/169

***5/170** Calculate the tension T required to steadily pull the cable over a roller support on the utility pole. Neglect the effects of friction at the support. The cable, which is horizontal at A, has a mass of 3 kg/m. Also determine the length of cable from A to B.



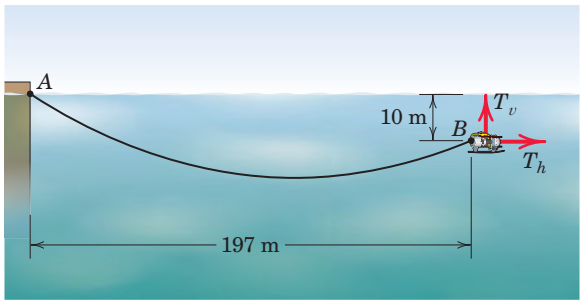
Problem 5/170

***5/171** A small remotely-controlled underwater robotic vehicle and its tether are positioned as shown. The neutrally buoyant vehicle has independent thrusters for horizontal and vertical control. The tether, designed to be very slightly buoyant, has a net upward force of 0.025 N per meter of its length acting on it. There are 60.5 m of cable between points A and B. Determine the horizontal and vertical forces which the vehicle must exert on the cable at B in order to maintain the configuration shown. Also, find the distance h. Assume that the tether between points A and B is entirely under water.



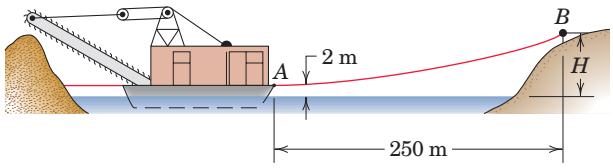
Problem 5/171

***5/172** The remotely-controlled robotic vehicle of Prob. 5/171 is now connected to a 200-m tether which has a slight negative buoyancy in that a net downward force of 0.025 N per meter of its length acts on it. Determine the horizontal and vertical components of thrust which must be produced in order to hold the vehicle in the position shown. The vehicle itself is neutrally buoyant.



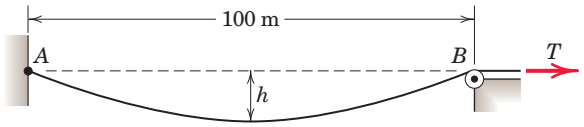
Problem 5/172

5/173 A floating dredge is anchored in position with a single stern cable which has a horizontal direction at the attachment A and extends a horizontal distance of 250 m to an anchorage B on shore. A tension of 300 kN is required in the cable at A. If the cable has a mass of 22 kg per meter of its length, compute the required height H of the anchorage above water level and find the length s of cable between A and B.



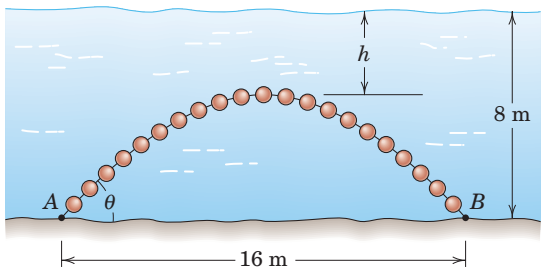
Problem 5/173

***5/174** A cable which weighs 50 newtons per meter of length is secured at point A and passes over the small pulley at B on the same level under a tension T. Determine the minimum value of T to support the cable and the corresponding deflection h.



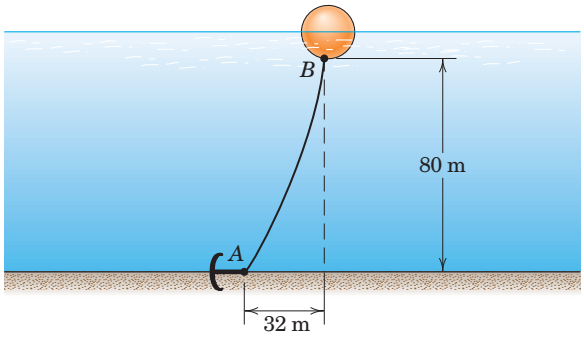
Problem 5/174

***5/175** A series of spherical floats are equally spaced and securely fastened to a flexible cable of length 20 m. Ends A and B are anchored 16 m apart to the bottom of a fresh-water lake at a depth of 8 m. The floats and cable have a combined weight of 100 N per meter of cable length, and the buoyancy of the water produces an upward force of 560 N per meter of cable length. Calculate the depth h below the surface to the top of the line of floats. Also find the angle θ made by the line of floats with the horizontal at A.



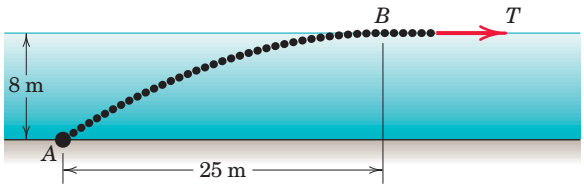
Problem 5/175

***5/176** A spherical buoy used to mark the course for a sailboat race is shown in the figure. There is a water current from left to right which causes a horizontal drag on the buoy; the effect of the current on the cable can be neglected. The length of the cable between points A and B is 87 m, and the effective cable mass is 2 kg/m when the buoyancy of the cable is accounted for. Determine the tensions at both A and B.



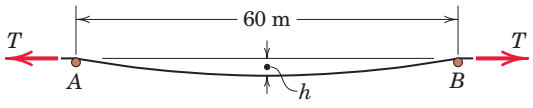
Problem 5/176

***5/177** Numerous small flotation devices are attached to the cable, and the difference between buoyancy and weight results in a net upward force of 30 newtons per meter of cable length. Determine the force T which must be applied to cause the cable configuration shown.



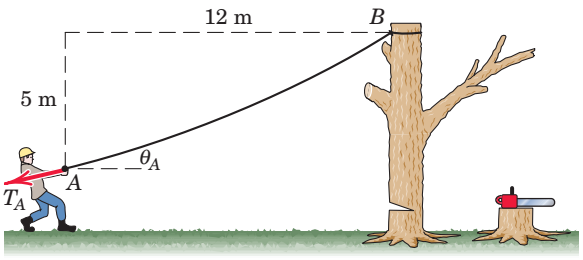
Problem 5/177

***5/178** A cable installation crew wishes to establish the dependence of the tension T on the cable sag h. Plot both the minimum tension T_0 and the tension T at the supports A and B as functions of h for $1 \le h \le 10$ m. The cable mass per unit length is 3 kg/m. State the values of T_0 and T for $h = 2$ m.



Problem 5/178

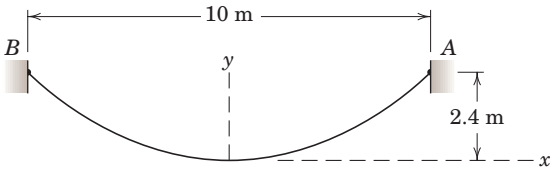
***5/179** The tree surgeon attempts to pull down the partially sawn-through tree trunk. He creates a tension $T_A = 200$ N in the rope, which has a mass of 0.6 kg per meter of its length. Determine the angle θ_A at which he pulls, the length L of rope between points A and B, and the tension T_B at point B.



Problem 5/179

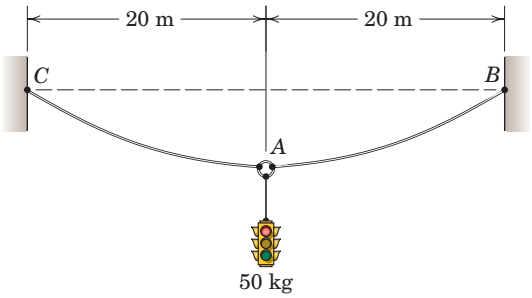
***5/180** Reconsider Prob. 5/179. If the length of the rope between points A and B is 13.02 m, determine the tension T_A which the tree surgeon must exert at A, the angle θ_A at which he pulls, the tension T_B at B, and the angle θ_B which the rope makes with the horizontal at B. The rope has a mass of 0.6 kg per meter of its length.

***5/181** A length of cable which has a mass of 1.2 kg/m is to have a sag of 2.4 m when suspended from the two points A and B on the same horizontal line 10 m apart. For comparison purposes, determine the length L of cable required and plot its configuration for the two cases of (a) assuming a parabolic shape and (b) using the proper catenary model. In order to more clearly distinguish between the two cases, also plot the difference $(y_C - y_P)$ as a function of x, where C and P refer to catenary and parabola, respectively. Use the approximation as stated in Eqs. (5/16) and (5/16a) as needed.



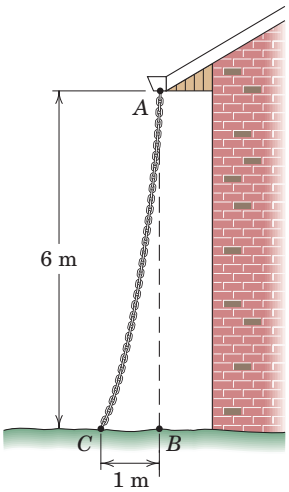
Problem 5/181

***5/182** A 50-kg traffic signal is suspended by two 21-m cables which have a mass of 1.2 kg per meter of length. Determine the vertical deflection δ of the junction ring A relative to its position before the signal is added.



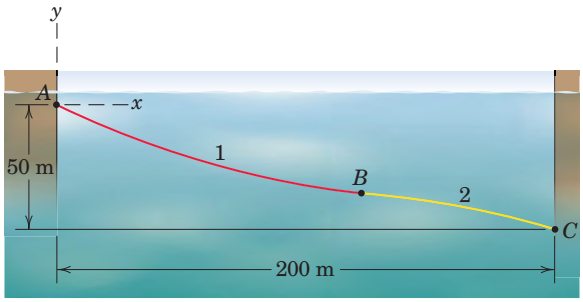
Problem 5/182

***5/183** For aesthetic reasons, chains are sometimes used instead of downspouts on small buildings in order to direct roof runoff water from the gutter down to ground level. The architect of the illustrated building specified a 6-m vertical chain from A to B, but the builder decided to use a 6.1-m chain from A to C as shown in order to place the water farther from the structure. By what percentage n did the builder increase the magnitude of the force exerted on the gutter at A over that figured by the architect? The chain weighs 100 N per meter of its length.



Problem 5/183

▶*5/184 The underwater cable is constructed of two portions. Portion 1 of length 130 m has a net downward force of 0.025 N per meter of its length acting on it. Portion 2 of length 77 m has a net upward force of 0.03 N per meter of its length acting on it. Plot the overall cable configuration. What are the coordinates of the junction point B? (*Note:* The difficulty level of this problem is well above average!)



Problem 5/184

5/9 Fluid Statics

So far in this chapter we have treated the action of forces on and between *solid* bodies. In this article we consider the equilibrium of bodies subjected to forces due to fluid pressures. A *fluid* is any continuous substance which, when at rest, is unable to support shear force. A shear force is one tangent to the surface on which it acts and is developed when differential velocities exist between adjacent layers of fluids. Thus, a fluid at rest can exert only normal forces on a bounding surface. Fluids may be either gaseous or liquid. The statics of fluids is generally called *hydrostatics* when the fluid is a liquid and *aerostatics* when the fluid is a gas.

Fluid Pressure

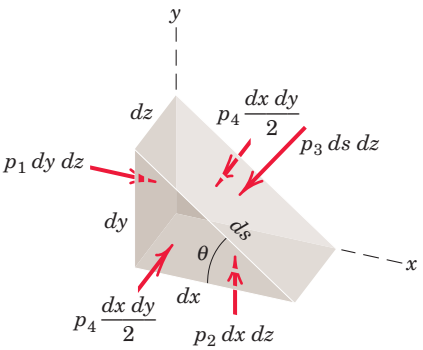


Figure 5/30

The pressure at any given point in a fluid is the same in all directions (Pascal's law). We may prove this by considering the equilibrium of an infinitesimal triangular prism of fluid as shown in Fig. 5/30. The fluid pressures normal to the faces of the element are p_1, p_2, p_3, and p_4 as shown. With force equal to pressure times area, the equilibrium of forces in the x- and y-directions gives

$$p_1\,dy\,dz = p_3\,ds\,dz\,\sin\theta \qquad p_2\,dx\,dz = p_3\,ds\,dz\,\cos\theta$$

Since $ds\,\sin\theta = dy$ and $ds\,\cos\theta = dx$, these questions require that

$$p_1 = p_2 = p_3 = p$$

By rotating the element through 90°, we see that p_4 is also equal to the other pressures. Thus, the pressure at any point in a fluid at rest is the same in all directions. In this analysis we need not account for the weight of the fluid element because, when the weight per unit volume (density ρ times g) is multiplied by the volume of the element, a differential quantity of third order results which disappears in the limit compared with the second-order pressure-force terms.

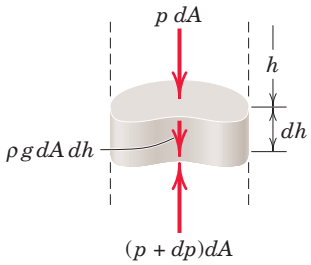


Figure 5/31

In all fluids at rest the pressure is a function of the vertical dimension. To determine this function, we consider the forces acting on a differential element of a vertical column of fluid of cross-sectional area dA, as shown in Fig. 5/31. The positive direction of vertical measurement h is taken downward. The pressure on the upper face is p, and that on the lower face is p plus the change in p, or $p + dp$. The weight of the element equals ρg multiplied by its volume. The normal forces on the lateral surface, which are horizontal and do not affect the balance of forces in the vertical direction, are not shown. Equilibrium of the fluid element in the h-direction requires

$$p\,dA + \rho g\,dA\,dh - (p + dp)\,dA = 0$$

$$dp = \rho g\,dh \qquad \textbf{(5/23)}$$

This differential relation shows us that the pressure in a fluid increases with depth or decreases with increased elevation. Equation 5/23 holds

for both liquids and gases, and agrees with our common observations of air and water pressures.

Fluids which are essentially incompressible are called *liquids*, and for most practical purposes we may consider their density ρ constant for every part of the liquid.* With ρ a constant, integration of Eq. 5/23 gives

$$p = p_0 + \rho g h \tag{5/24}$$

The pressure p_0 is the pressure on the surface of the liquid where $h = 0$. If p_0 is due to atmospheric pressure and the measuring instrument records only the increment above atmospheric pressure,† the measurement gives what is called *gage pressure*. It is computed from $p = \rho g h$.

The common unit for pressure in SI units is the kilopascal (kPa), which is the same as a kilonewton per square meter (10^3 N/m^2). In computing pressure, if we use Mg/m^3 for ρ, m/s^2 for g, and m for h, then the product $\rho g h$ gives us pressure in kPa directly. For example, the pressure at a depth of 10 m in fresh water is

$$p = \rho g h = \left(1.0\,\frac{\text{Mg}}{\text{m}^3}\right)\left(9.81\,\frac{\text{m}}{\text{s}^2}\right)(10\text{ m}) = 98.1\left(10^3\,\frac{\text{kg}\cdot\text{m}}{\text{s}^2}\,\frac{1}{\text{m}^2}\right)$$

$$= 98.1\text{ kN/m}^2 = 98.1\text{ kPa}$$

In the U.S. customary system, fluid pressure is generally expressed in pounds per square inch (lb/in.^{2}) or occasionally in pounds per square foot (lb/ft^2). Thus, at a depth of 10 ft in fresh water the pressure is

$$p = \rho g h = \left(62.4\,\frac{\text{lb}}{\text{ft}^3}\right)\left(\frac{1}{1728}\,\frac{\text{ft}^3}{\text{in.}^3}\right)(120\text{ in.}) = 4.33\text{ lb/in.}^2$$

Hydrostatic Pressure on Submerged Rectangular Surfaces

A body submerged in a liquid, such as a gate valve in a dam or the wall of a tank, is subjected to fluid pressure acting normal to its surface and distributed over its area. In problems where fluid forces are appreciable, we must determine the resultant force due to the distribution of pressure on the surface and the position at which this resultant acts. For systems open to the atmosphere, the atmospheric pressure p_0 acts over all surfaces and thus yields a zero resultant. In such cases, then, we need to consider only the gage pressure $p = \rho g h$, which is the increment above atmospheric pressure.

Consider the special but common case of the action of hydrostatic pressure on the surface of a rectangular plate submerged in a liquid. Figure 5/32*a* shows such a plate 1-2-3-4 with its top edge horizontal and with the plane of the plate making an arbitrary angle θ with the vertical plane. The horizontal surface of the liquid is represented by the x-y' plane. The fluid pressure (gage) acting normal to the plate at point 2 is

*See Table D/1, Appendix D, for table of densities.

†Atmospheric pressure at sea level may be taken to be 101.3 kPa or 14.7 lb/in.^{2}

represented by the arrow 6-2 and equals ρg times the vertical distance from the liquid surface to point 2. This same pressure acts at all points along the edge 2-3. At point 1 on the lower edge, the fluid pressure equals ρg times the depth of point 1, and this pressure is the same at all points along edge 1-4. The variation of pressure p over the area of the plate is governed by the linear depth relationship and therefore it is represented by the arrow p, shown in Fig. 5/32*b*, which varies linearly from the value 6-2 to the value 5-1. The resultant force produced by this pressure distribution is represented by R, which acts at some point P called the *center of pressure*.

The conditions which prevail at the vertical section 1-2-6-5 in Fig. 5/32*a* are identical to those at section 4-3-7-8 and at every other vertical section normal to the plate. Thus, we may analyze the problem from the two-dimensional view of a vertical section as shown in Fig. 5/32*b* for section 1-2-6-5. For this section the pressure distribution is trapezoidal. If b is the horizontal width of the plate measured normal to the plane of the figure (dimension 2-3 in Fig. 5/32*a*), an element of plate area over which

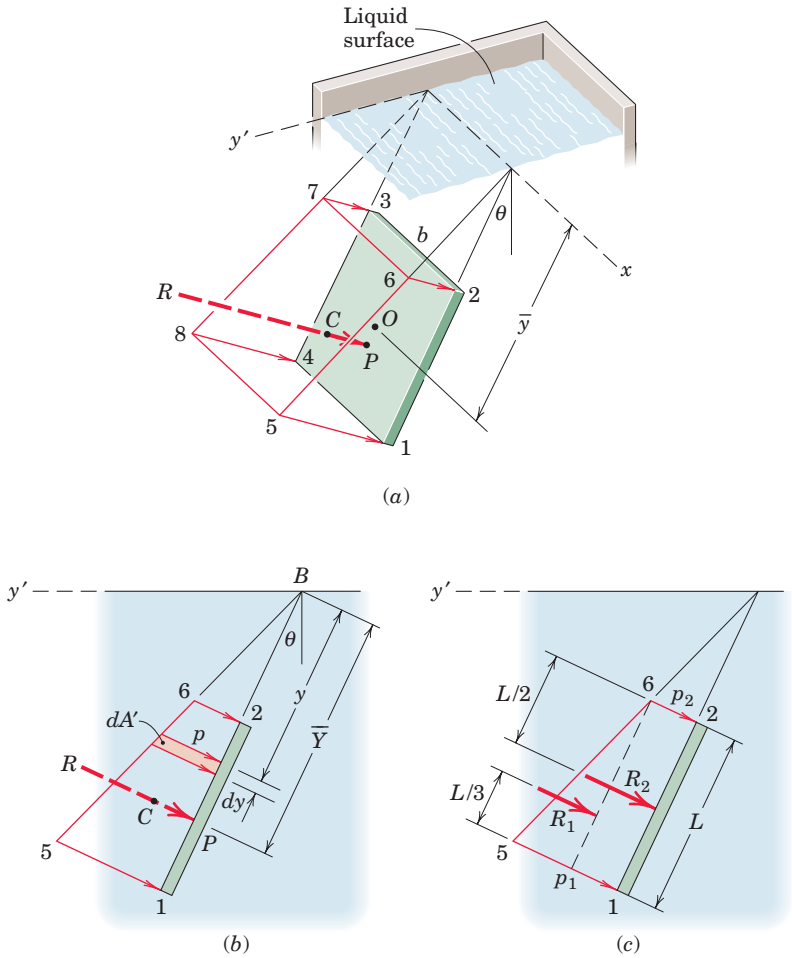


Figure 5/32

the pressure $p = \rho gh$ acts is $dA = b\,dy$, and an increment of the resultant force is $dR = p\,dA = bp\,dy$. But $p\,dy$ is merely the shaded increment of trapezoidal area dA', so that $dR = b\,dA'$. We may therefore express the resultant force acting on the entire plate as the trapezoidal area 1-2-6-5 times the width b of the plate,

$$R = b\int dA' = bA'$$

Be careful not to confuse the physical area A of the plate with the geometrical area A' defined by the trapezoidal distribution of pressure.

The trapezoidal area representing the pressure distribution is easily expressed by using its average altitude. The resultant force R may therefore be written in terms of the average pressure $p_{av} = \frac{1}{2}(p_1 + p_2)$ times the plate area A. The average pressure is also the pressure which exists at the average depth, measured to the centroid O of the plate. An alternative expression for R is therefore

$$R = p_{av}A = \rho g\bar{h}A$$

where $\bar{h} = \bar{y}\cos\theta$.

We obtain the line of action of the resultant force R from the principle of moments. Using the x-axis (point B in Fig. 5/32b) as the moment axis yields $R\bar{Y} = \int y(pb\,dy)$. Substituting $p\,dy = dA'$ and $R = bA'$ and canceling b give

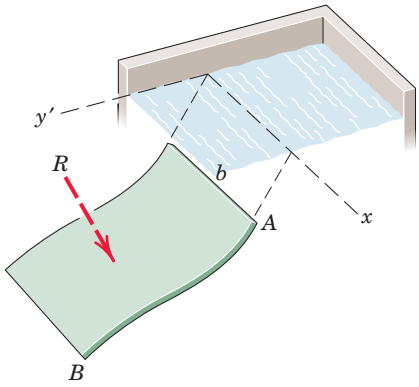
$$\bar{Y} = \frac{\int y\,dA'}{\int dA'}$$

which is simply the expression for the centroidal coordinate of the trapezoidal area A'. In the two-dimensional view, therefore, the resultant R passes through the centroid C of the trapezoidal area defined by the pressure distribution in the vertical section. Clearly $\bar{Y}$ also locates the centroid C of the truncated prism 1-2-3-4-5-6-7-8 in Fig. 5/32a through which the resultant passes.

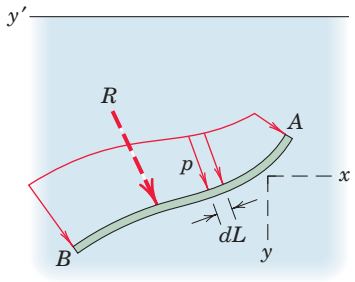
For a trapezoidal distribution of pressure, we may simplify the calculation by dividing the trapezoid into a rectangle and a triangle, Fig. 5/32c, and separately considering the force represented by each part. The force represented by the rectangular portion acts at the center O of the plate and is $R_2 = p_2A$, where A is the area 1-2-3-4 of the plate. The force R_1 represented by the triangular increment of pressure distribution is $\frac{1}{2}(p_1 - p_2)A$ and acts through the centroid of the triangular portion shown.

Hydrostatic Pressure on Cylindrical Surfaces

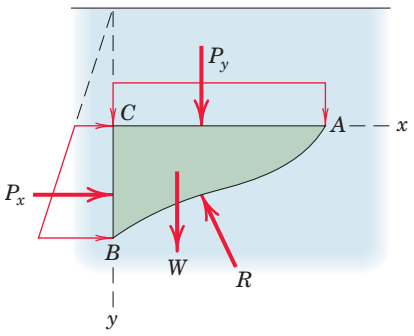
The determination of the resultant R due to distributed pressure on a submerged curved surface involves more calculation than for a flat surface. For example, consider the submerged cylindrical surface shown in Fig. 5/33a where the elements of the curved surface are parallel to the horizontal surface x-y' of the liquid. Vertical sections perpendicular to the surface all disclose the same curve AB and the same pressure distribution.



(a)



(b)



(c)

Figure 5/33

Thus, the two-dimensional representation in Fig. 5/33*b* may be used. To find R by a direct integration, we need to integrate the x- and y-components of dR along the curve AB, since dR continuously changes direction. Thus,

$$R_x = b\int (p\,dL)_x = b\int p\,dy \qquad \text{and} \qquad R_y = b\int (p\,dL)_y = b\int p\,dx$$

A moment equation would now be required if we wished to establish the position of R.

A second method for finding R is usually much simpler. Consider the equilibrium of the block of liquid ABC directly above the surface, shown in Fig. 5/33*c*. The resultant R then appears as the equal and opposite reaction of the surface on the block of liquid. The resultants of the pressures along AC and CB are P_y and P_x, respectively, and are easily obtained. The weight W of the liquid block is calculated from the area ABC of its section multiplied by the constant dimension b and by ρg. The weight W passes through the centroid of area ABC. The equilibrant R is then determined completely from the equilibrium equations which we apply to the free-body diagram of the fluid block.

Hydrostatic Pressure on Flat Surfaces of Any Shape

Figure 5/34*a* shows a flat plate of any shape submerged in a liquid. The horizontal surface of the liquid is the plane x-y', and the plane of the plate makes an angle θ with the vertical. The force acting on a differential strip of area dA parallel to the surface of the liquid is $dR = p\,dA = \rho g h\,dA$. The pressure p has the same magnitude throughout the length of the strip, because there is no change of

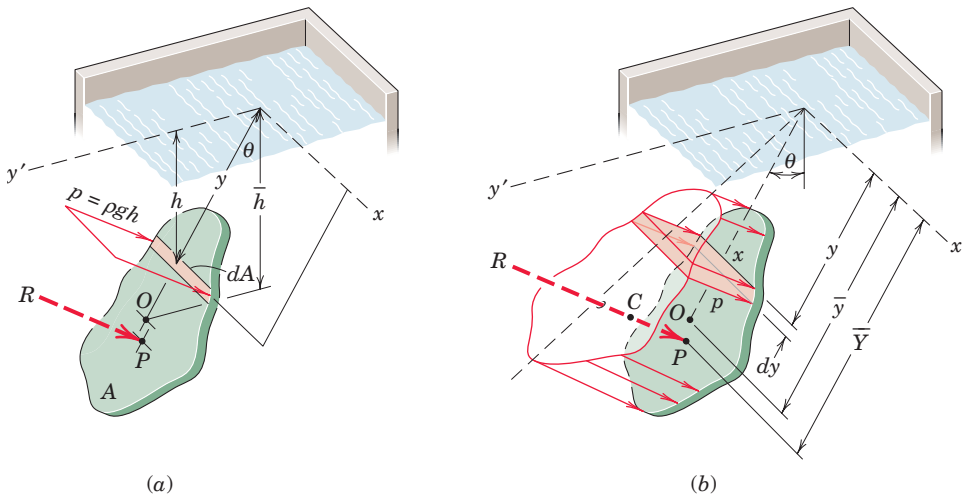


Figure 5/34

depth along the strip. We obtain the total force acting on the exposed area A by integration, which gives

$$R = \int dR = \int p\, dA = \rho g \int h\, dA$$

Substituting the centroidal relation $\bar{h}A = \int h\, dA$ gives us

$$R = \rho g \bar{h} A \qquad \textbf{(5/25)}$$

The quantity $\rho g \bar{h}$ is the pressure which exists at the depth of the centroid O of the area and is the average pressure over the area.

We may also represent the resultant R geometrically by the volume V' of the figure shown in Fig. 5/34*b*. Here the fluid pressure p is represented as a dimension normal to the plate regarded as a base. We see that the resulting volume is a truncated right cylinder. The force dR acting on the differential area $dA = x\, dy$ is represented by the elemental volume $p\, dA$ shown by the shaded slice, and the total force is represented by the total volume of the cylinder. We see from Eq. 5/25 that the average altitude of the truncated cylinder is the average pressure $\rho g \bar{h}$ which exists at a depth corresponding to the centroid O of the area exposed to pressure.

For problems where the centroid O or the volume V' is not readily apparent, a direct integration may be performed to obtain R. Thus,

$$R = \int dR = \int p\, dA = \int \rho g h x\, dy$$

where the depth h and the length x of the horizontal strip of differential area must be expressed in terms of y to carry out the integration.

After the resultant is obtained, we must determine its location. Using the principle of moments with the x-axis of Fig. 5/34*b* as the moment axis, we obtain

$$R\bar{Y} = \int y\, dR \qquad \text{or} \qquad \bar{Y} = \frac{\int y(px\, dy)}{\int px\, dy} \qquad \textbf{(5/26)}$$

This second relation satisfies the definition of the coordinate $\bar{Y}$ to the centroid of the volume V' of the pressure-area truncated cylinder. We conclude, therefore, that the resultant R passes through the centroid C of the volume described by the plate area as base and the linearly varying pressure as the perpendicular coordinate. The point P at which R is applied to the plate is the center of pressure. Note that the center of pressure P and the centroid O of the plate area are *not* the same.

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Glen Canyon Dam at Lake Powell, Arizona

Buoyancy

Archimedes is credited with discovering the *principle of buoyancy*. This principle is easily explained for any fluid, gaseous or liquid, in equilibrium. Consider a portion of the fluid defined by an imaginary closed

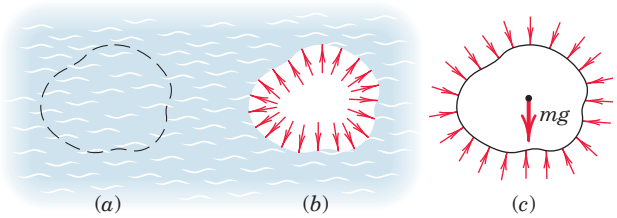

Figure 5/35

surface, as illustrated by the irregular dashed boundary in Fig. 5/35*a*. If the body of the fluid could be sucked out from within the closed cavity and replaced simultaneously by the forces which it exerted on the boundary of the cavity, Fig. 5/35*b*, the equilibrium of the surrounding fluid would not be disturbed. Furthermore, a free-body diagram of the fluid portion before removal, Fig. 5/35*c*, shows that the resultant of the pressure forces distributed over its surface must be equal and opposite to its weight mg and must pass through the center of mass of the fluid element. If we replace the fluid element by a body of the same dimensions, the surface forces acting on the body held in this position will be identical to those acting on the fluid element. Thus, the resultant force exerted on the surface of an object immersed in a fluid is equal and opposite to the weight of fluid displaced and passes through the center of mass of the displaced fluid. This resultant force is called the force of *buoyancy*

$$F = \rho g V \qquad \textbf{(5/27)}$$

where ρ is the density of the fluid, g is the acceleration due to gravity, and V is the volume of the fluid displaced. In the case of a liquid whose density is constant, the center of mass of the displaced liquid coincides with the centroid of the displaced volume.

Thus when the density of an object is less than the density of the fluid in which it is fully immersed, there is an imbalance of force in the vertical direction, and the object rises. When the immersing fluid is a liquid, the object continues to rise until it comes to the surface of the liquid and then comes to rest in an equilibrium position, assuming that the density of the new fluid above the surface is less than the density of the object. In the case of the surface boundary between a liquid and a gas, such as water and air, the effect of the gas pressure on that portion of the floating object above the liquid is balanced by the added pressure in the liquid due to the action of the gas on its surface.

An important problem involving buoyancy is the determination of the stability of a floating object, such as a ship hull shown in cross section in an upright position in Fig. 5/36*a*. Point B is the centroid of the displaced volume and is called the *center of buoyancy*. The resultant of the forces exerted on the hull by the water pressure is the buoyancy force F which passes through B and is equal and opposite to the weight W of the ship. If the ship is caused to list through an angle α, Fig. 5/36*b*,

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The designers of high-performance sailboats must consider both air-pressure distributions on the sails and water-pressure distributions on the hull.

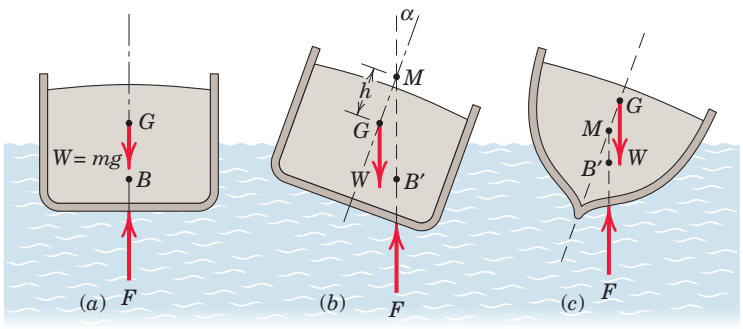

Figure 5/36

the shape of the displaced volume changes, and the center of buoyancy shifts to B'.

The point of intersection of the vertical line through B' with the centerline of the ship is called the *metacenter* M, and the distance h of M from the center of mass G is called the *metacentric height*. For most hull shapes h remains practically constant for angles of list up to about 20°. When M is above G, as in Fig. 5/36*b*, there is a righting moment which tends to bring the ship back to its upright position. If M is below G, as for the hull of Fig. 5/36*c*, the moment accompanying the list is in the direction to increase the list. This is clearly a condition of instability and must be avoided in the design of any ship.

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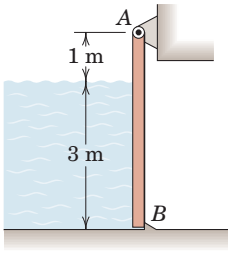
Wind-tunnel testing of a scale model can be useful in predicting the performance of the full-size aircraft.

SAMPLE PROBLEM 5/19

A rectangular plate, shown in vertical section AB, is 4 m high and 6 m wide (normal to the plane of the paper) and blocks the end of a fresh-water channel 3 m deep. The plate is hinged about a horizontal axis along its upper edge through A and is restrained from opening by the fixed ridge B which bears horizontally against the lower edge of the plate. Find the force B exerted on the plate by the ridge.



Solution. The free-body diagram of the plate is shown in section and includes the vertical and horizontal components of the force at A, the unspecified weight $W = mg$ of the plate, the unknown horizontal force B, and the resultant R of the triangular distribution of pressure against the vertical face.

The density of fresh water is $\rho = 1.000$ Mg/m^3 so that the average pressure is

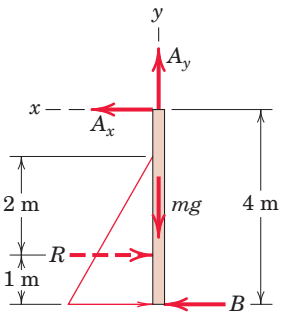
(1) $[p_{av} = \rho g\bar{h}]$ $\qquad p_{av} = 1.000(9.81)(\frac{3}{2}) = 14.72$ kPa

The resultant R of the pressure forces against the plate becomes

$[R = p_{av}A]$ $\qquad R = (14.72)(3)(6) = 265$ kN

This force acts through the centroid of the triangular distribution of pressure, which is 1 m above the bottom of the plate. A zero moment summation about A establishes the unknown force B. Thus,

$[\Sigma M_A = 0]$ $\qquad 3(265) - 4B = 0$ $\qquad B = 198.7$ kN $\qquad$ *Ans.*



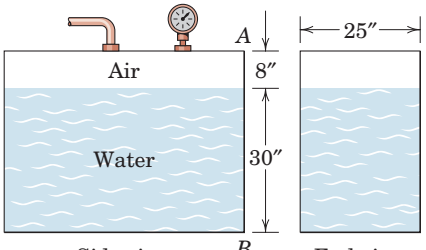
Helpful Hint

(1) Note that the units of pressure ρgh are

$$\left(10^3 \frac{\text{kg}}{\text{m}^3}\right)\left(\frac{\text{m}}{\text{s}^2}\right)(\text{m}) = \left(10^3 \frac{\text{kg}\cdot\text{m}}{\text{s}^2}\right)\left(\frac{1}{\text{m}^2}\right) = \text{kN/m}^2 = \text{kPa}.$$

SAMPLE PROBLEM 5/20

The air space in the closed fresh-water tank is maintained at a pressure of 0.80 lb/in.^{2} (above atmospheric). Determine the resultant force R exerted by the air and water on the end of the tank.



Solution. The pressure distribution on the end surface is shown, where $p_0 = 0.80$ lb/in.^{2} The specific weight of fresh water is $\mu = \rho g = 62.4/1728 = 0.0361$ lb/in.^{3} so that the increment of pressure Δp due to the water is

$$\Delta p = \mu\, \Delta h = 0.0361(30) = 1.083 \text{ lb/in.}^2$$

(1) The resultant forces R_1 and R_2 due to the rectangular and triangular distributions of pressure, respectively, are

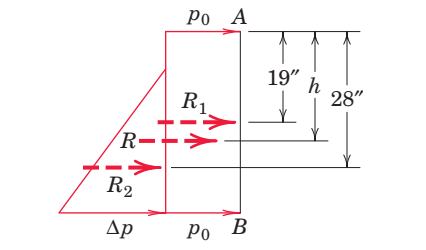
$$R_1 = p_0 A_1 = 0.80(38)(25) = 760 \text{ lb}$$

$$R_2 = \Delta p_{av} A_2 = \frac{1.083}{2}(30)(25) = 406 \text{ lb}$$

The resultant is then $R = R_1 + R_2 = 760 + 406 = 1166$ lb. $\qquad$ *Ans.*

We locate R by applying the moment principle about A noting that R_1 acts through the center of the 38-in. depth and that R_2 acts through the centroid of the triangular pressure distribution 20 in. below the surface of the water and $20 + 8 = 28$ in. below A. Thus,

$[Rh = \Sigma M_A]$ $\qquad 1166h = 760(19) + 406(28)$ $\qquad h = 22.1$ in. $\qquad$ *Ans.*

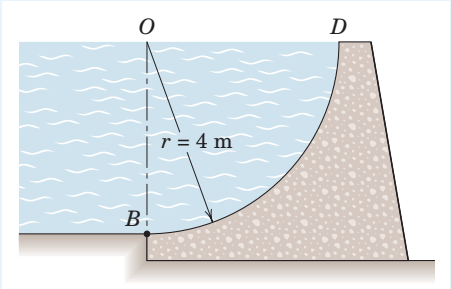


Helpful Hint

(1) Dividing the pressure distribution into these two parts is decidedly the simplest way in which to make the calculation.

SAMPLE PROBLEM 5/21

Determine completely the resultant force R exerted on the cylindrical dam surface by the water. The density of fresh water is 1.000 Mg/m^3, and the dam has a length b, normal to the paper, of 30 m.



Solution. The circular block of water BDO is isolated and its free-body diagram is drawn. The force P_x is

(1) $$P_x = \rho g \bar{h} A = \frac{\rho g r}{2} br = \frac{(1.000)(9.81)(4)}{2}(30)(4) = 2350 \text{ kN}$$

The weight W of the water passes through the mass center G of the quarter-circular section and is

$$mg = \rho g V = (1.000)(9.81)\frac{\pi(4)^2}{4}(30) = 3700 \text{ kN}$$

Equilibrium of the section of water requires

$$[\Sigma F_x = 0] \qquad R_x = P_x = 2350 \text{ kN}$$

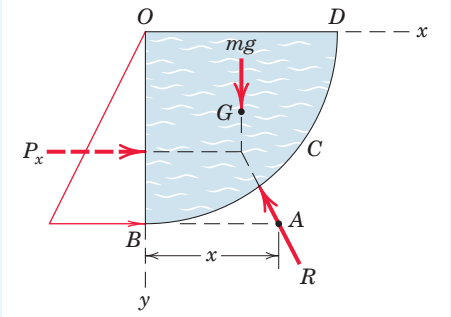
$$[\Sigma F_y = 0] \qquad R_y = mg = 3700 \text{ kN}$$

The resultant force R exerted by the fluid on the dam is equal and opposite to that shown acting on the fluid and is

$$[R = \sqrt{R_x^2 + R_y^2}] \qquad R = \sqrt{(2350)^2 + (3700)^2} = 4380 \text{ kN} \qquad \textit{Ans.}$$

The x-coordinate of the point A through which R passes may be found from the principle of moments. Using B as a moment center gives

$$P_x \frac{r}{3} + mg\frac{4r}{3\pi} - R_y x = 0, \qquad x = \frac{2350\left(\frac{4}{3}\right) + 3700\left(\frac{16}{3\pi}\right)}{3700} = 2.55 \text{ m} \qquad \textit{Ans.}$$



(2) ***Alternative Solution.*** The force acting on the dam surface may be obtained by a direct integration of the components

$$dR_x = p\,dA\cos\theta \qquad \text{and} \qquad dR_y = p\,dA\sin\theta$$

where $p = \rho g h = \rho g r \sin\theta$ and $dA = b(r\,d\theta)$. Thus,

$$R_x = \int_0^{\pi/2} \rho g r^2 b \sin\theta\cos\theta\,d\theta = -\rho g r^2 b\left[\frac{\cos 2\theta}{4}\right]_0^{\pi/2} = \tfrac{1}{2}\rho g r^2 b$$

$$R_y = \int_0^{\pi/2} \rho g r^2 b \sin^2\theta\,d\theta = \rho g r^2 b\left[\frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = \tfrac{1}{4}\pi\rho g r^2 b$$

Thus, $R = \sqrt{R_x^2 + R_y^2} = \frac{1}{2}\rho g r^2 b\sqrt{1 + \pi^2/4}$. Substituting the numerical values gives

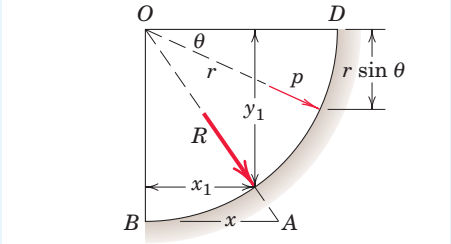
$$R = \tfrac{1}{2}(1.000)(9.81)(4^2)(30)\sqrt{1 + \pi^2/4} = 4380 \text{ kN} \qquad \textit{Ans.}$$

Since dR always passes through point O, we see that R also passes through O and, therefore, the moments of R_x and R_y about O must cancel. So we write $R_x y_1 = R_y x_1$, which gives us

$$x_1/y_1 = R_x/R_y = (\tfrac{1}{2}\rho g r^2 b)/(\tfrac{1}{4}\pi\rho g r^2 b) = 2/\pi$$

By similar triangles we see that

$$x/r = x_1/y_1 = 2/\pi \qquad \text{and} \qquad x = 2r/\pi = 2(4)/\pi = 2.55 \text{ m} \qquad \textit{Ans.}$$



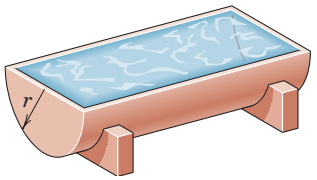
Helpful Hints

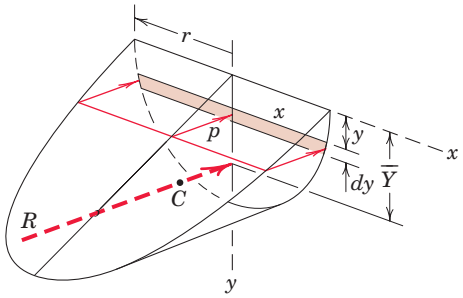
(1) See note (1) in Sample Problem 5/19 if there is any question about the units for $\rho g\bar{h}$.

(2) This approach by integration is feasible here mainly because of the simple geometry of the circular arc.

SAMPLE PROBLEM 5/22

Determine the resultant force R exerted on the semicircular end of the water tank shown in the figure if the tank is filled to capacity. Express the result in terms of the radius r and the water density ρ.





Solution I. We will obtain R first by a direct integration. With a horizontal strip of area $dA = 2x\,dy$ acted on by the pressure $p = \rho g y$, the increment of the resultant force is $dR = p\,dA$ so that

$$R = \int p\,dA = \int \rho g y(2x\,dy) = 2\rho g \int_0^r y\sqrt{r^2 - y^2}\,dy$$

Integrating gives $R = \frac{2}{3}\rho g r^3$ *Ans.*

The location of R is determined by using the principle of moments. Taking moments about the x-axis gives

$$[R\bar{Y} = \int y\,dR] \qquad \frac{2}{3}\rho g r^3 \bar{Y} = 2\rho g \int_0^r y^2\sqrt{r^2 - y^2}\,dy$$

Integrating gives $\frac{2}{3}\rho g r^3 \bar{Y} = \frac{\rho g r^4}{4}\frac{\pi}{2}$ and $\bar{Y} = \frac{3\pi r}{16}$ *Ans.*

Solution II. We may use Eq. 5/25 directly to find R, where the average pressure is $\rho g\bar{h}$ and $\bar{h}$ is the coordinate to the centroid of the area over which the pressure acts. For a semicircular area, $\bar{h} = 4r/(3\pi)$.

$$[R = \rho g \bar{h} A] \qquad R = \rho g \frac{4r}{3\pi}\frac{\pi r^2}{2} = \frac{2}{3}\rho g r^3 \quad \textit{Ans.}$$

which is the volume of the pressure-area figure.

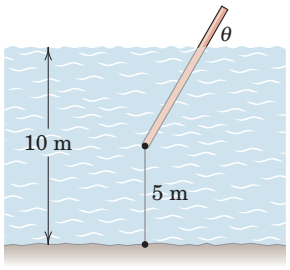
The resultant R acts through the centroid C of the volume defined by the pressure-area figure. Calculation of the centroidal distance $\bar{Y}$ involves the same integral obtained in *Solution I*.

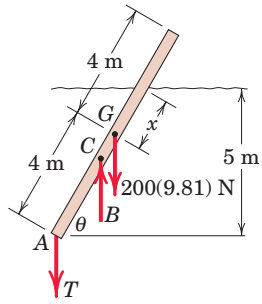
Helpful Hint

1. Be very careful not to make the mistake of assuming that R passes through the centroid of the area over which the pressure acts.

SAMPLE PROBLEM 5/23

A buoy in the form of a uniform 8-m pole 0.2 m in diameter has a mass of 200 kg and is secured at its lower end to the bottom of a fresh-water lake with 5 m of cable. If the depth of the water is 10 m, calculate the angle θ made by the pole with the horizontal.





Solution. The free-body diagram of the buoy shows its weight acting through G, the vertical tension T in the anchor cable, and the buoyancy force B which passes through centroid C of the submerged portion of the buoy. Let x be the distance from G to the waterline. The density of fresh water is $\rho = 10^3$ kg/m^3, so that the buoyancy force is

$$[B = \rho g V] \qquad B = 10^3(9.81)\pi(0.1)^2(4 + x)\ \text{N}$$

Moment equilibrium, $\Sigma M_A = 0$, about A gives

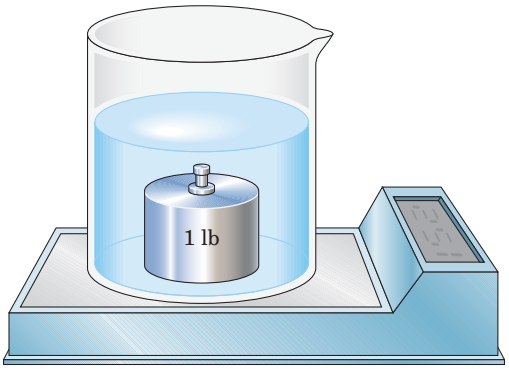
$$200(9.81)(4\cos\theta) - [10^3(9.81)\pi(0.1)^2(4 + x)]\frac{4 + x}{2}\cos\theta = 0$$

Thus, $x = 3.14$ m and $\theta = \sin^{-1}\left(\frac{5}{4 + 3.14}\right) = 44.5°$ *Ans.*

PROBLEMS

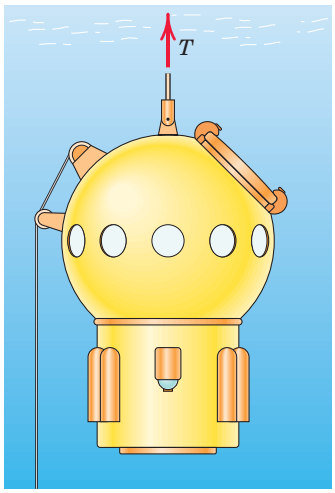
Introductory Problems

5/185 A beaker of fresh water is in place on the scale when the 1-lb stainless-steel cylinder is added to the beaker. What is the normal force which the cylinder exerts on the bottom of the beaker? By how much does the scale reading increase as the cylinder is added? Explain your answer.



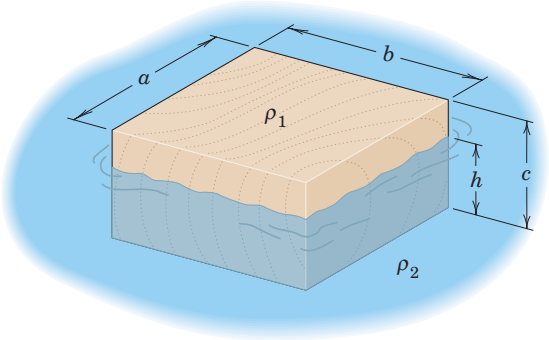
Problem 5/185

5/186 The submersible diving chamber has a total mass of 6.7 Mg including personnel, equipment, and ballast. When the chamber is lowered to a depth of 1.2 km in the ocean, the cable tension is 8 kN. Compute the total volume V displaced by the chamber.



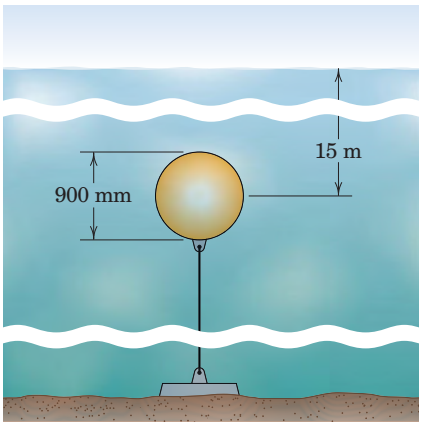
Problem 5/186

5/187 A rectangular block of density ρ_1 floats in a liquid of density ρ_2. Determine the ratio $r = h/c$, where h is the submerged depth of block. Evaluate r for an oak block floating in fresh water and for steel floating in mercury.



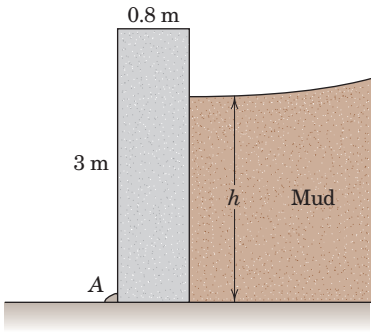
Problem 5/187

5/188 A 150-kg spherical steel buoy 900 mm in diameter is to be anchored in salt water 15 m below the surface. Find the mass m of scrap iron to be sealed inside the buoy in order that the tension in the anchor cable where it attaches to the buoy will not exceed 1.5 kN. For negligible change in water density, does the buoyancy depend on the depth of submergence?



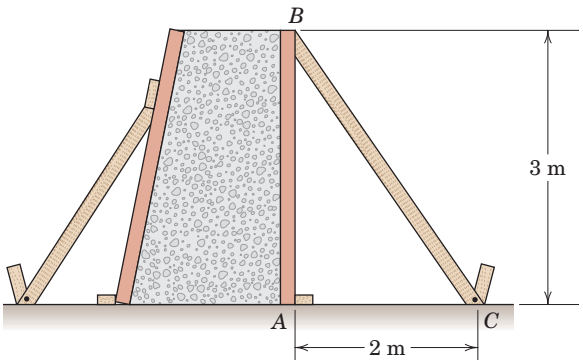
Problem 5/188

5/189 Calculate the depth h of mud for which the 3-m concrete retaining wall is on the verge of tipping about its forward edge A. The density of mud may be taken to be 1760 kg/m^3 and that of concrete to be 2400 kg/m^3.



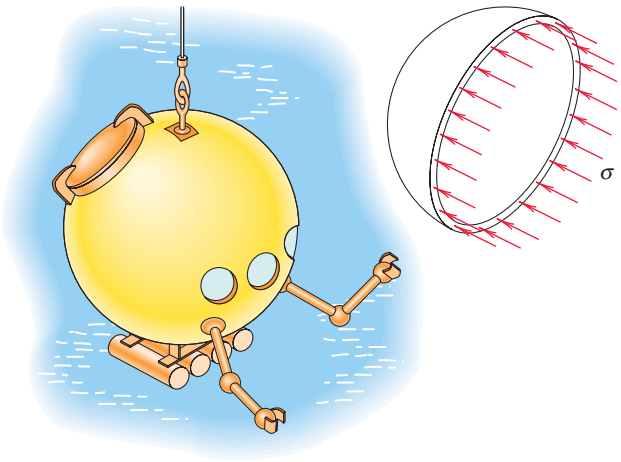
Problem 5/189

5/190 The forms for a small concrete retaining wall are shown in section. There is a brace BC for every 1.5 m of wall length. Assuming that the joints at A, B, and C act as hinged connections, compute the compression in each brace BC. Wet concrete may be treated as a liquid with a density of 2400 kg/m^3.



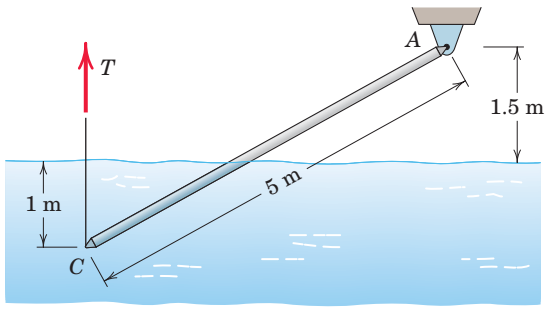
Problem 5/190

5/191 A deep-submersible diving chamber designed in the form of a spherical shell 1500 mm in diameter is ballasted with lead so that its weight slightly exceeds its buoyancy. Atmospheric pressure is maintained within the sphere during an ocean dive to a depth of 3 km. The thickness of the shell is 25 mm. For this depth calculate the compressive stress σ which acts on a diametral section of the shell, as indicated in the right-hand view.



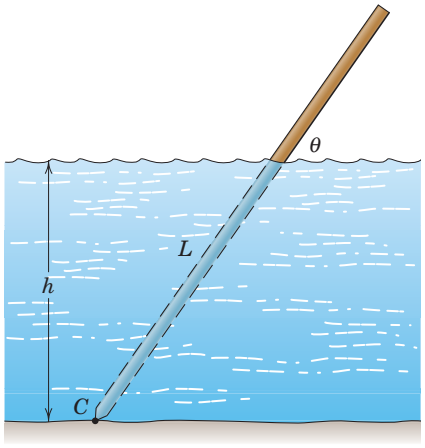
Problem 5/191

5/192 The uniform 62-kg pole of 150-mm diameter is hinged at A, and its lower end is immersed in fresh water. Determine the tension T in the vertical cable required to maintain C at a depth of 1 m.



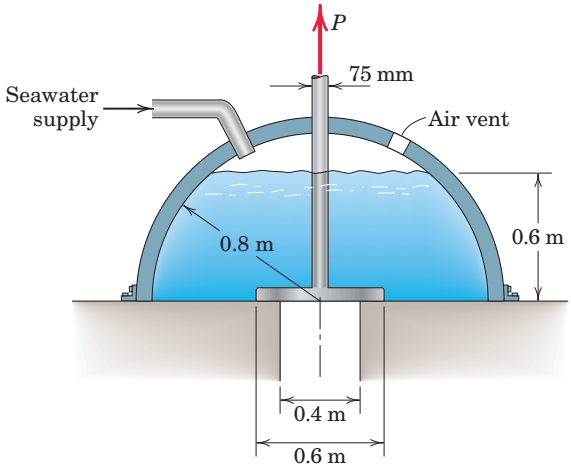
Problem 5/192

5/193 One end of a uniform pole of length L and density ρ' is secured at C to the bottom of a tank of liquid of density ρ and depth h. For the conditions $\rho' < \rho$ and $h < L$, find the angle θ assumed by the pole.



Problem 5/193

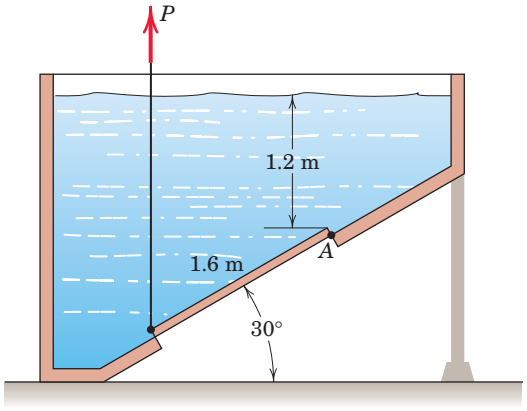
5/194 When the seawater level inside the hemispherical chamber reaches the 0.6-m level shown in the figure, the plunger is lifted, allowing a surge of sea water to enter the vertical pipe. For this fluid level (a) determine the average pressure σ supported by the seal area of the valve before force is applied to lift the plunger and (b) determine the force P (in addition to the force needed to support its weight) required to lift the plunger. Assume atmospheric pressure in all airspaces and in the seal area when contact ceases under the action of P.



Problem 5/194

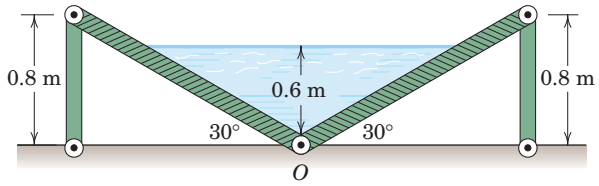
Representative Problems

5/195 The cross section of a fresh-water tank with a slanted bottom is shown. A rectangular door 1.6 m by 0.8 m (normal to the plane of the figure) in the bottom of the tank is hinged at A and is opened against the pressure of the water by the cable under a tension P as shown. Calculate P.



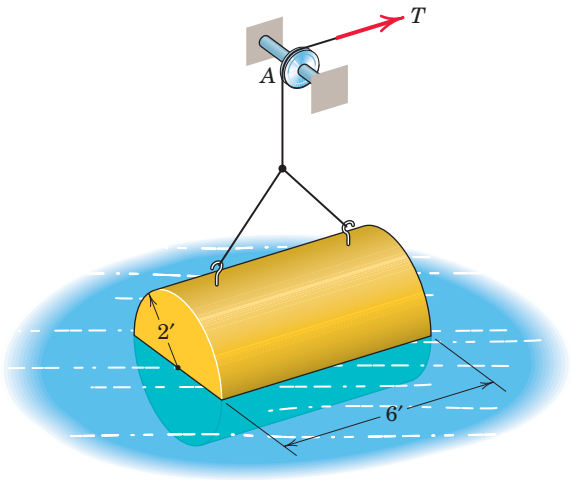
Problem 5/195

5/196 The sides of a V-shaped fresh-water trough, shown in section, are hinged along their common intersection through O and are supported on each side by vertical struts hinged at each end and spaced every 2 m. Calculate the compression C in each strut. Neglect the weights of the members.



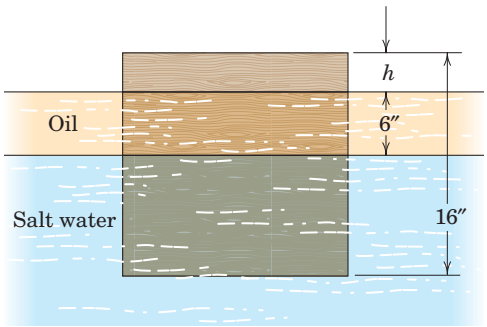
Problem 5/196

5/197 The solid concrete cylinder 6 ft long and 4 ft in diameter is supported in a half-submerged position in fresh water by a cable which passes over a fixed pulley at A. Compute the tension T in the cable. The cylinder is waterproofed by a plastic coating. (Consult Table D/1, Appendix D, as needed.)



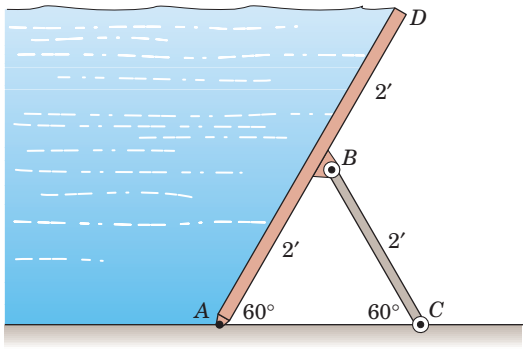
Problem 5/197

5/198 A block of wood in the form of a waterproofed 16-in. cube is floating in a tank of salt water with a 6-in. layer of oil floating on the water. Assume that the cube floats in the attitude shown, and calculate the height h of the block above the surface of the oil. The specific weights of oil, salt water, and wood are 56, 64, and 50 lb/ft^3, respectively.



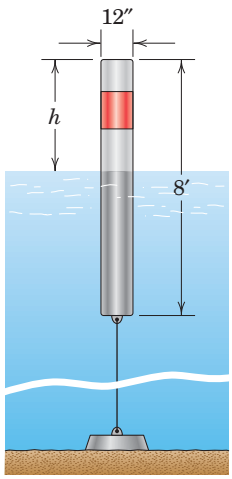
Problem 5/198

5/199 A fresh-water channel 10 ft wide (normal to the plane of the paper) is blocked at its end by a rectangular barrier, shown in section ABD. Supporting struts BC are spaced every 2 ft along the 10-ft width. Determine the compression C in each strut. Neglect the weights of the members.



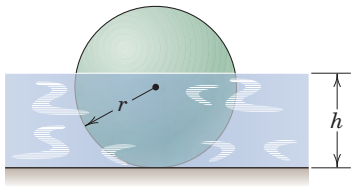
Problem 5/199

5/200 A channel-marker buoy consists of an 8-ft hollow steel cylinder 12 in. in diameter weighing 180 lb and anchored to the bottom with a cable as shown. If $h = 2$ ft at high tide, calculate the tension T in the cable. Also find the value of h when the cable goes slack as the tide drops. The specific weight of sea water is 64 lb/ft^3. Assume the buoy is weighted at its base so that it remains vertical.



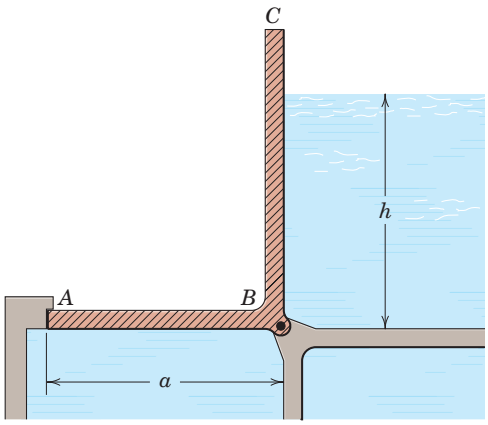
Problem 5/200

5/201 A homogeneous solid sphere of radius r is resting on the bottom of a tank containing a liquid of density ρ_l, which is greater than the density ρ_s of the sphere. As the tank is filled, a depth h is reached at which the sphere begins to float. Determine the expression for the density ρ_s of the sphere.



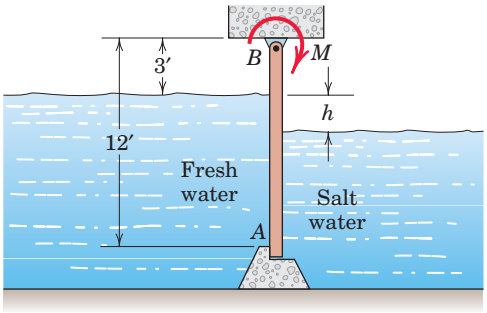
Problem 5/201

5/202 The hinged gate ABC closes an opening of width b (perpendicular to the paper) in a water channel. The water has free access to the underside as well as the right side of the gate. When the water level rises above a certain value of h, the gate will open. Determine the critical value of h. Neglect the mass of the gate.



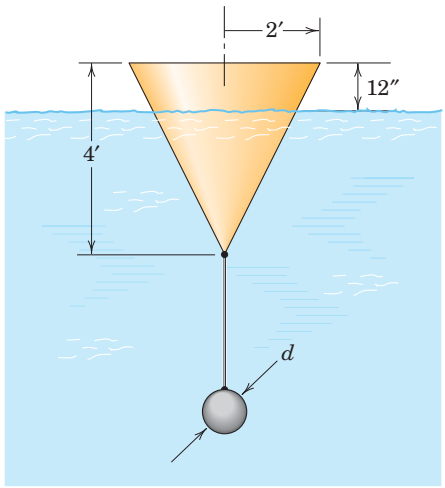
Problem 5/202

5/203 The rectangular gate shown in section is 10 ft long (perpendicular to the paper) and is hinged about its upper edge B. The gate divides a channel leading to a fresh-water lake on the left and a salt-water tidal basin on the right. Calculate the torque M on the shaft of the gate at B required to prevent the gate from opening when the salt-water level drops to $h = 3$ ft.



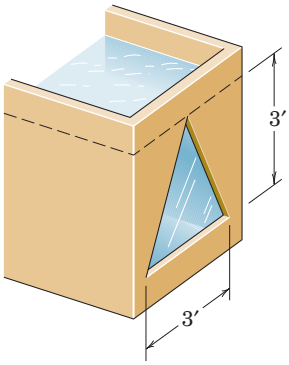
Problem 5/203

5/204 The conical surface and circular base of the salt-water buoy are fabricated from thin steel plate which weighs 10.2 pounds per square foot. Calculate the diameter d of the spherical lead ball suspended from the vertex of the buoy so that the base will be 12 in. above the surface of the water.



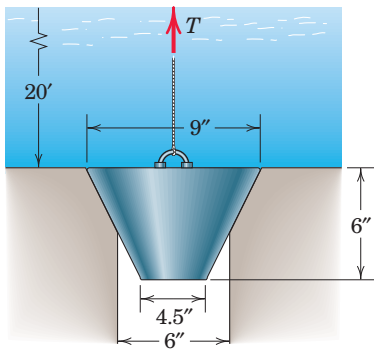
Problem 5/204

5/205 Determine the total force R exerted on the triangular window by the fresh water in the tank. The water level is even with the top of the window. Also determine the distance H from R to the water level.



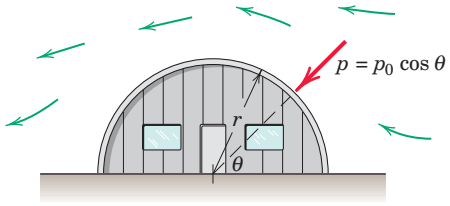
Problem 5/205

5/206 The cast-iron plug seals the drainpipe of an open fresh-water tank which is filled to a depth of 20 ft. Determine the tension T required to remove the plug from its tapered hole. Atmospheric pressure exists in the drainpipe and in the seal area as the plug is being removed. Neglect mechanical friction between the plug and its supporting surface.



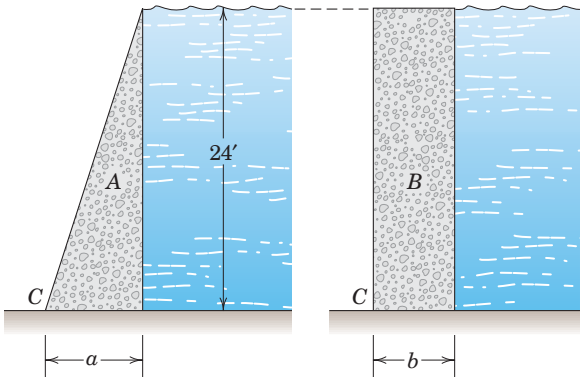
Problem 5/206

5/207 The Quonset hut is subjected to a horizontal wind, and the pressure p against the circular roof is approximated by $p_0 \cos \theta$. The pressure is positive on the windward side of the hut and is negative on the leeward side. Determine the total horizontal shear Q on the foundation per unit length of roof measured normal to the paper.



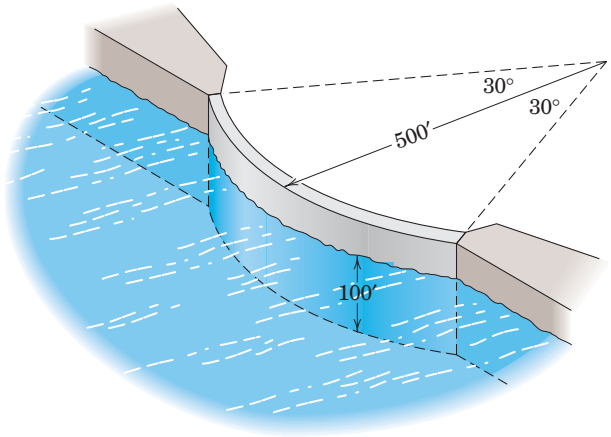
Problem 5/207

5/208 The triangular and rectangular sections are being considered for the design of a small fresh-water concrete dam. From the standpoint of resistance to overturning about C, which section will require less concrete, and how much less per foot of dam length? Concrete weighs 150 lb/ft^3.



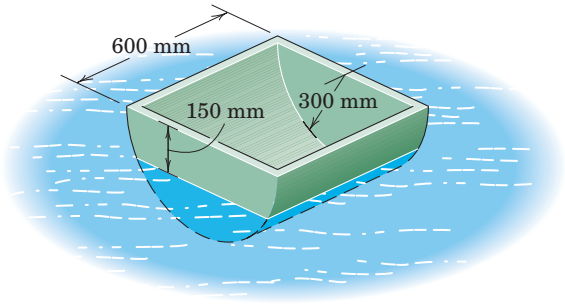
Problem 5/208

5/209 The upstream side of an arched dam has the form of a vertical cylindrical surface of 500-ft radius and subtends an angle of 60°. If the fresh water is 100 ft deep, determine the total force R exerted by the water on the dam face.



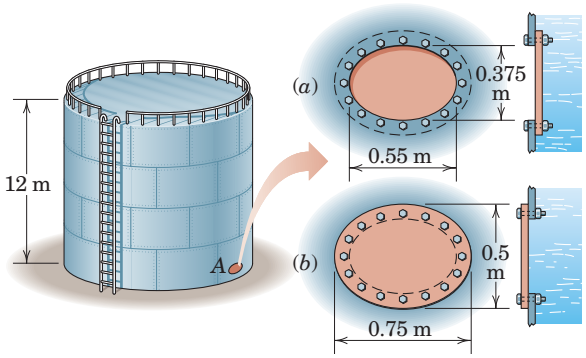
Problem 5/209

5/210 The semicylindrical steel shell with closed ends has a mass of 26.6 kg. Determine the mass m of the lead ballast which must be placed in the shell so that it floats in fresh water at its half-radius depth of 150 mm.



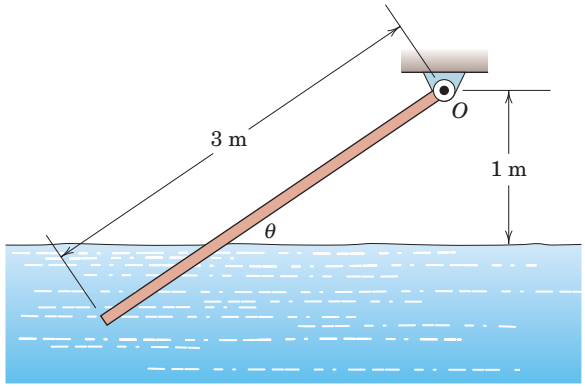
Problem 5/210

5/211 The small access hole A allows maintenance workers to enter the storage tank at ground level when it is empty. Two designs, (a) and (b), are shown for the hole cover. If the tank is full of fresh water, estimate the average pressure σ in the seal area of design (a) and the average increase ΔT in the initial tension in each of the 16 bolts of design (b). You may take the pressure over the hole area to be constant, and the pressure in the seal area of design (b) may be assumed to be atmospheric.



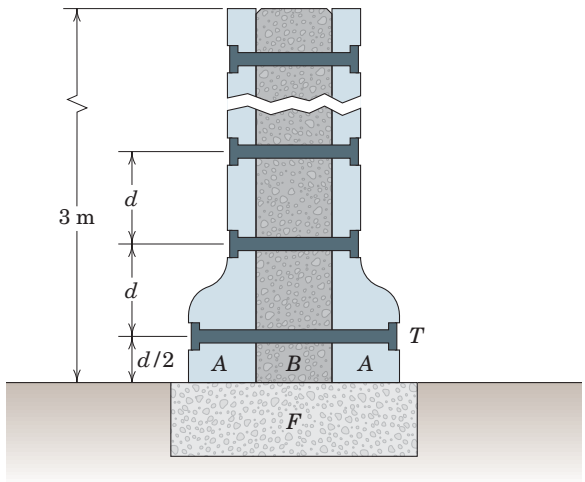
Problem 5/211

5/212 The 3-m plank shown in section has a density of 800 kg/m^3 and is hinged about a horizontal axis through its upper edge O. Calculate the angle θ assumed by the plank with the horizontal for the level of fresh water shown.



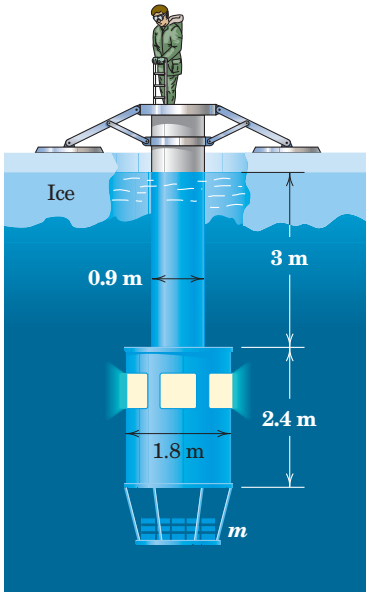
Problem 5/212

5/213 The elements of a new method for constructing concrete foundation walls for new houses are shown in the figure. Once the footing F is in place, polystyrene forms A are erected and a thin concrete mixture B is poured between the forms. Ties T prevent the forms from separating. After the concrete cures, the forms are left in place for insulation. As a design exercise, make a conservative estimate for the uniform tie spacing d if the tension in each tie is not to exceed 6.5 kN. The horizontal tie spacing is the same as the vertical spacing. State any assumptions. The density of wet concrete is 2400 kg/m^3.



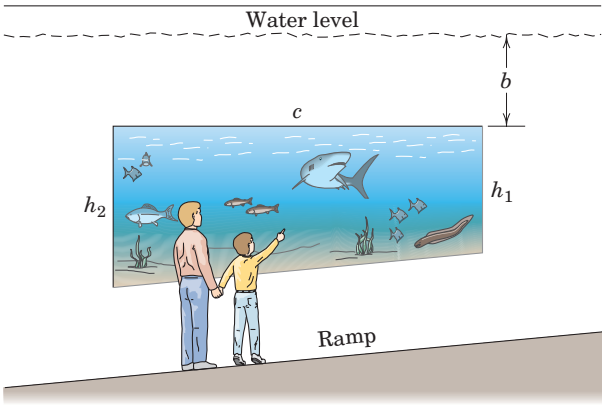
Problem 5/213

5/214 A structure designed for observation of sea life beneath the ice in polar waters consists of the cylindrical viewing chamber connected to the surface by the cylindrical shaft open at the top for ingress and egress. Ballast is carried in the rack below the chamber. To ensure a stable condition for the structure, it is necessary that its legs bear on the ice with a force that is at least 15 percent of the total buoyancy force of the submerged structure. If the structure less ballast has a mass of 5.7 Mg, calculate the required mass m of lead ballast. The density of lead is 11.37 Mg/m^3.



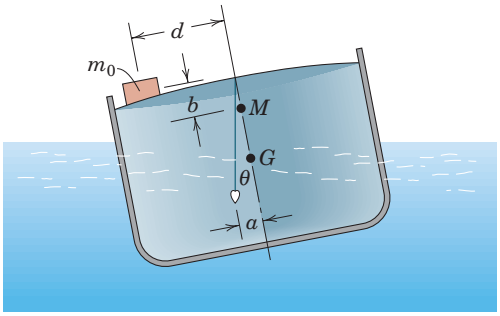
Problem 5/214

5/215 The trapezoidal viewing window in a sea-life aquarium has the dimensions shown. With the aid of appropriate diagrams and coordinates, describe two methods by which the resultant force R on the glass due to water pressure, and the vertical location of R, could be found if numerical values were supplied.



Problem 5/215

▶ **5/216** The accurate determination of the vertical position of the center of mass G of a ship is difficult to achieve by calculation. It is more easily obtained by a simple inclining experiment on the loaded ship. With reference to the figure, a known external mass m_0 is placed a distance d from the centerline, and the angle of list θ is measured by means of the deflection of a plumb bob. The displacement of the ship and the location of the metacenter M are known. Calculate the metacentric height $\overline{GM}$ for a 12 000-t ship inclined by a 27-t mass placed 7.8 m from the centerline if a 6-m plumb line is deflected a distance $a = 0.2$ m. The mass m_0 is at a distance $b = 1.8$ m above M. [Note that the metric ton (t) equals 1000 kg and is the same as the megagram (Mg).]



Problem 5/216

5/10 Chapter Review

In Chapter 5 we have studied various common examples of forces distributed throughout volumes, over areas, and along lines. In all these problems we often need to determine the resultant of the distributed forces and the location of the resultant.

Finding Resultants of Distributed Forces

To find the resultant and line of action of a distributed force:

1. Begin by multiplying the intensity of the force by the appropriate element of volume, area, or length in terms of which the intensity is expressed. Then sum (integrate) the incremental forces over the region involved to obtain their resultant.
2. To locate the line of action of the resultant, use the principle of moments. Evaluate the sum of the moments, about a convenient axis, of all of the increments of force. Equate this sum to the moment of the resultant about the same axis. Then solve for the unknown moment arm of the resultant.

Gravitational Forces

When force is distributed throughout a mass, as in the case of gravitational attraction, the intensity is the force of attraction ρg per unit of volume, where ρ is the density and g is the gravitational acceleration. For bodies whose density is constant, we saw in Section A that ρg cancels when the moment principle is applied. This leaves us with a strictly geometric problem of finding the centroid of the figure, which coincides with the mass center of the physical body whose boundary defines the figure.

1. For flat plates and shells which are homogeneous and have constant thickness, the problem becomes one of finding the properties of an area.
2. For slender rods and wires of uniform density and constant cross section, the problem becomes one of finding the properties of a line segment.

Integration of Differential Relationships

For problems which require the integration of differential relationships, keep in mind the following considerations.

1. Select a coordinate system which provides the simplest description of the boundaries of the region of integration.
2. Eliminate higher-order differential quantities whenever lower-order differential quantities will remain.
3. Choose a first-order differential element in preference to a second-order element and a second-order element in preference to a third-order element.

4. Whenever possible, choose a differential element which avoids discontinuities within the region of integration.

Distributed Forces in Beams, Cables, and Fluids

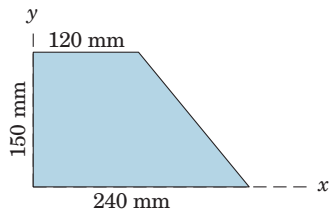
In Section B we used these guidelines along with the principles of equilibrium to solve for the effects of distributed forces in beams, cables, and fluids. Remember that:

1. For beams and cables the force intensity is expressed as force per unit length.
2. For fluids the force intensity is expressed as force per unit area, or pressure.

Although beams, cables, and fluids are physically quite different applications, their problem formulations share the common elements cited above.

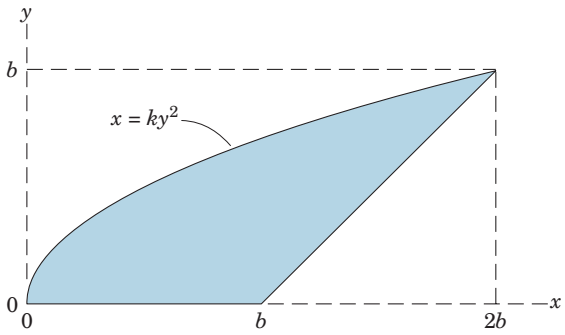
REVIEW PROBLEMS

5/217 Determine the coordinates of the centroid of the trapezoidal area shown.



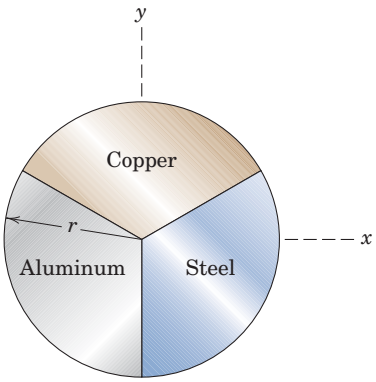
Problem 5/217

5/218 Locate the centroid of the shaded area.



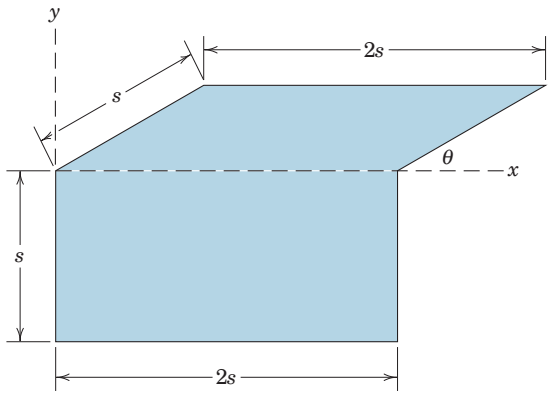
Problem 5/218

5/219 The disk of uniform thickness is composed of equal sectors of the materials shown. Determine the location of the mass center of the disk.



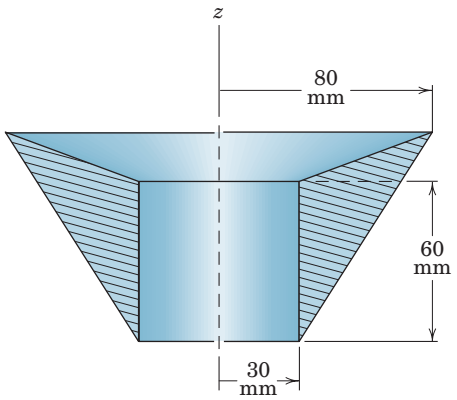
Problem 5/219

5/220 Determine the x- and y-coordinates of the centroid of the shaded area. Evaluate your expressions for $\theta = 0$ and $\theta = 90°$.



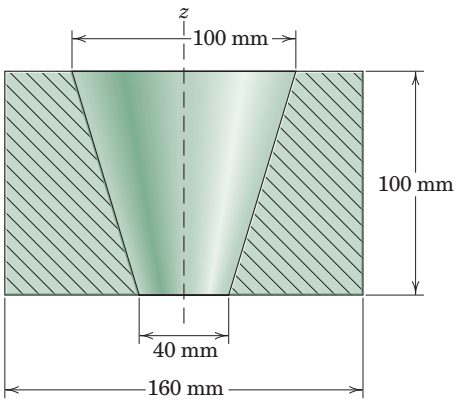
Problem 5/220

5/221 The cross section of an aluminum casting in the form of a solid of revolution about the z-axis is shown. Calculate the mass m of the casting.



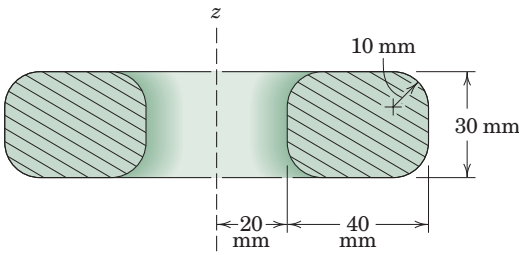
Problem 5/221

5/222 The cross section shown is for a complete cast-iron body of revolution about the z-axis. Compute its mass m.



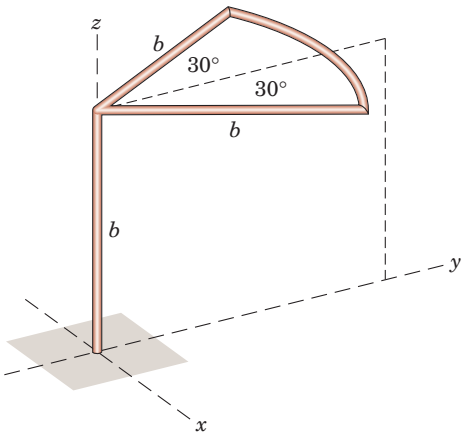
Problem 5/222

5/223 Determine the volume V and total surface area A of the complete ring which is shown in section. All four corner radii of the cross section are 10 mm.



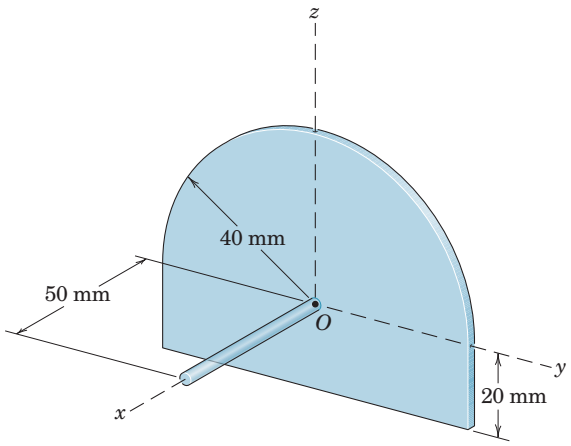
Problem 5/223

5/224 The assembly consists of four rods cut from the same bar stock. The curved member forms a circular arc of radius b. Determine the y- and z-coordinates of the mass center of the assembly.



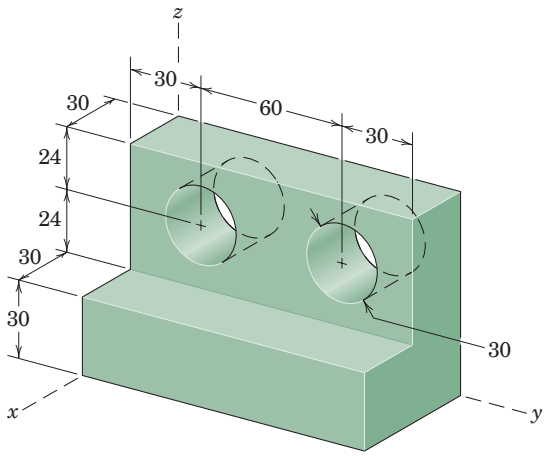
Problem 5/224

5/225 Determine the coordinates of the mass center of the body constructed of steel plate and a steel rod. The plate thickness is 2 mm. The rod diameter is 4 mm and its length is 50 mm. The origin O is at the midthickness of the plate, and the rod is welded to the face of the plate. What is the mass of the body?



Problem 5/225

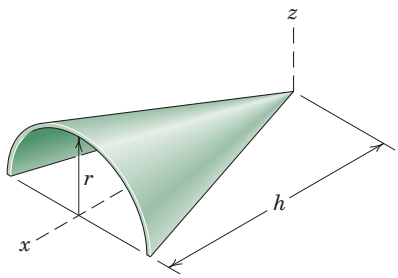
5/226 Determine the mass and the centroid location of the cast-aluminum body.



Dimensions in millimeters

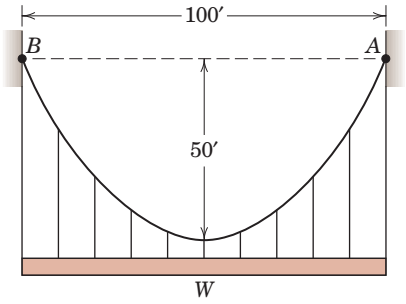
Problem 5/226

5/227 Determine the position of the center of mass of the homogeneous thin conical shell shown.



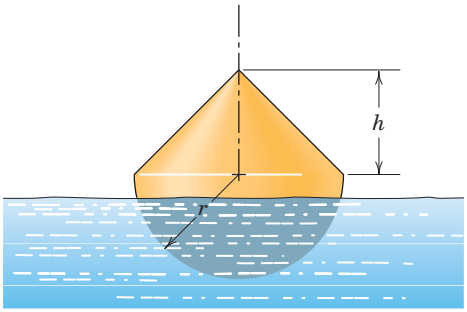
Problem 5/227

5/228 A cable is suspended from points A and B on the same horizontal line and supports a total load W uniformly distributed along the horizontal. Determine the length S of the cable. (Recall that the convergence of the series of Eq. 5/16a requires that the sag-to-span ratio be less than 1/4.)



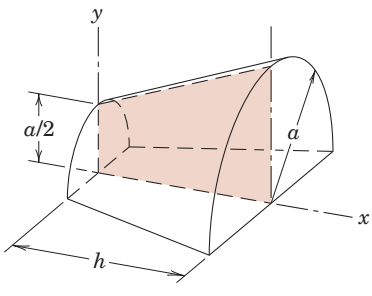
Problem 5/228

5/229 A solid floating object is composed of a hemisphere and a cone of equal radius r made from the same homogeneous material. If the object floats with the center of the hemisphere above the water surface, find the maximum altitude h which the cone may have before the object will no longer float in the upright position illustrated.



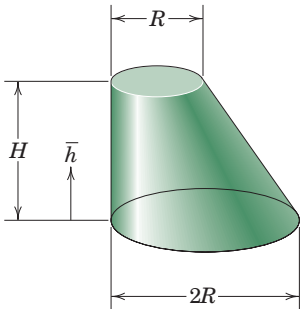
Problem 5/229

5/230 Determine the y-coordinate of the centroid of the volume obtained by revolving the shaded area about the x-axis through 180°.



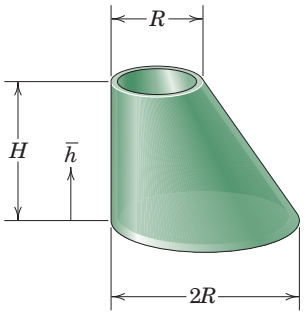
Problem 5/230

5/231 The tapered body has a horizontal cross section which is circular. Determine the height $\bar{h}$ of its mass center above the base of the homogeneous body.



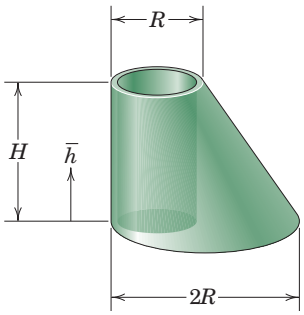
Problem 5/231

5/232 Shown in the figure is a body similar to that of Prob. 5/231, only now the body is a thin shell with open top and bottom. Determine the height $\bar{h}$ of its mass center above the base of the homogeneous body and compare your result with the printed answer for Prob. 5/231.



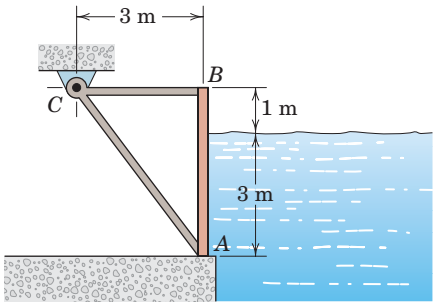
Problem 5/232

5/233 The homogeneous body shown is constructed by beginning with the solid body of Prob. 5/231 and then removing a cylindrical volume of diameter just less than R as shown, leaving a thin wall at the left portion of the body as well as at the very top. Determine the height $\bar{h}$ of the mass center above the base of the body.



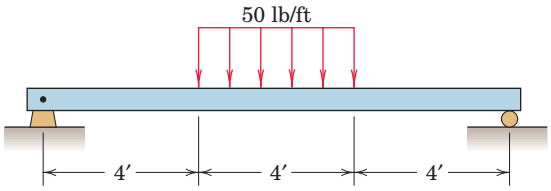
Problem 5/233

5/234 The figure shows the cross section of a rectangular gate 4 m high and 6 m long (perpendicular to the paper) which blocks a fresh-water channel. The gate has a mass of 8.5 Mg and is hinged about a horizontal axis through C. Compute the vertical force P exerted by the foundation on the lower edge A of the gate. Neglect the mass of the frame to which the gate is attached.



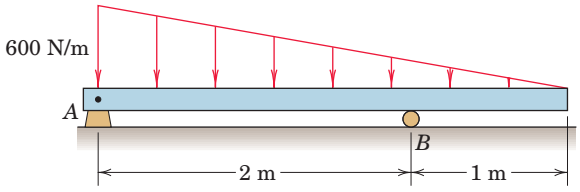
Problem 5/234

5/235 Draw the shear and moment diagrams for the beam, which supports the uniform load of 50 lb per foot of beam length distributed over its midsection. Determine the maximum bending moment and its location.



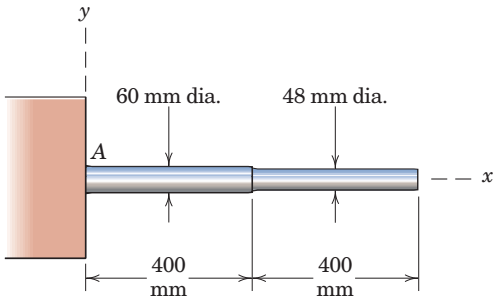
Problem 5/235

5/236 Determine the maximum bending moment M_{max} for the loaded beam and specify the distance x to the right of end A where M_{max} exists.



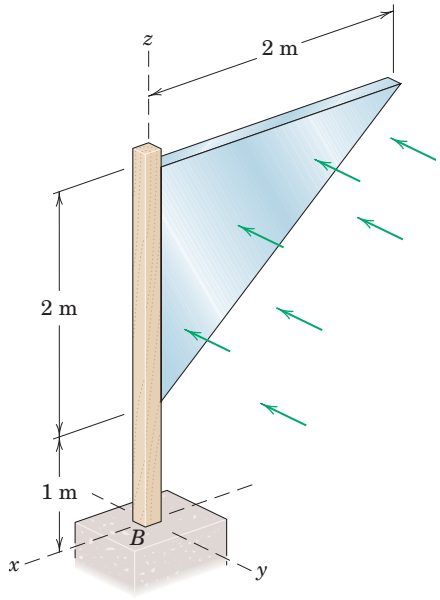
Problem 5/236

5/237 The horizontal steel shaft with two diameters is welded to the fixed block at A. Construct the diagrams of shear and moment in the shaft due to its weight. Determine the values of the shear force and bending moment at the discontinuity in diameter.



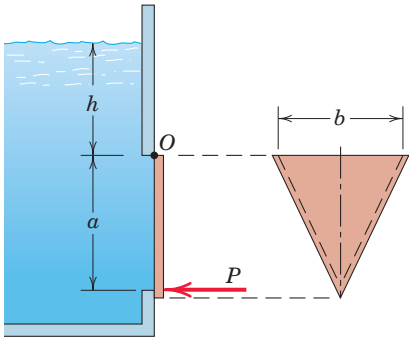
Problem 5/237

5/238 The triangular sign is attached to the post embedded in the concrete base at B. Calculate the shear force V, the bending moment M, and the torsional moment T in the post at B during a storm where the wind velocity normal to the sign reaches 100 km/h. The air pressure (called stagnation pressure) against the vertical surface corresponding to this velocity is 1.4 kPa.



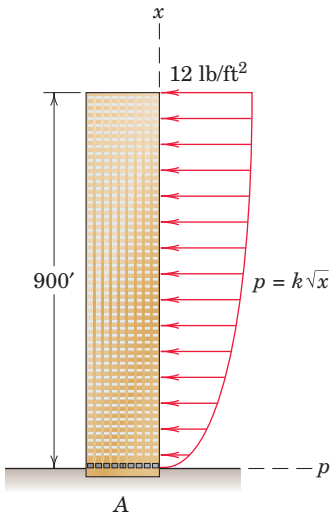
Problem 5/238

5/239 A flat plate seals a triangular opening in the vertical wall of a tank of liquid of density ρ. The plate is hinged about the upper edge O of the triangle. Determine the force P required to hold the gate in a closed position against the pressure of the liquid.



Problem 5/239

5/240 As part of a preliminary design study, the effects of wind loads on a 900-ft building are investigated. For the parabolic distribution of wind pressure shown in the figure, compute the force and moment reactions at the base A of the building due to the wind load. The depth of the building (perpendicular to the paper) is 200 ft.

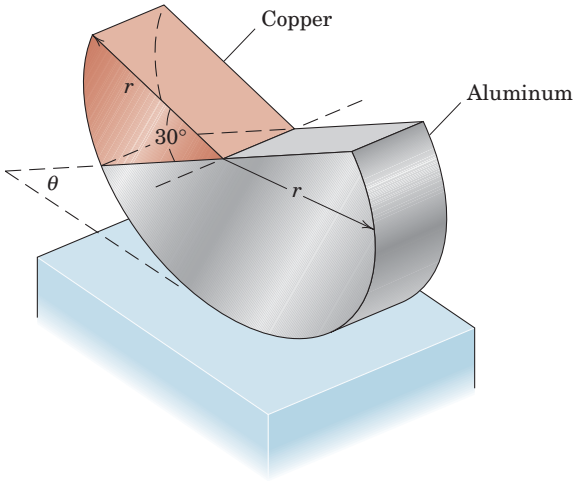


Problem 5/240

▶**5/241** Regard the tall building of Prob. 5/240 as a uniform upright beam. Determine and plot the shear force and bending moment in the structure as functions of the height x above the ground. Evaluate your expressions at $x = 450$ ft.

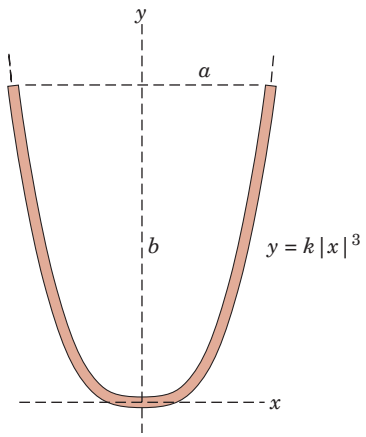
*Computer-Oriented Problems

***5/242** The 30° cylindrical sector is made of copper and is attached as shown to the semicylinder made of aluminum. Determine the angle θ for the equilibrium position of the cylinder resting on a horizontal surface.



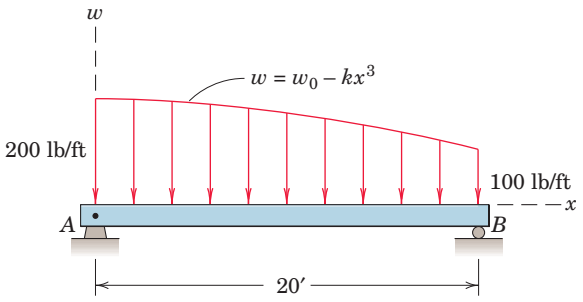
Problem 5/242

***5/243** Set up the integral expression for the y-coordinate of the centroid of the uniform slender rod bent into the shape shown. Then, for the values $a = 2$ and $b = 8$, numerically evaluate your integral expression.



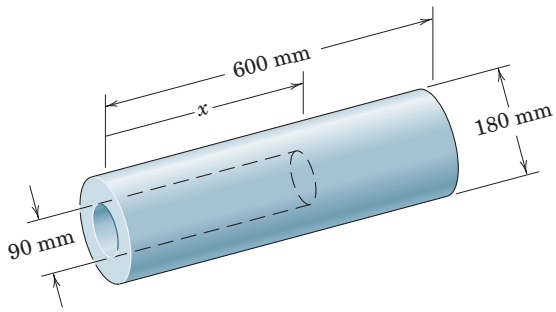
Problem 5/243

*** 5/244** Construct the shear and moment diagrams for the loaded beam of Prob. 5/120, repeated here. Determine the maximum values of the shear and moment and their locations on the beam.



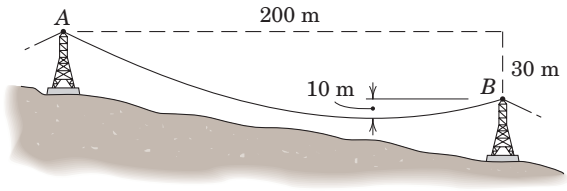
Problem 5/244

*** 5/245** A homogeneous charge of solid propellant for a rocket is in the shape of the circular cylinder formed with a concentric hole of depth x. For the dimensions shown, plot $\bar{X}$, the x-coordinate of the mass center of the propellant, as a function of the depth x of the hole from $x = 0$ to $x = 600$ mm. Determine the maximum value of $\bar{X}$ and show that it is equal to the corresponding value of x.



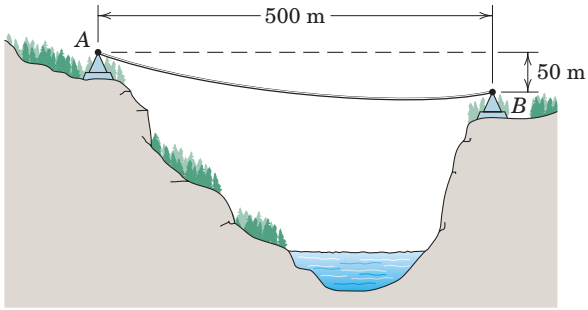
Problem 5/245

*** 5/246** A length of power cable is suspended from the two towers as shown. The cable has a mass of 20 kg per meter of cable. If the maximum allowable cable tension is 75 kN, determine the mass ρ of ice per meter which can form on the cable without the maximum allowable tension being exceeded. If additional stretch in the cable is neglected, does the addition of the ice change the cable configuration?



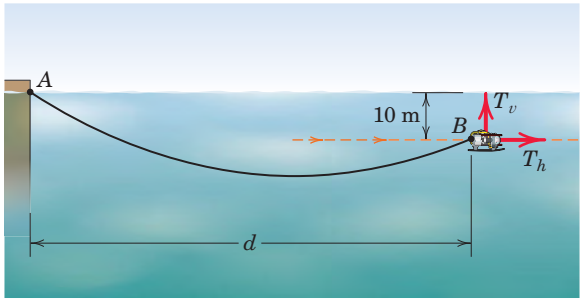
Problem 5/246

*** 5/247** As a preliminary step in the construction of a tramway across a scenic river gorge, a 505-m cable with a mass of 12 kg/m is strung between points A and B. Determine the horizontal distance x to the right of point A to the lowest point on the cable and compute the tensions at points A and B.



Problem 5/247

*** 5/248** The small remotely-controlled robotic vehicle of Probs. 5/171 and 5/172 is shown again here. The 200-m tether has a slight negative buoyancy in that a net downward force of 0.025 N per meter of its length acts on it. Using its onboard variable horizontal and vertical thrusters, the vehicle maintains the constant 10-m depth as it moves slowly to the right. If the maximum horizontal thrust is 10 N and the maximum vertical thrust is 7 N, determine the maximum permissible value of the distance d and state which thruster is the limiting one.



Problem 5/248

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Large changes in tension can occur when a line is wrapped around a fixed cylinder.

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Friction

6

CHAPTER OUTLINE

6/1 Introduction

Section A Frictional Phenomena

6/2 Types of Friction

6/3 Dry Friction

Section B Applications of Friction in Machines

6/4 Wedges

6/5 Screws

6/6 Journal Bearings

6/7 Thrust Bearings; Disk Friction

6/8 Flexible Belts

6/9 Rolling Resistance

6/10 Chapter Review

6/1 Introduction

In the preceding chapters we have usually assumed that the forces of action and reaction between contacting surfaces act normal to the surfaces. This assumption characterizes the interaction between smooth surfaces and was illustrated in Example 2 of Fig. 3/1. Although this ideal assumption often involves only a relatively small error, there are many problems in which we must consider the ability of contacting surfaces to support tangential as well as normal forces. Tangential forces generated between contacting surfaces are called *friction forces* and occur to some degree in the interaction between all real surfaces. Whenever a tendency exists for one contacting surface to slide along another surface, the friction forces developed are always in a direction to oppose this tendency.

In some types of machines and processes we want to minimize the retarding effect of friction forces. Examples are bearings of all types, power screws, gears, the flow of fluids in pipes, and the propulsion of aircraft and missiles through the atmosphere. In other situations we wish to maximize the effects of friction, as in brakes, clutches, belt drives, and wedges. Wheeled vehicles depend on friction for both starting and stopping, and ordinary walking depends on friction between the shoe and the ground.

Friction forces are present throughout nature and exist in all machines no matter how accurately constructed or carefully lubricated. A machine or process in which friction is small enough to be neglected is said to be *ideal*. When friction must be taken into account, the machine or process is termed *real*. In all cases where there is sliding motion between parts, the friction forces result in a loss of energy which is dissipated in the form of heat. Wear is another effect of friction.

SECTION A FRICTIONAL PHENOMENA

6/2 Types of Friction

In this article we briefly discuss the types of frictional resistance encountered in mechanics. The next article contains a more detailed account of the most common type of friction, dry friction.

(a) Dry Friction. Dry friction occurs when the unlubricated surfaces of two solids are in contact under a condition of sliding or a tendency to slide. A friction force tangent to the surfaces of contact occurs both during the interval leading up to impending slippage and while slippage takes place. The direction of this friction force always opposes the motion or impending motion. This type of friction is also called *Coulomb friction*. The principles of dry or Coulomb friction were developed largely from the experiments of Coulomb in 1781 and from the work of Morin from 1831 to 1834. Although we do not yet have a comprehensive theory of dry friction, in Art. 6/3 we describe an analytical model sufficient to handle the vast majority of problems involving dry friction. This model forms the basis for most of this chapter.

(b) Fluid Friction. Fluid friction occurs when adjacent layers in a fluid (liquid or gas) are moving at different velocities. This motion causes frictional forces between fluid elements, and these forces depend on the relative velocity between layers. When there is no relative velocity, there is no fluid friction. Fluid friction depends not only on the velocity gradients within the fluid but also on the viscosity of the fluid, which is a measure of its resistance to shearing action between fluid layers. Fluid friction is treated in the study of fluid mechanics and will not be discussed further in this book.

(c) Internal Friction. Internal friction occurs in all solid materials which are subjected to cyclical loading. For highly elastic materials the recovery from deformation occurs with very little loss of energy due to internal friction. For materials which have low limits of elasticity and which undergo appreciable plastic deformation during loading, a considerable amount of internal friction may accompany this deformation. The mechanism of internal friction is associated with the action of shear deformation, which is discussed in references on materials science. Because this book deals primarily with the external effects of forces, we will not discuss internal friction further.

6/3 Dry Friction

The remainder of this chapter describes the effects of dry friction acting on the exterior surfaces of rigid bodies. We will now explain the mechanism of dry friction with the aid of a very simple experiment.

Mechanism of Dry Friction

Consider a solid block of mass m resting on a horizontal surface, as shown in Fig. 6/1*a*. We assume that the contacting surfaces have some roughness. The experiment involves the application of a horizontal force P which continuously increases from zero to a value sufficient to move the block and give it an appreciable velocity. The free-body diagram of the block for any value of P is shown in Fig. 6/1*b*, where the tangential friction force exerted by the plane on the block is labeled F. This friction force acting on the body will *always* be in a direction to oppose motion or the tendency toward motion of the body. There is also a normal force N which in this case equals mg, and the total force R exerted by the supporting surface on the block is the resultant of N and F.

A magnified view of the irregularities of the mating surfaces, Fig. 6/1*c*, helps us to visualize the mechanical action of friction. Support is necessarily intermittent and exists at the mating humps. The direction of each of the reactions on the block, R_1, R_2, R_3, etc. depends not only on the geometric profile of the irregularities but also on the extent of local deformation at each contact point. The total normal force N is the sum

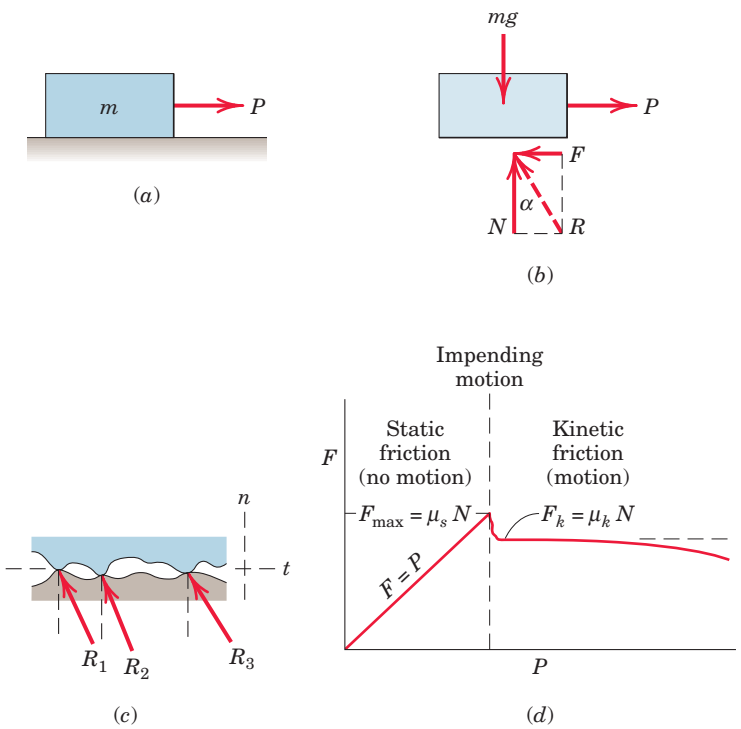

Figure 6/1

of the n-components of the R's, and the total frictional force F is the sum of the t-components of the R's. When the surfaces are in relative motion, the contacts are more nearly along the tops of the humps, and the t-components of the R's are smaller than when the surfaces are at rest relative to one another. This observation helps to explain the well-known fact that the force P necessary to maintain motion is generally less than that required to start the block when the irregularities are more nearly in mesh.

If we perform the experiment and record the friction force F as a function of P, we obtain the relation shown in Fig. 6/1d. When P is zero, equilibrium requires that there be no friction force. As P is increased, the friction force must be equal and opposite to P as long as the block does not slip. During this period the block is in equilibrium, and all forces acting on the block must satisfy the equilibrium equations. Finally, we reach a value of P which causes the block to slip and to move in the direction of the applied force. At this same time the friction force decreases slightly and abruptly. It then remains essentially constant for a time but then decreases still more as the velocity increases.

Static Friction

The region in Fig. 6/1d up to the point of slippage or impending motion is called the range of *static friction*, and in this range the value of the friction force is determined by the *equations of equilibrium*. This friction force may have any value from zero up to and including the maximum value. For a given pair of mating surfaces the experiment shows that this maximum value of static friction $F_{\max}$ is proportional to the normal force N. Thus, we may write

$$F_{\max} = \mu_s N \tag{6/1}$$

where μ_s is the proportionality constant, called the *coefficient of static friction*.

Be aware that Eq. 6/1 describes only the *limiting* or *maximum* value of the static friction force and *not* any lesser value. Thus, the equation applies only to cases where motion is impending with the friction force at its peak value. For a condition of static equilibrium when motion is *not* impending, the static friction force is

$$F < \mu_s N$$

Kinetic Friction

After slippage occurs, a condition of *kinetic friction* accompanies the ensuing motion. Kinetic friction force is usually somewhat less than the maximum static friction force. The kinetic friction force F_k is also proportional to the normal force. Thus,

$$F_k = \mu_k N \tag{6/2}$$

where μ_k is the *coefficient of kinetic friction*. It follows that μ_k is generally less than μ_s. As the velocity of the block increases, the kinetic friction decreases somewhat, and at high velocities, this decrease may be significant. Coefficients of friction depend greatly on the exact condition of the surfaces, as well as on the relative velocity, and are subject to considerable uncertainty.

Because of the variability of the conditions governing the action of friction, in engineering practice it is frequently difficult to distinguish between a static and a kinetic coefficient, especially in the region of transition between impending motion and motion. Well-greased screw threads under mild loads, for example, often exhibit comparable frictional resistance whether they are on the verge of turning or whether they are in motion.

In the engineering literature we frequently find expressions for maximum static friction and for kinetic friction written simply as $F = \mu N$. It is understood from the problem at hand whether maximum static friction or kinetic friction is described. Although we will frequently distinguish between the static and kinetic coefficients, in other cases no distinction will be made, and the friction coefficient will be written simply as μ. In those cases you must decide which of the friction conditions, maximum static friction for impending motion or kinetic friction, is involved. We emphasize again that many problems involve a static friction force which is less than the maximum value at impending motion, and therefore under these conditions the friction relation Eq. 6/1 cannot be used.

Figure 6/1*c* shows that rough surfaces are more likely to have larger angles between the reactions and the n-direction than are smoother surfaces. Thus, for a pair of mating surfaces, a friction coefficient reflects the roughness, which is a geometric property of the surfaces. With this geometric model of friction, we describe mating surfaces as "smooth" when the friction forces they can support are negligibly small. It is meaningless to speak of a coefficient of friction for a single surface.

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This tree surgeon depends on the friction between the rope and the mechanical devices through which the rope can slip.

Friction Angles

The direction of the resultant R in Fig. 6/1*b* measured from the direction of N is specified by $\tan \alpha = F/N$. When the friction force reaches its limiting static value $F_{\max}$, the angle α reaches a maximum value ϕ_s. Thus,

$$\tan \phi_s = \mu_s$$

When slippage is occurring, the angle α has a value ϕ_k corresponding to the kinetic friction force. In like manner,

$$\tan \phi_k = \mu_k$$

In practice we often see the expression $\tan \phi = \mu$, in which the coefficient of friction may refer to either the static or the kinetic case,

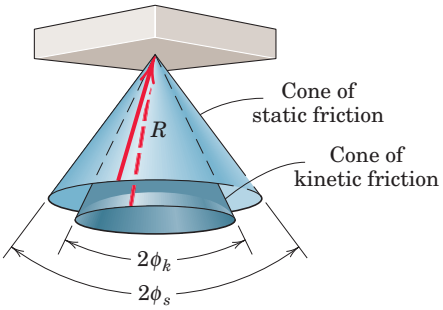


Figure 6/2

depending on the particular problem. The angle ϕ_s is called the *angle of static friction*, and the angle ϕ_k is called the *angle of kinetic friction*. The friction angle for each case clearly defines the limiting direction of the total reaction R between two contacting surfaces. If motion is impending, R must be one element of a right-circular cone of vertex angle $2\phi_s$, as shown in Fig. 6/2. If motion is not impending, R is within the cone. This cone of vertex angle $2\phi_s$ is called the *cone of static friction* and represents the locus of possible directions for the reaction R at impending motion. If motion occurs, the angle of kinetic friction applies, and the reaction must lie on the surface of a slightly different cone of vertex angle $2\phi_k$. This cone is the *cone of kinetic friction*.

Factors Affecting Friction

Further experiment shows that the friction force is essentially independent of the apparent or projected area of contact. The true contact area is much smaller than the projected value, since only the peaks of the contacting surface irregularities support the load. Even relatively small normal loads result in high stresses at these contact points. As the normal force increases, the true contact area also increases as the material undergoes yielding, crushing, or tearing at the points of contact.

A comprehensive theory of dry friction must go beyond the mechanical explanation presented here. For example, there is evidence that molecular attraction may be an important cause of friction under conditions where the mating surfaces are in very close contact. Other factors which influence dry friction are the generation of high local temperatures and adhesion at contact points, relative hardness of mating surfaces, and the presence of thin surface films of oxide, oil, dirt, or other substances.

Some typical values of coefficients of friction are given in Table D/1, Appendix D. These values are only approximate and are subject to considerable variation, depending on the exact conditions prevailing. They may be used, however, as typical examples of the magnitudes of frictional effects. To make a reliable calculation involving friction, the appropriate friction coefficient should be determined by experiments which duplicate the surface conditions of the application as closely as possible.

KEY CONCEPTS

Types of Friction Problems

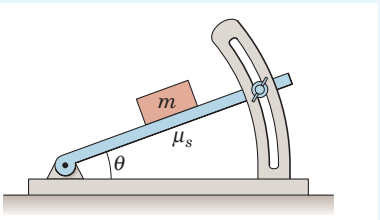
We can now recognize the following three types of problems encountered in applications involving dry friction. The first step in solving a friction problem is to identify its type.

1. In the *first* type of problem, the condition of impending motion is known to exist. Here a body which is in equilibrium is on the verge of slipping, and the friction force equals the limiting static friction $F_{\max} = \mu_s N$. The equations of equilibrium will, of course, also hold.
2. In the *second* type of problem, neither the condition of impending motion nor the condition of motion is known to exist. To determine the actual friction conditions, we first assume static equilibrium and then solve for the friction force F necessary for equilibrium. Three outcomes are possible:
   (*a*) $F < (F_{\max} = \mu_s N)$: Here the friction force necessary for equilibrium can be supported, and therefore the body is in static equilibrium as assumed. We emphasize that the *actual* friction force F is *less than* the limiting value $F_{\max}$ given by Eq. 6/1 and that F is determined *solely* by the equations of equilibrium.
   (*b*) $F = (F_{\max} = \mu_s N)$: Since the friction force F is at its maximum value $F_{\max}$, motion impends, as discussed in problem type (1). The assumption of static equilibrium is valid.
   (*c*) $F > (F_{\max} = \mu_s N)$: Clearly this condition is impossible, because the surfaces cannot support more force than the maximum $\mu_s N$. The assumption of equilibrium is therefore invalid, and motion occurs. The friction force F is equal to $\mu_k N$ from Eq. 6/2.
3. In the *third* type of problem, relative motion is known to exist between the contacting surfaces, and thus the kinetic coefficient of friction clearly applies. For this problem type, Eq. 6/2 always gives the kinetic friction force directly.

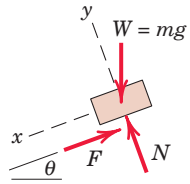
The foregoing discussion applies to all dry contacting surfaces and, to a limited extent, to moving surfaces which are partially lubricated.

SAMPLE PROBLEM 6/1

Determine the maximum angle θ which the adjustable incline may have with the horizontal before the block of mass m begins to slip. The coefficient of static friction between the block and the inclined surface is μ_s.



Solution. The free-body diagram of the block shows its weight $W = mg$, the normal force N, and the friction force F exerted by the incline on the block. The friction force acts in the direction to oppose the slipping which would occur if no friction were present.



1 Equilibrium in the x- and y-directions requires

$$[\Sigma F_x = 0] \qquad mg \sin\theta - F = 0 \qquad F = mg \sin\theta$$

$$[\Sigma F_y = 0] \qquad -mg \cos\theta + N = 0 \qquad N = mg \cos\theta$$

Dividing the first equation by the second gives $F/N = \tan\theta$. Since the maximum angle occurs when $F = F_{\max} = \mu_s N$, for impending motion we have

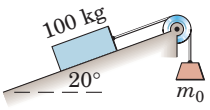
2 $$\mu_s = \tan\theta_{\max} \qquad \text{or} \qquad \theta_{\max} = \tan^{-1}\mu_s \qquad \textit{Ans.}$$

Helpful Hints

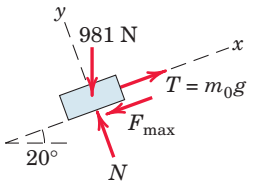
1. We choose reference axes along and normal to the direction of F to avoid resolving both F and N into components.
2. This problem describes a very simple way to determine a static coefficient of friction. The maximum value of θ is known as the *angle of repose*.

SAMPLE PROBLEM 6/2

Determine the range of values which the mass m_0 may have so that the 100-kg block shown in the figure will neither start moving up the plane nor slip down the plane. The coefficient of static friction for the contact surfaces is 0.30.



Solution. The maximum value of m_0 will be given by the requirement for motion impending up the plane. The friction force on the block therefore acts down the plane, as shown in the free-body diagram of the block for Case I in the figure. With the weight $mg = 100(9.81) = 981$ N, the equations of equilibrium give

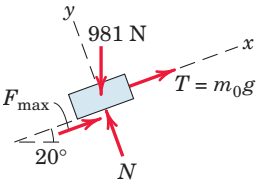


Case I

$$[\Sigma F_y = 0] \qquad N - 981\cos 20° = 0 \qquad N = 922 \text{ N}$$

$$[F_{\max} = \mu_s N] \qquad F_{\max} = 0.30(922) = 277 \text{ N}$$

$$[\Sigma F_x = 0] \qquad m_0(9.81) - 277 - 981\sin 20° = 0 \qquad m_0 = 62.4 \text{ kg} \qquad \textit{Ans.}$$



Case II

The minimum value of m_0 is determined when motion is impending down the
1 plane. The friction force on the block will act up the plane to oppose the tendency to move, as shown in the free-body diagram for Case II. Equilibrium in the x-direction requires

$$[\Sigma F_x = 0] \qquad m_0(9.81) + 277 - 981\sin 20° = 0 \qquad m_0 = 6.01 \text{ kg} \qquad \textit{Ans.}$$

Thus, m_0 may have any value from 6.01 to 62.4 kg, and the block will remain at rest.

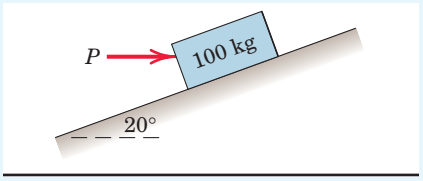
In both cases equilibrium requires that the resultant of $F_{\max}$ and N be concurrent with the 981-N weight and the tension T.

Helpful Hint

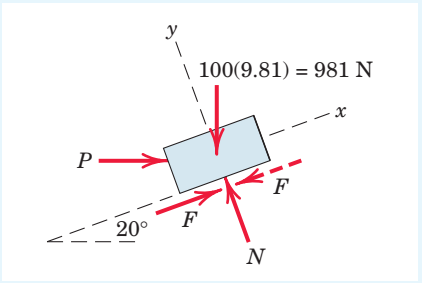
1. We see from the results of Sample Problem 6/1 that the block would slide down the incline without the restraint of attachment to m_0 since $\tan 20° > 0.30$. Thus, a value of m_0 will be required to maintain equilibrium.

SAMPLE PROBLEM 6/3

Determine the magnitude and direction of the friction force acting on the 100-kg block shown if, first, $P = 500$ N and, second, $P = 100$ N. The coefficient of static friction is 0.20, and the coefficient of kinetic friction is 0.17. The forces are applied with the block initially at rest.



Solution. There is no way of telling from the statement of the problem whether the block will remain in equilibrium or whether it will begin to slip following the application of P. It is therefore necessary that we make an assumption, so we will take the friction force to be up the plane, as shown by the solid arrow. From the free-body diagram a balance of forces in both x- and y-directions gives



$$[\Sigma F_x = 0] \qquad P \cos 20° + F - 981 \sin 20° = 0$$

$$[\Sigma F_y = 0] \qquad N - P \sin 20° - 981 \cos 20° = 0$$

Case I. $P = 500$ N

Substitution into the first of the two equations gives

$$F = -134.3 \text{ N}$$

The negative sign tells us that *if* the block is in equilibrium, the friction force acting on it is in the direction opposite to that assumed and therefore is down the plane, as represented by the dashed arrow. We cannot reach a conclusion on the magnitude of F, however, until we verify that the surfaces are capable of supporting 134.3 N of friction force. This may be done by substituting $P = 500$ N into the second equation, which gives

$$N = 1093 \text{ N}$$

The maximum static friction force which the surfaces can support is then

$$[F_{\max} = \mu_s N] \qquad F_{\max} = 0.20(1093) = 219 \text{ N}$$

Since this force is greater than that required for equilibrium, we conclude that the assumption of equilibrium was correct. The answer is, then,

$$F = 134.3 \text{ N down the plane} \qquad \textit{Ans.}$$

Case II. $P = 100$ N

Substitution into the two equilibrium equations gives

$$F = 242 \text{ N} \qquad N = 956 \text{ N}$$

But the maximum possible static friction force is

$$[F_{\max} = \mu_s N] \qquad F_{\max} = 0.20(956) = 191.2 \text{ N}$$

It follows that 242 N of friction cannot be supported. Therefore, equilibrium cannot (1) exist, and we obtain the correct value of the friction force by using the kinetic coefficient of friction accompanying the motion down the plane. Hence, the answer is

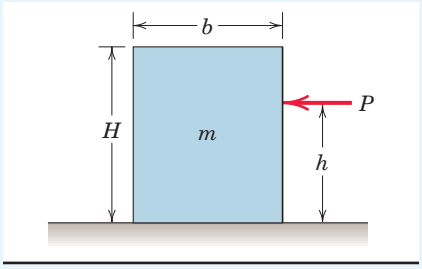
$$[F_k = \mu_k N] \qquad F = 0.17(956) = 162.5 \text{ N up the plane} \qquad \textit{Ans.}$$

Helpful Hint

(1) We should note that even though ΣF_x is no longer equal to zero, equilibrium does exist in the y-direction, so that $\Sigma F_y = 0$. Therefore, the normal force N is 956 N whether or not the block is in equilibrium.

SAMPLE PROBLEM 6/4

The homogeneous rectangular block of mass m, width b, and height H is placed on the horizontal surface and subjected to a horizontal force P which moves the block along the surface with a constant velocity. The coefficient of kinetic friction between the block and the surface is μ_k. Determine (a) the greatest value which h may have so that the block will slide without tipping over and (b) the location of a point C on the bottom face of the block through which the resultant of the friction and normal forces acts if $h = H/2$.



Solution. (a) With the block on the verge of tipping, we see that the entire reaction between the plane and the block will necessarily be at A. The free-body diagram of the block shows this condition. Since slipping occurs, the friction force is the limiting value $\mu_k N$, and the angle θ becomes $\theta = \tan^{-1}\mu_k$. The resultant of F_k and N passes through a point B through which P must also pass, since three coplanar forces in equilibrium are concurrent. Hence, from the geometry of the block (1)

$$\tan\theta = \mu_k = \frac{b/2}{h} \qquad h = \frac{b}{2\mu_k} \qquad \textit{Ans.}$$

If h were greater than this value, moment equilibrium about A would not be satisfied, and the block would tip over.

Alternatively, we may find h by combining the equilibrium requirements for the x- and y-directions with the moment-equilibrium equation about A. Thus,

$$[\Sigma F_y = 0] \qquad N - mg = 0 \qquad N = mg$$

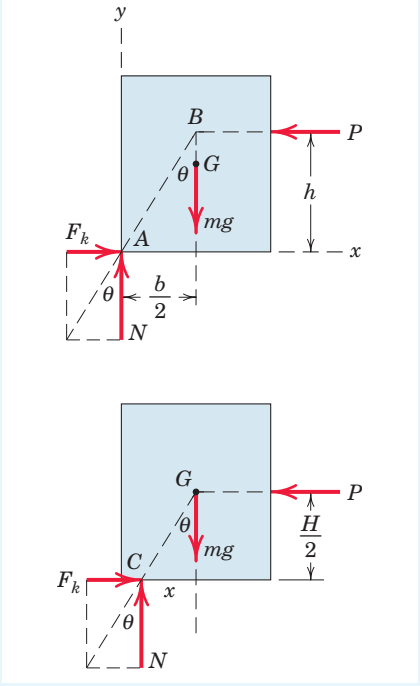
$$[\Sigma F_x = 0] \qquad F_k - P = 0 \qquad P = F_k = \mu_k N = \mu_k mg$$

$$[\Sigma M_A = 0] \qquad Ph - mg\frac{b}{2} = 0 \qquad h = \frac{mgb}{2P} = \frac{mgb}{2\mu_k mg} = \frac{b}{2\mu_k} \qquad \textit{Ans.}$$

(b) With $h = H/2$ we see from the free-body diagram for case (b) that the resultant of F_k and N passes through a point C which is a distance x to the left of the vertical centerline through G. The angle θ is still $\theta = \phi = \tan^{-1}\mu_k$ as long as the block is slipping. Thus, from the geometry of the figure we have (2)

$$\frac{x}{H/2} = \tan\theta = \mu_k \qquad \text{so} \qquad x = \mu_k H/2 \qquad \textit{Ans.}$$

If we were to replace μ_k by the static coefficient μ_s, then our solutions would describe the conditions under which the block is (a) on the verge of tipping and (b) on the verge of slipping, both from a rest position.



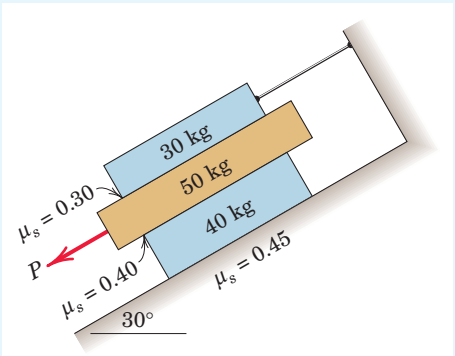
Helpful Hints

(1) Recall that the equilibrium equations apply to a body moving with a constant velocity (zero acceleration) just as well as to a body at rest.

(2) Alternatively, we could equate the moments about G to zero, which would give us $F(H/2) - Nx = 0$. Thus, with $F_k = \mu_k N$ we get $x = \mu_x H/2$.

SAMPLE PROBLEM 6/5

The three flat blocks are positioned on the 30° incline as shown, and a force P parallel to the incline is applied to the middle block. The upper block is prevented from moving by a wire which attaches it to the fixed support. The coefficient of static friction for each of the three pairs of mating surfaces is shown. Determine the maximum value which P may have before any slipping takes place.



Solution. The free-body diagram of each block is drawn. The friction forces are assigned in the directions to oppose the relative motion which would occur if (1) no friction were present. There are two possible conditions for impending motion. Either the 50-kg block slips and the 40-kg block remains in place, or the 50- and 40-kg blocks move together with slipping occurring between the 40-kg block and the incline.

The normal forces, which are in the y-direction, may be determined without reference to the friction forces, which are all in the x-direction. Thus,

$[\Sigma F_y = 0]$ (30-kg) $N_1 - 30(9.81)\cos 30° = 0$ $N_1 = 255$ N

(50-kg) $N_2 - 50(9.81)\cos 30° - 255 = 0$ $N_2 = 680$ N

(40-kg) $N_3 - 40(9.81)\cos 30° - 680 = 0$ $N_3 = 1019$ N

We will assume arbitrarily that only the 50-kg block slips, so that the 40-kg block remains in place. Thus, for impending slippage at both surfaces of the 50-kg block, we have

$[F_{\max} = \mu_s N]$ $F_1 = 0.30(255) = 76.5$ N $F_2 = 0.40(680) = 272$ N

The assumed equilibrium of forces at impending motion for the 50-kg block gives

$[\Sigma F_x = 0]$ $P - 76.5 - 272 + 50(9.81)\sin 30° = 0$ $P = 103.1$ N

We now check on the validity of our initial assumption. For the 40-kg block with $F_2 = 272$ N the friction force F_3 would be given by

$[\Sigma F_x = 0]$ $272 + 40(9.81)\sin 30° - F_3 = 0$ $F_3 = 468$ N

But the maximum possible value of F_3 is $F_3 = \mu_s N_3 = 0.45(1019) = 459$ N. Thus, 468 N cannot be supported and our initial assumption was wrong. We conclude, therefore, that slipping occurs first between the 40-kg block and the incline. With the corrected value $F_3 = 459$ N, equilibrium of the 40-kg block for its impending motion requires

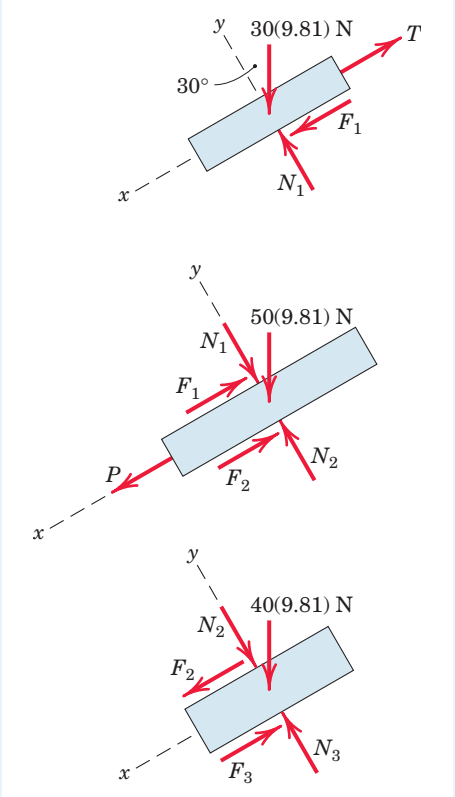
(2) $[\Sigma F_x = 0]$ $F_2 + 40(9.81)\sin 30° - 459 = 0$ $F_2 = 263$ N

Equilibrium of the 50-kg block gives, finally,

$[\Sigma F_x = 0]$ $P + 50(9.81)\sin 30° - 263 - 76.5 = 0$

$P = 93.8$ N *Ans.*

Thus, with $P = 93.8$ N, motion impends for the 50-kg and 40-kg blocks as a unit.



Helpful Hints

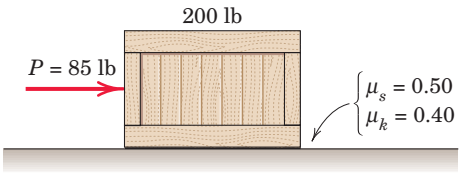
(1) In the absence of friction the middle block, under the influence of P, would have a greater movement than the 40-kg block, and the friction force F_2 will be in the direction to oppose this motion as shown.

(2) We see now that F_2 is less than $\mu_s N_2 = 272$ N.

PROBLEMS

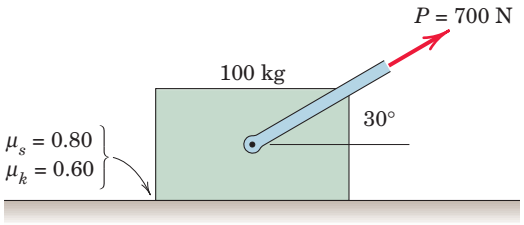
Introductory Problems

6/1 The 85-lb force P is applied to the 200-lb crate, which is stationary before the force is applied. Determine the magnitude and direction of the friction force F exerted by the horizontal surface on the crate.



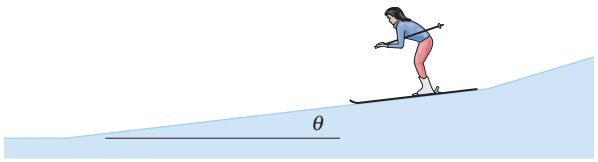
Problem 6/1

6/2 The 700-N force is applied to the 100-kg block, which is stationary before the force is applied. Determine the magnitude and direction of the friction force F exerted by the horizontal surface on the block.



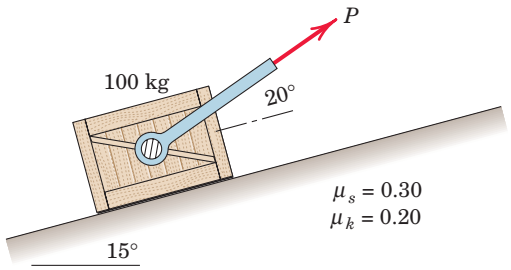
Problem 6/2

6/3 The designer of a ski resort wishes to have a portion of a beginner's slope on which the skier's speed will remain fairly constant. Tests indicate the average coefficients of friction between skis and snow to be $\mu_s = 0.10$ and $\mu_k = 0.08$. What should be the slope angle θ of the constant-speed section?



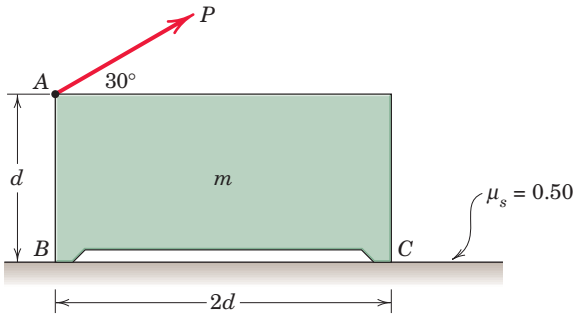
Problem 6/3

6/4 The coefficients of static and kinetic friction between the 100-kg block and the inclined plane are 0.30 and 0.20, respectively. Determine (*a*) the friction force F acting on the block when P is applied with a magnitude of 200 N to the block at rest, (*b*) the force P required to initiate motion up the incline from rest, and (*c*) the friction force F acting on the block if $P = 600$ N.



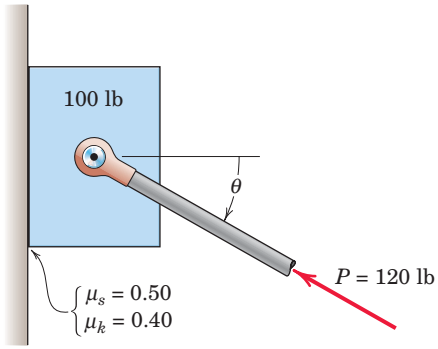
Problem 6/4

6/5 The magnitude of force P is slowly increased. Does the homogeneous box of mass m slip or tip first? State the value of P which would cause each occurrence. Neglect any effect of the size of the small feet.



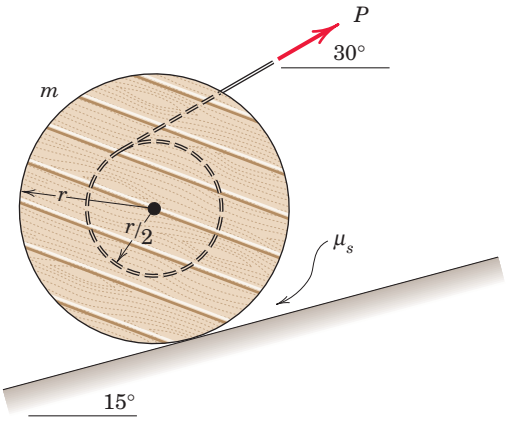
Problem 6/5

6/6 Determine the magnitude and direction of the friction force which the vertical wall exerts on the 100-lb block if (*a*) $\theta = 15°$ and (*b*) $\theta = 30°$.



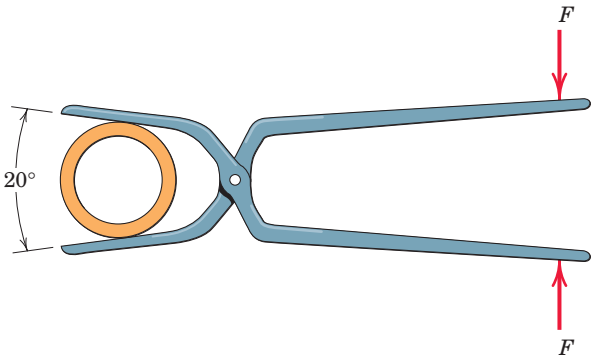
Problem 6/6

6/7 Determine the minimum coefficient of static friction μ_s which will allow the drum with fixed inner hub to be rolled up the 15° incline at a steady speed without slipping. What are the corresponding values of the force P and the friction force F?



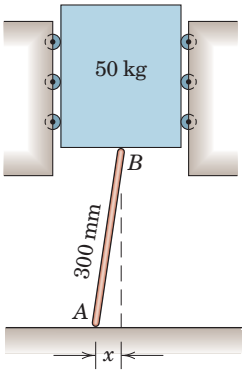
Problem 6/7

6/8 The tongs are designed to handle hot steel tubes which are being heat-treated in an oil bath. For a 20° jaw opening, what is the minimum coefficient of static friction between the jaws and the tube which will enable the tongs to grip the tube without slipping?



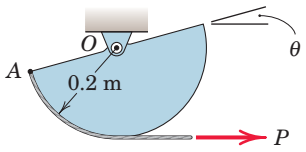
Problem 6/8

6/9 The light bar is used to support the 50-kg block in its vertical guides. If the coefficient of static friction is 0.30 at the upper end of the bar and 0.40 at the lower end of the bar, find the friction force acting at each end for $x = 75$ mm. Also find the maximum value of x for which the bar will not slip.



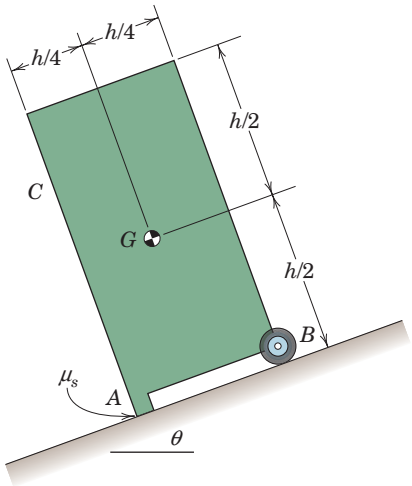
Problem 6/9

6/10 The semicircular 5-kg plate rotates about a fixed bearing at O that can exert a maximum frictional moment of 1.2 N·m on the plate. The horizontal force P is applied to a cord wrapped around the plate periphery and pinned at A. (*a*) With $P = 0$, determine the maximum angle θ for which equilibrium is possible. (*b*) From the position determined in (*a*) calculate the force P required to rotate the body counterclockwise.



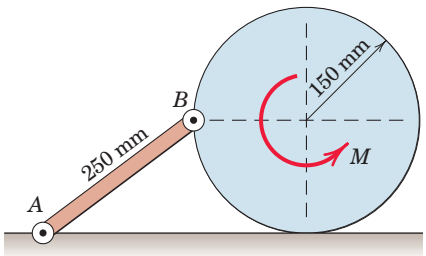
Problem 6/10

6/11 The homogeneous body with two small feet and two ideal wheels is at rest on the rough incline. (*a*) Assume no slippage and determine the maximum value of the angle θ for which the body does not overturn about feet *A*. (*b*) If $\mu_s = 0.40$, determine the maximum value of the angle θ for which the body does not slip. (*c*) If $\mu_s = 0.40$ and side *C* of the body now faces up the incline, determine the maximum value of the angle θ for which the body does not slip.



Problem 6/11

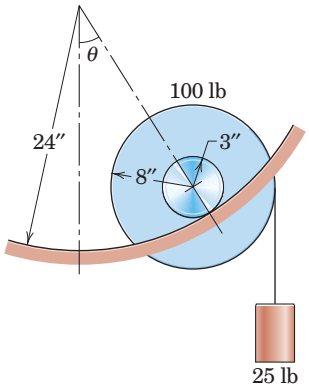
6/12 The strut *AB* of negligible mass is hinged to the horizontal surface at *A* and to the uniform 25-kg wheel at *B*. Determine the minimum couple *M* applied to the wheel which will cause it to slip if the coefficient of static friction between the wheel and the surface is 0.40.



Problem 6/12

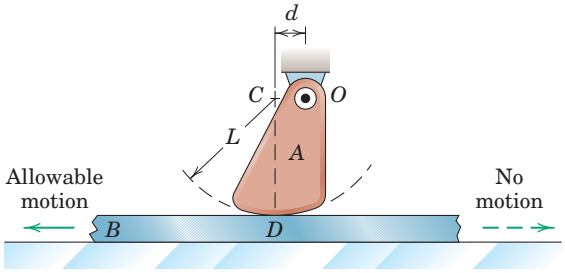
Representative Problems

6/13 The 100-lb wheel rolls on its hub up the circular incline under the action of the 25-lb weight attached to a cord around the rim. Determine the angle θ at which the wheel comes to rest, assuming that friction is sufficient to prevent slippage. What is the minimum coefficient of friction which will permit this position to be reached with no slipping?



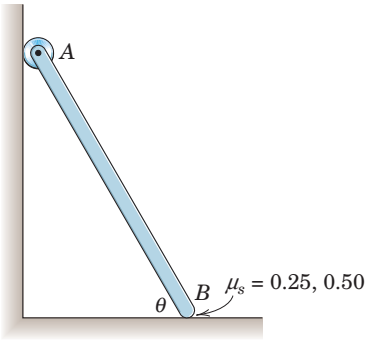
Problem 6/13

6/14 The illustration shows the design of a cam-type locking device, which, in the presence of sufficient friction, limits the movement of body *B* to be to the left only; rightward movement is prevented. The surface of the cam near point *D* is circular with center at *C*. Given the distance *L*, specify the cam offset *d* so that the device will work if the coefficient of static friction μ_s is 0.20 or greater.



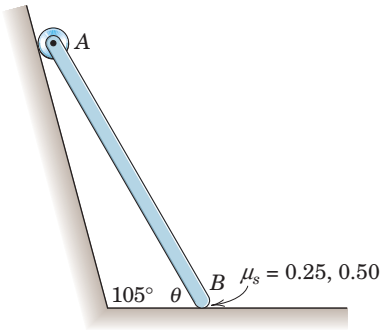
Problem 6/14

6/15 The uniform slender bar has an ideal roller at its upper end *A*. If the coefficient of static friction at *B* is $\mu_s = 0.25$, determine the minimum angle θ for which equilibrium is possible. Repeat for $\mu_s = 0.50$.



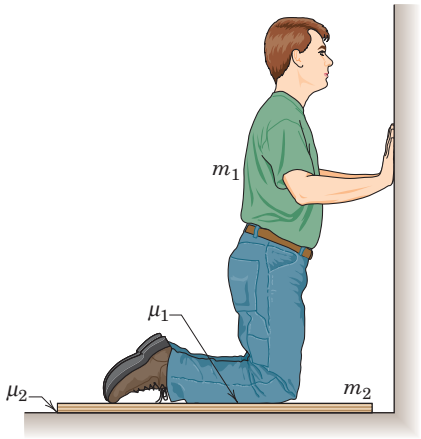
Problem 6/15

6/16 The upper end of the bar of Prob. 6/15 is now leaned against a nonvertical wall. Determine the minimum value of the angle θ for which equilibrium is possible for $\mu_s = 0.25$ and for $\mu_s = 0.50$.



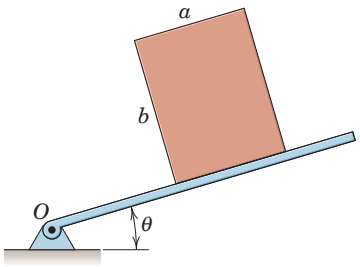
Problem 6/16

6/17 A person (mass m_1) who is kneeling on a plywood panel (mass m_2) wishes to push himself and the panel away from the right-hand wall. Determine a condition on μ_2 which would allow him to do this, as opposed to his slipping relative to a stationary panel. The two applicable coefficients of static friction are μ_1 and μ_2. Evaluate μ_2 for $m_1 = 80$ kg, $m_2 = 10$ kg, and $\mu_1 = 0.60$.



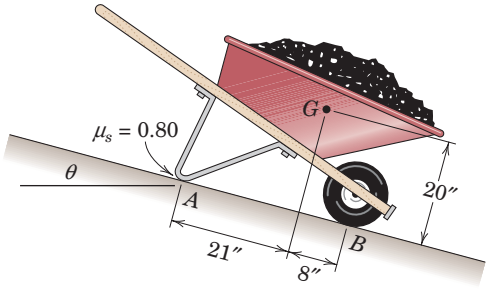
Problem 6/17

6/18 The homogeneous rectangular block of mass m rests on the inclined plane which is hinged about a horizontal axis through O. If the coefficient of static friction between the block and the plane is μ, specify the conditions which determine whether the block tips before it slips or slips before it tips as the angle θ is gradually increased.



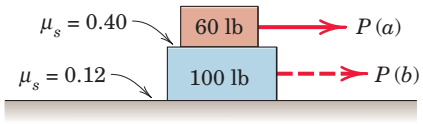
Problem 6/18

6/19 A loaded wheelbarrow is placed on a rough incline. The combined weight of the wheelbarrow and its load acts at the center of gravity G. (*a*) Determine the maximum angle θ for which the wheelbarrow will not slip. (*b*) Repeat if the wheelbarrow faces uphill. Neglect all friction associated with the front wheel B.



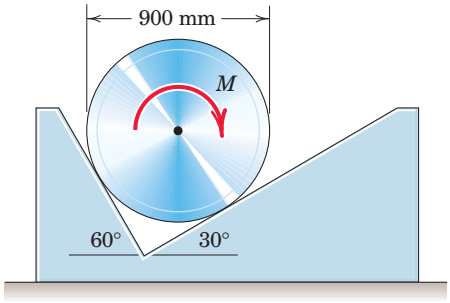
Problem 6/19

6/20 The force P is applied to (*a*) the 60-lb block and (*b*) the 100-lb block. For each case, determine the magnitude of P required to initiate motion.



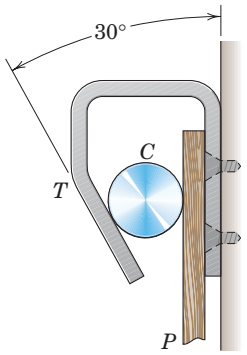
Problem 6/20

6/21 Calculate the magnitude of the clockwise couple M required to turn the 50-kg cylinder in the supporting block shown. The coefficient of kinetic friction is 0.30.



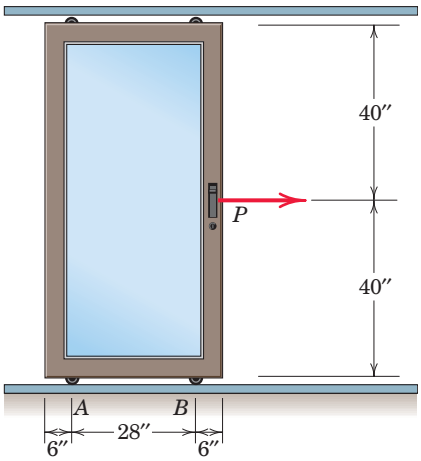
Problem 6/21

6/22 The inverted track T with freely floating cylinder C comprise a system which is designed to hold paper or other thin materials P in place. The coefficient of static friction is μ for all interfaces. What minimum value of μ ensures that the device will work no matter how heavy the supported material P is?



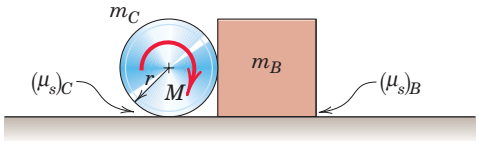
Problem 6/22

6/23 The sliding glass door rolls on the two small lower wheels A and B. Under normal conditions the upper wheels do not touch their horizontal guide. (a) Compute the force P required to slide the door at a steady speed if wheel A becomes "frozen" and does not turn in its bearing. (b) Rework the problem if wheel B becomes frozen instead of wheel A. The coefficient of kinetic friction between a frozen wheel and the supporting surface is 0.30, and the center of mass of the 140-lb door is at its geometric center. Neglect the small diameter of the wheels.



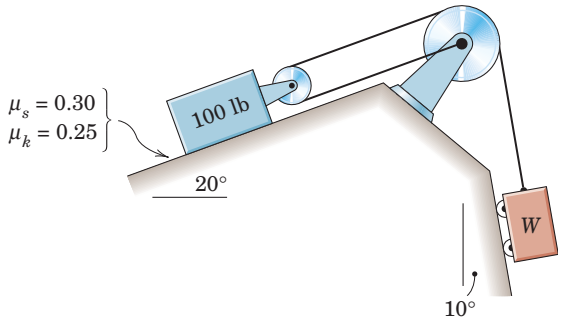
Problem 6/23

6/24 A clockwise couple M is applied to the circular cylinder as shown. Determine the value of M required to initiate motion for the conditions $m_B = 3$ kg, $m_C = 6$ kg, $(\mu_s)_B = 0.50$, $(\mu_s)_C = 0.40$, and $r = 0.2$ m. Friction between the cylinder C and the block B is negligible.



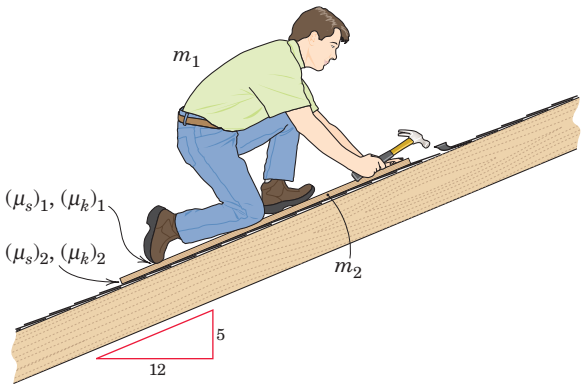
Problem 6/24

6/25 Determine the range of weights W for which the 100-lb block is in equilibrium. All wheels and pulleys have negligible friction.



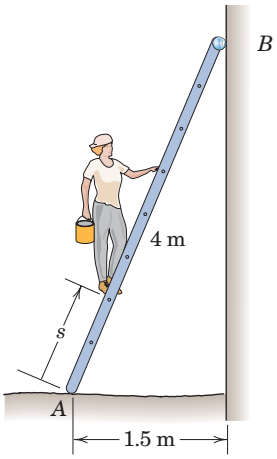
Problem 6/25

6/26 A 100-kg roofer crouches on an 18-kg sheet of plywood as he is about to effect a shingle repair. Determine the friction force which the roof surface exerts on the plywood if the coefficients of static friction are (a) $(\mu_s)_1 = 0.60$ and $(\mu_s)_2 = 0.50$ and (b) $(\mu_s)_1 = 0.50$ and $(\mu_s)_2 = 0.40$. In all cases, the coefficient of kinetic friction is 75% of the static value.



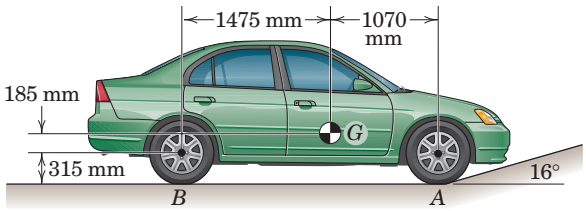
Problem 6/26

6/27 Determine the distance s to which the 90-kg painter can climb without causing the 4-m ladder to slip at its lower end A. The top of the 15-kg ladder has a small roller, and at the ground the coefficient of static friction is 0.25. The mass center of the painter is directly above her feet.



Problem 6/27

6/28 The 1600-kg car is just beginning to negotiate the 16° ramp. If the car has rear-wheel drive, determine the minimum coefficient of static friction required at B.

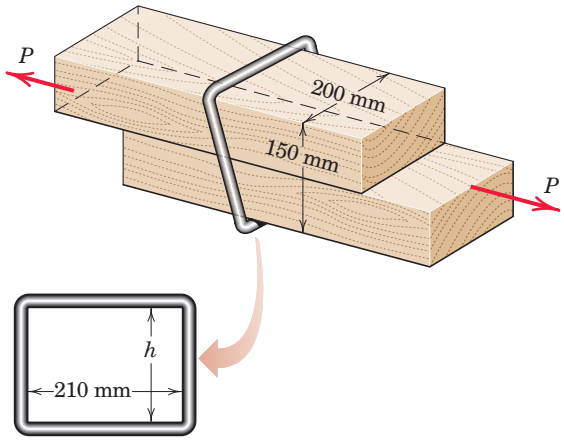


Problem 6/28

6/29 Repeat Prob. 6/28, but now the car has front-wheel drive.

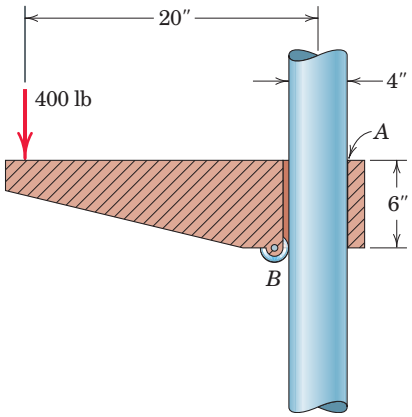
6/30 Repeat Prob. 6/28, but now the car has all-wheel drive. Assume that slipping occurs at A and B simultaneously.

6/31 The rectangular steel yoke is used to prevent slippage between the two boards under tensile loads P. If the coefficients of static friction between the yoke and the board surfaces and between the boards are all 0.30, determine the maximum value of h for which there is no slipping. For P = 800 N, determine the corresponding normal force N between the two boards if motion impends at all surfaces.



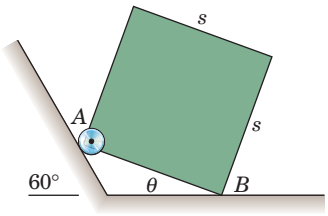
Problem 6/31

6/32 The figure shows the design in section of a loaded bracket which is supported on the fixed shaft by the roller at B and by friction at the corner A. The coefficient of static friction is 0.40. Neglect the weight of the bracket and show that the bracket as designed will remain in place. Find the friction force F.



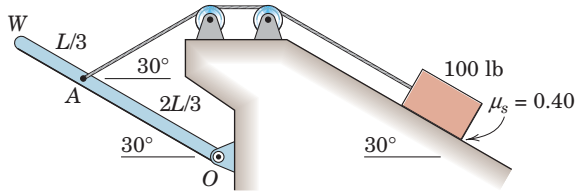
Problem 6/32

6/33 The homogeneous square body is positioned as shown. If the coefficient of static friction at B is 0.40, determine the critical value of the angle θ below which slipping will occur. Neglect friction at A.



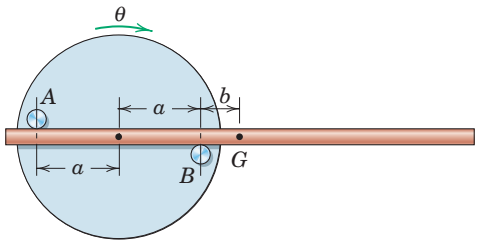
Problem 6/33

6/34 Determine the range of weights W of the uniform slender bar for which the system will be in equilibrium. Neglect friction at all bearings.



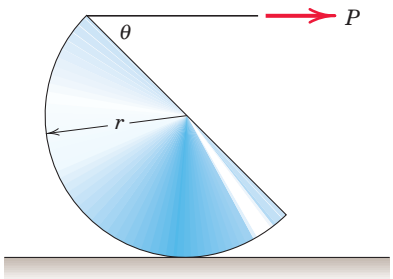
Problem 6/34

6/35 The uniform rod with center of mass at G is supported by the pegs A and B, which are fixed in the wheel. If the coefficient of friction between the rod and pegs is μ, determine the angle θ through which the wheel may be slowly turned about its horizontal axis through O, starting from the position shown, before the rod begins to slip. Neglect the diameter of the rod compared with the other dimensions.



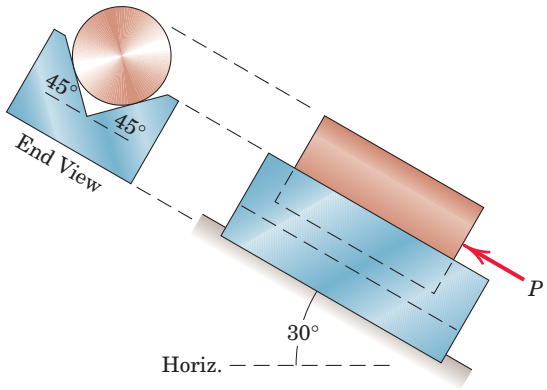
Problem 6/35

6/36 The solid semicylinder of mass m and radius r is rolled through an angle θ by the horizontal force P. If the coefficient of static friction is μ_s, determine the angle θ at which the cylinder begins to slip on the horizontal surface as P is gradually increased. What value of μ_s would permit θ to reach 90°?



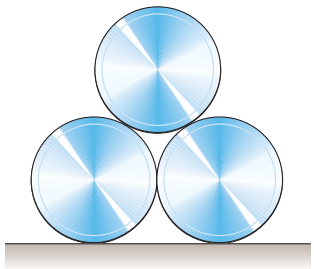
Problem 6/36

6/37 The 10-kg solid cylinder is resting in the inclined V-block. If the coefficient of static friction between the cylinder and the block is 0.50, determine (*a*) the friction force F acting on the cylinder at each side before force P is applied and (*b*) the value of P required to start sliding the cylinder up the incline.



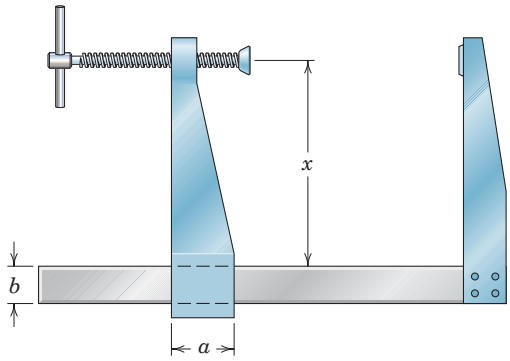
Problem 6/37

6/38 The three identical rollers are stacked on a horizontal surface as shown. If the coefficient of static friction μ_s is the same for all pairs of contacting surfaces, find the minimum value of μ_s for which the rollers will not slip.



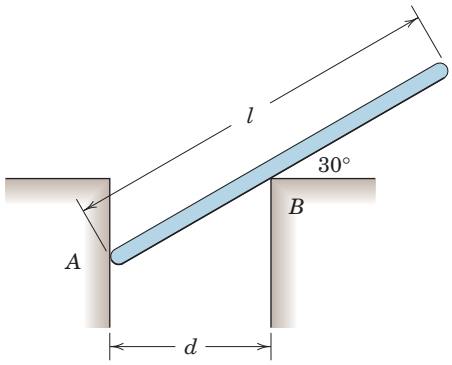
Problem 6/38

6/39 The movable left-hand jaw of the bar clamp can be slid along the frame to increase the capacity of the clamp. To prevent slipping of the jaw on the frame when the clamp is under load, the dimension x must exceed a certain minimum value. For given values of a and b and a static friction coefficient μ_s, specify this design minimum value of x to prevent slipping of the jaw.



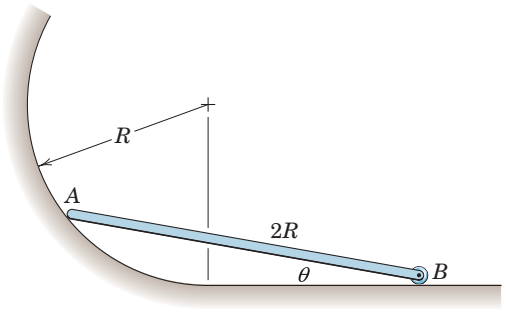
Problem 6/39

6/40 The uniform slender bar of length l is placed in the opening of width d at the 30° angle shown. For what range of l/d will the bar remain in static equilibrium? The coefficient of static friction at A and B is $\mu_s = 0.40$.



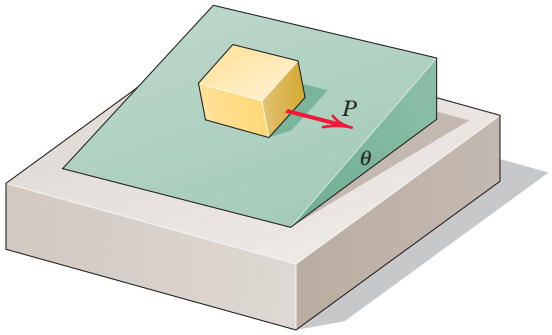
Problem 6/40

6/41 Determine the maximum value of the angle θ for which the uniform slender rod will remain in equilibrium. The coefficient of static friction at A is $\mu_A = 0.80$, and friction associated with the small roller at B may be neglected.



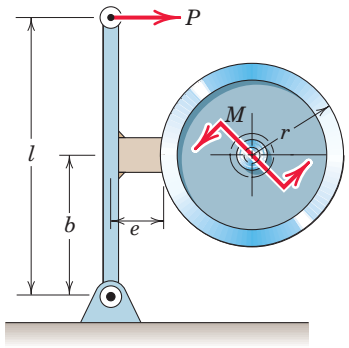
Problem 6/41

6/42 A uniform block of mass m is at rest on an incline θ. Determine the maximum force P which can be applied to the block in the direction shown before slipping begins. The coefficient of static friction between the block and the incline is μ_s. Also determine the angle β between the horizontal direction of P and the direction of initial movement of the block.



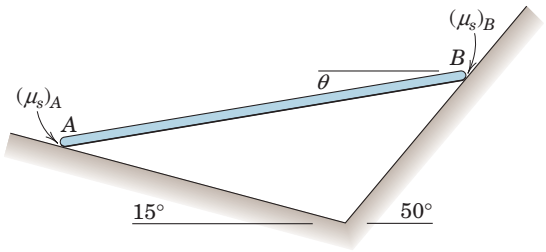
Problem 6/42

6/43 The single-lever block brake prevents rotation of the flywheel under a counterclockwise torque M. Find the force P required to prevent rotation if the coefficient of static friction is μ_s. Explain what would happen if the geometry permitted b to equal $\mu_s e$.



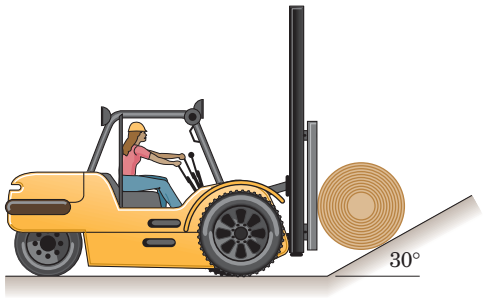
Problem 6/43

6/44 Determine the minimum value of θ for which equilibrium is possible if (*a*) $(\mu_s)_A = 0.60$ and $(\mu_s)_B = 0.20$ and (*b*) $(\mu_s)_A = 0.40$ and $(\mu_s)_B = 0.10$.



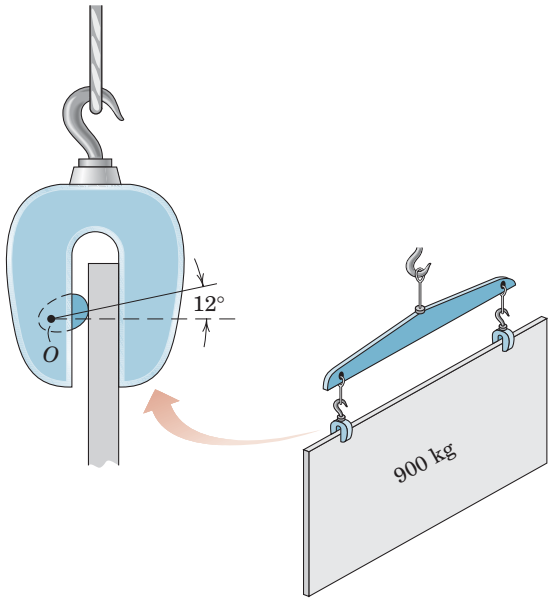
Problem 6/44

6/45 The industrial truck is used to move the solid 1200-kg roll of paper up the 30° incline. If the coefficients of static and kinetic friction between the roll and the vertical barrier of the truck and between the roll and the incline are both 0.40, compute the required tractive force P between the tires of the truck and the horizontal surface.



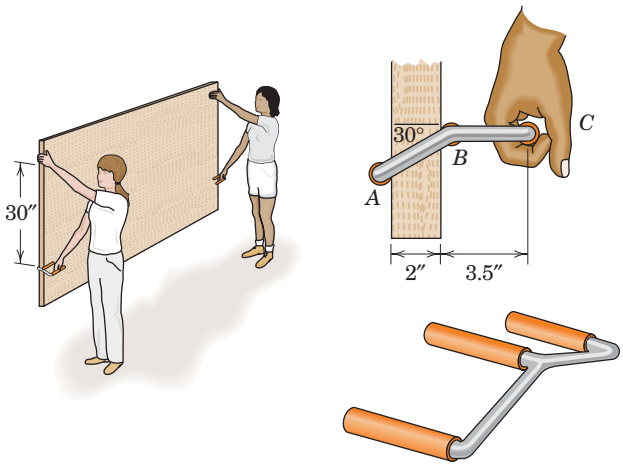
Problem 6/45

6/46 Heavy-duty plate clamps with wedging cams are designed to be used in pairs to lift and transport large steel plates. As each cam begins to take effect, the plate slips down slightly against the vertical surface of the clamp and, therefore, generates limiting static friction at that interface. If the coefficient of static friction for all contacting surfaces is 0.30, calculate the force R supported by the pin at O for each of the two clamps symmetrically placed on the 900-kg plate.



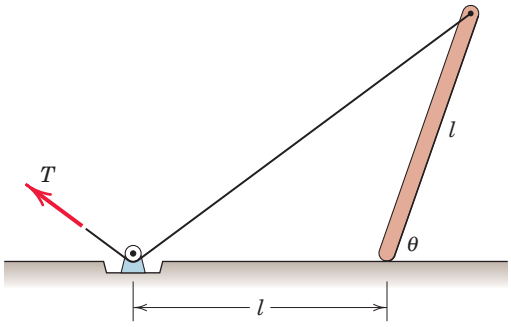
Problem 6/46

6/47 Two workers are carrying a 2-in.-thick panel by means of panel carriers, one of which is shown in the detail figures. The vertical panel is steadied by equal horizontal forces applied by the left hands of the workers. Determine the minimum coefficient of static friction between the panel and the carriers for which there will be no slippage. The carrier grips at *A*, *B*, and *C* do not rotate on the carrier frame. Note that each worker must apply both a vertical and a horizontal force to the carrier handle. Assume that each worker supports half the weight of the panel.



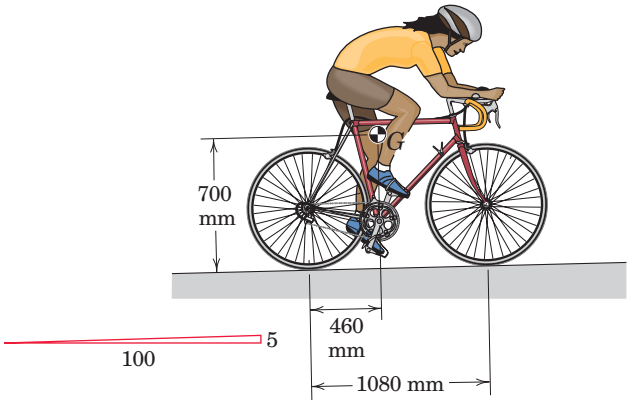
Problem 6/47

6/48 The uniform slender rod is slowly lowered from the upright position ($\theta = 90°$) by means of the cord attached to its upper end and passing under the small fixed pulley. If the rod is observed to slip at its lower end when $\theta = 40°$, determine the coefficient of static friction at the horizontal surface.



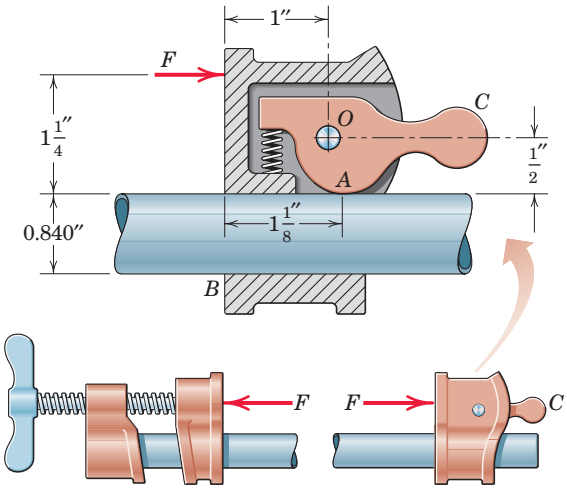
Problem 6/48

6/49 A woman pedals her bicycle up a 5-percent grade on a slippery road at a steady speed. The woman and bicycle have a combined mass of 82 kg with mass center at G. If the rear wheel is on the verge of slipping, determine the coefficient of friction μ_s between the rear tire and the road. If the coefficient of friction were doubled, what would be the friction force F acting on the rear wheel? (Why may we neglect friction under the front wheel?)



Problem 6/49

6/50 The pipe-clamp ends are designed to fit a standard $\frac{1}{2}$-in. steel water pipe (outside diameter = 0.840 in.). The right-hand fitting slides loosely along the pipe when the lever C is held up to release its contact with the pipe, as can be seen from the expanded view. Under a clamping load F, the right-hand fitting contacts the pipe at A and B only, as its loose fit allows it to rotate slightly clockwise on the pipe. Determine (a) the minimum coefficients of static friction μ_s between the contacting surfaces at A and B so that the end fitting will not slip under load and (b) the force R supported by the pin at O under a clamping force F = 800 lb. The spring force under the lever projection is negligible.



Problem 6/50

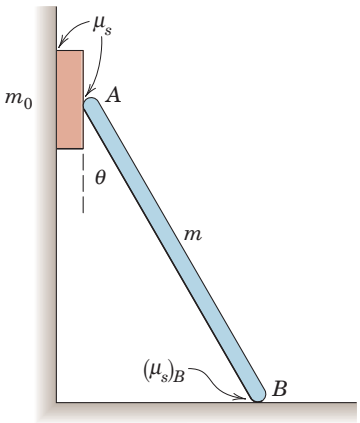
6/51 A block of mass m_0 is placed between the vertical wall and the upper end A of the uniform slender bar of mass m. If the coefficient of static friction is μ_s between the block and the wall and also between the block and the bar, determine a general expression for the minimum value $\theta_{\min}$ of the angle θ for which the block will remain in equilibrium. Evaluate your expression for the conditions $\mu_s = 0.50$ and

(*a*) $\dfrac{m}{m_0} = 0.1$,

(*b*) $\dfrac{m}{m_0} = 1$, and

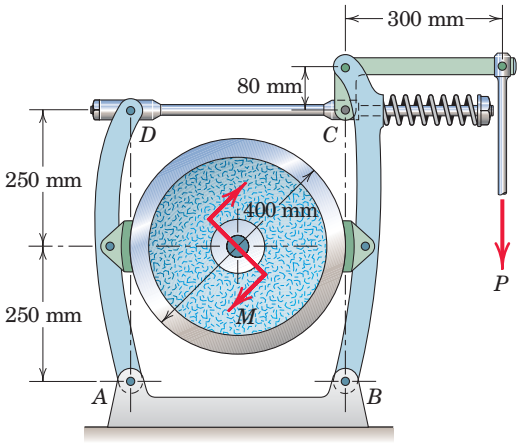
(*c*) $\dfrac{m}{m_0} = 10$.

For each case, state the minimum coefficient of static friction $(\mu_s)_B$ necessary to prevent slippage at B.



Problem 6/51

6/52 The double-block brake shown is applied to the flywheel by means of the action of the spring. To release the brake, a force P is applied to the control rod. In the operating position with $P = 0$, the spring is compressed 30 mm. Select a spring with an appropriate constant (stiffness) k which will provide sufficient force to brake the flywheel under the torque $M = 100$ N·m if the applicable coefficient of friction for both brake shoes is 0.20. Neglect the dimensions of the shoes.



Problem 6/52

SECTION B APPLICATIONS OF FRICTION IN MACHINES

In Section B we investigate the action of friction in various machine applications. Because the conditions in these applications are normally either limiting static or kinetic friction, we will use the variable μ (rather than μ_s or μ_k) in general. Depending on whether motion is impending or actually occurring, μ can be interpreted as either the static or kinetic coefficient of friction.

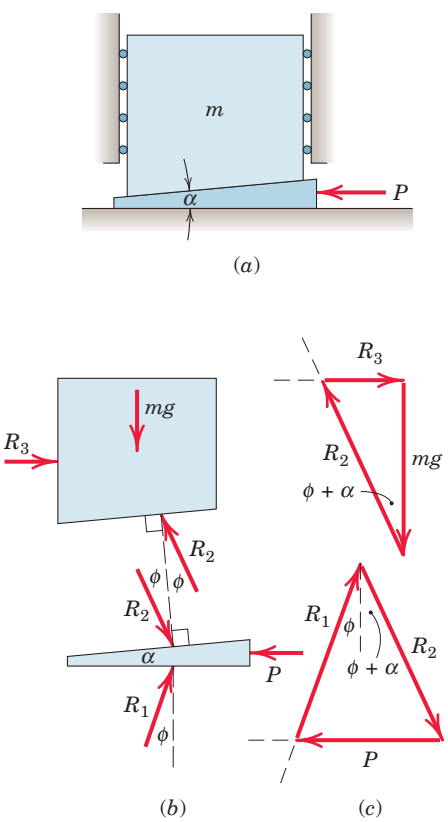
6/4 Wedges

A wedge is one of the simplest and most useful machines. A wedge is used to produce small adjustments in the position of a body or to apply large forces. Wedges largely depend on friction to function. When sliding of a wedge is impending, the resultant force on each sliding surface of the wedge will be inclined from the normal to the surface by an amount equal to the friction angle. The component of the resultant along the surface is the friction force, which is always in the direction to oppose the motion of the wedge relative to the mating surfaces.

Figure 6/3*a* shows a wedge used to position or lift a large mass m, where the vertical loading is mg. The coefficient of friction for each pair of surfaces is $\mu = \tan \phi$. The force P required to start the wedge is found from the equilibrium triangles of the forces on the load and on the wedge. The free-body diagrams are shown in Fig. 6/3*b*, where the reactions are inclined at an angle ϕ from their respective normals and are in the direction to oppose the motion. We neglect the mass of the wedge. From the free-body diagrams we write the force equilibrium conditions by equating to zero the sum of the force vectors acting on each body. The solutions of these equations are shown in part *c* of the figure, where R_2 is found first in the upper diagram using the known value of mg. The force P is then found from the lower triangle once the value of R_2 has been established.

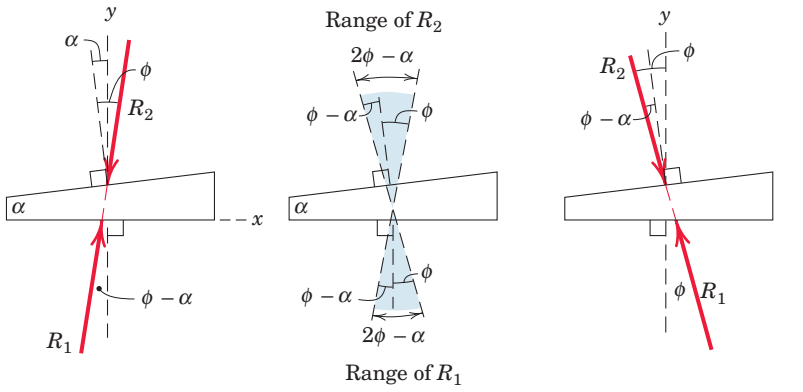
If P is removed and the wedge remains in place, equilibrium of the wedge requires that the equal reactions R_1 and R_2 be collinear as shown in Fig. 6/4, where the wedge angle α is taken to be less than ϕ. Part *a* of the figure represents impending slippage at the upper surface, and part *c* of the figure represents impending slippage at the lower surface. In order for the wedge to slide out of its space, slippage must occur at *both* surfaces simultaneously; otherwise, the wedge is *self-locking*, and there is a finite range of possible intermediate angular positions of R_1 and R_2 for which the wedge will remain in place. Figure 6/4*b* illustrates this range and shows that simultaneous slippage is not possible if $\alpha < 2\phi$. You are encouraged to construct additional diagrams for the case where $\alpha > \phi$ and verify that the wedge is self-locking as long as $\alpha < 2\phi$.

If the wedge is self-locking and is to be withdrawn, a pull P on the wedge will be required. To oppose the new impending motion, the reactions R_1 and R_2 must act on the opposite sides of their normals from those when the wedge was inserted. The solution can be obtained as



Forces to raise load

Figure 6/3



(*a*) Slipping impending at upper surface

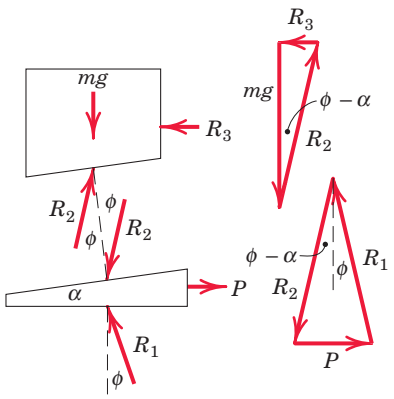
(*b*) Range of $R_1 = R_2$ for no slip

(*c*) Slipping impending at lower surface

Figure 6/4

with the case of raising the load. The free-body diagrams and vector polygons for this condition are shown in Fig. 6/5.

Wedge problems lend themselves to graphical solutions as indicated in the three figures. The accuracy of a graphical solution is easily held within tolerances consistent with the uncertainty of friction coefficients. Algebraic solutions may also be obtained from the trigonometry of the equilibrium polygons.



Forces to lower load

Figure 6/5

6/5 Screws

Screws are used for fastening and for transmitting power or motion. In each case the friction developed in the threads largely determines the action of the screw. For transmitting power or motion the square thread is more efficient than the V-thread, and the analysis here is confined to the square thread.

Force Analysis

Consider the square-threaded jack, Fig. 6/6, under the action of the axial load W and a moment M applied about the axis of the screw. The screw has a lead L (advancement per revolution) and a mean radius r. The force R exerted by the thread of the jack frame on a small representative portion of the screw thread is shown on the free-body diagram of the screw. Similar reactions exist on all segments of the screw thread where contact occurs with the thread of the base.

If M is just sufficient to turn the screw, the thread of the screw will slide around and up on the fixed thread of the frame. The angle ϕ made by R with the normal to the thread is the angle of friction, so that $\tan \phi = \mu$. The moment of R about the vertical axis of the screw is $Rr \sin (\alpha + \phi)$, and the total moment due to all reactions on the threads is $\Sigma Rr \sin (\alpha + \phi)$. Since $r \sin (\alpha + \phi)$ appears in each term, we may factor it out. The moment equilibrium equation for the screw becomes

$$M = [r \sin (\alpha + \phi)] \Sigma R$$

Equilibrium of forces in the axial direction further requires that

$$W = \Sigma R \cos (\alpha + \phi) = [\cos (\alpha + \phi)] \Sigma R$$

Combining the expressions for M and W gives

$$M = Wr \tan (\alpha + \phi) \qquad \textbf{(6/3)}$$

To determine the helix angle α, unwrap the thread of the screw for one complete turn and note that $\alpha = \tan^{-1} (L/2\pi r)$.

We may use the unwrapped thread of the screw as an alternative model to simulate the action of the entire screw, as shown in Fig. 6/7*a*. The equivalent force required to push the movable thread up the fixed incline is $P = M/r$, and the triangle of force vectors gives Eq. 6/3 immediately.

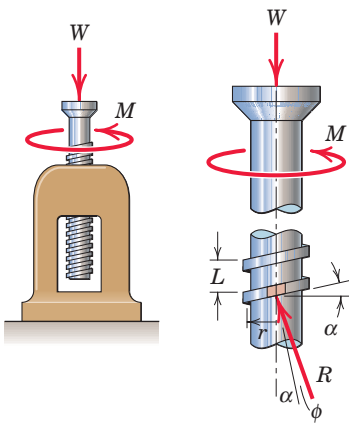
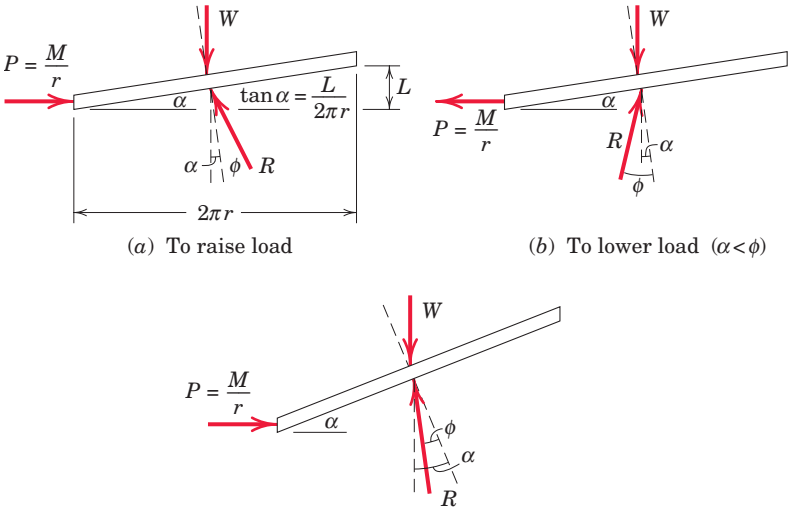


Figure 6/6



(*a*) To raise load

(*b*) To lower load ($\alpha < \phi$)

(*c*) To lower load ($\alpha > \phi$)

Figure 6/7

Conditions for Unwinding

If the moment M is removed, the friction force changes direction so that ϕ is measured to the other side of the normal to the thread. The screw will remain in place and be self-locking provided that $\alpha < \phi$, and will be on the verge of unwinding if $\alpha = \phi$.

To lower the load by unwinding the screw, we must reverse the direction of M as long as $\alpha < \phi$. This condition is illustrated in Fig. 6/7*b* for our simulated thread on the fixed incline. An equivalent force $P = M/r$ must be applied to the thread to pull it down the incline. From the triangle of vectors we therefore obtain the moment required to lower the screw, which is

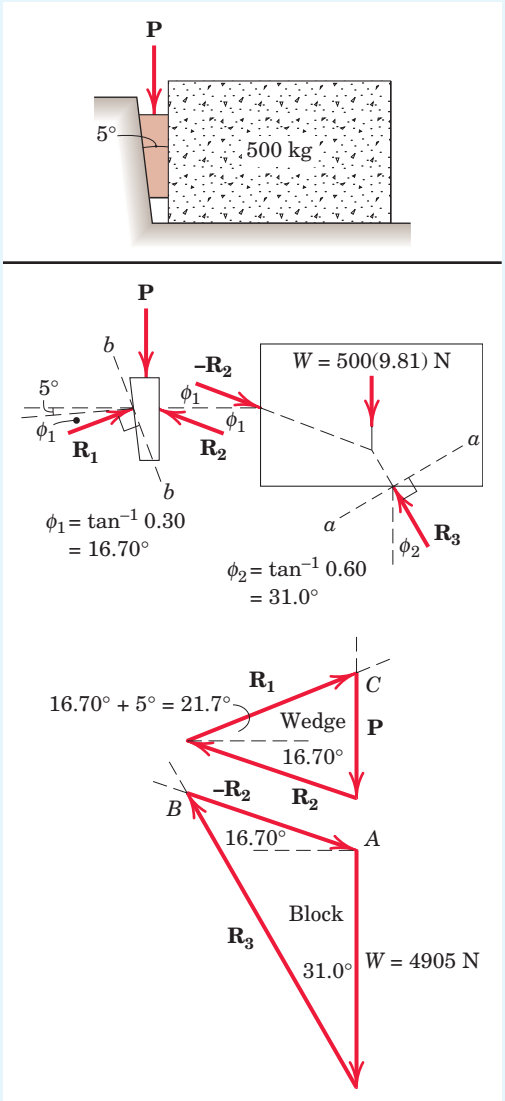
$$M = Wr \tan (\phi - \alpha) \qquad \textbf{(6/3}\boldsymbol{a}\textbf{)}$$

If $\alpha > \phi$, the screw will unwind by itself, and Fig. 6/7*c* shows that the moment required to prevent unwinding is

$$M = Wr \tan (\alpha - \phi) \qquad \textbf{(6/3}\boldsymbol{b}\textbf{)}$$

SAMPLE PROBLEM 6/6

The horizontal position of the 500-kg rectangular block of concrete is adjusted by the 5° wedge under the action of the force **P**. If the coefficient of static friction for both wedge surfaces is 0.30 and if the coefficient of static friction between the block and the horizontal surface is 0.60, determine the least force P required to move the block.



Solution. The free-body diagrams of the wedge and the block are drawn with the reactions $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$ inclined with respect to their normals by the amount of the friction angles for impending motion. The friction angle for limiting static friction is given by $\phi = \tan^{-1} \mu$. Each of the two friction angles is computed and shown on the diagram. (1)

We start our vector diagram expressing the equilibrium of the block at a convenient point A and draw the only known vector, the weight **W** of the block. Next we add $\mathbf{R}_3$, whose 31.0° inclination from the vertical is now known. The vector $-\mathbf{R}_2$, whose 16.70° inclination from the horizontal is also known, must close the polygon for equilibrium. Thus, point B on the lower polygon is determined by the intersection of the known directions of $\mathbf{R}_3$ and $-\mathbf{R}_2$, and their magnitudes become known.

For the wedge we draw $\mathbf{R}_2$, which is now known, and add $\mathbf{R}_1$, whose direction is known. The directions of $\mathbf{R}_1$ and **P** intersect at C, thus giving us the solution for the magnitude of **P**.

Algebraic solution. The simplest choice of reference axes for calculation purposes is, for the block, in the direction a-a normal to $\mathbf{R}_3$ and, for the wedge, in the direction b-b normal to $\mathbf{R}_1$. The angle between $\mathbf{R}_2$ and the a-direction is $16.70° + 31.0° = 47.7°$. Thus, for the block (2)

$$[\Sigma F_a = 0] \qquad 500(9.81) \sin 31.0° - R_2 \cos 47.7° = 0$$

$$R_2 = 3750 \text{ N}$$

For the wedge the angle between $\mathbf{R}_2$ and the b-direction is $90° - (2\phi_1 + 5°) = 51.6°$, and the angle between **P** and the b-direction is $\phi_1 + 5° = 21.7°$. Thus,

$$[\Sigma F_b = 0] \qquad 3750 \cos 51.6° - P \cos 21.7° = 0$$

$$P = 2500 \text{ N} \qquad \textit{Ans.}$$

Graphical solution. The accuracy of a graphical solution is well within the uncertainty of the friction coefficients and provides a simple and direct result. By laying off the vectors to a reasonable scale following the sequence described, we obtain the magnitudes of **P** and the **R**'s easily by scaling them directly from the diagrams.

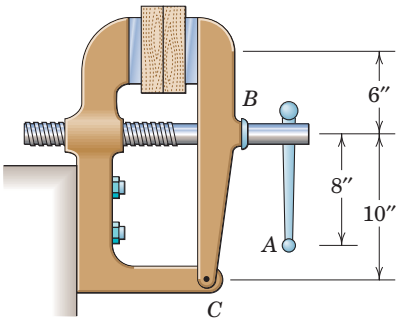
Helpful Hints

(1) Be certain to note that the reactions are inclined from their normals in the direction to oppose the motion. Also, we note the equal and opposite reactions $\mathbf{R}_2$ and $-\mathbf{R}_2$.

(2) It should be evident that we avoid simultaneous equations by eliminating reference to $\mathbf{R}_3$ for the block and $\mathbf{R}_1$ for the wedge.

SAMPLE PROBLEM 6/7

The single-threaded screw of the vise has a mean diameter of 1 in. and has 5 square threads per inch. The coefficient of static friction in the threads is 0.20. A 60-lb pull applied normal to the handle at A produces a clamping force of 1000 lb between the jaws of the vise. (a) Determine the frictional moment M_B, developed at B, due to the thrust of the screw against the body of the jaw. (b) Determine the force Q applied normal to the handle at A required to loosen the vise.



Solution. From the free-body diagram of the jaw we first obtain the tension T in the screw.

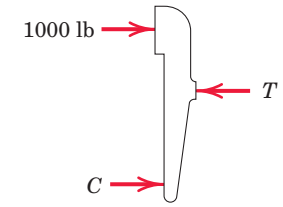
$$[\Sigma M_C = 0] \qquad 1000(16) - 10T = 0 \qquad T = 1600 \text{ lb}$$

The helix angle α and the friction angle ϕ for the thread are given by

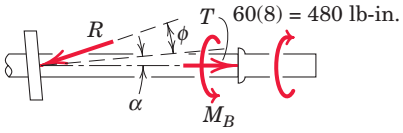
1 $$\alpha = \tan^{-1}\frac{L}{2\pi r} = \tan^{-1}\frac{1/5}{2\pi(0.5)} = 3.64°$$

$$\phi = \tan^{-1}\mu = \tan^{-1}0.20 = 11.31°$$

where the mean radius of the thread is $r = 0.5$ in.



(a) To tighten. The isolated screw is simulated by the free-body diagram shown where all of the forces acting on the threads of the screw are represented by a single force R inclined at the friction angle ϕ from the normal to the thread. The moment applied about the screw axis is $60(8) = 480$ lb-in. in the clockwise direction as seen from the front of the vise. The frictional moment M_B due to the friction forces acting on the collar at B is in the counterclockwise direction to oppose the impending motion. From Eq. 6/3 with T substituted for W the net moment acting on the screw is

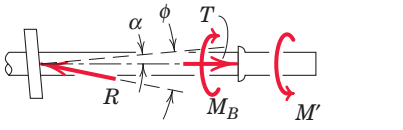


(a) To tighten

$$M = Tr\tan(\alpha + \phi)$$

$$480 - M_B = 1600(0.5)\tan(3.64° + 11.31°)$$

$$M_B = 266 \text{ lb-in.} \qquad \textit{Ans.}$$



(b) To loosen

(b) To loosen. The free-body diagram of the screw on the verge of being loosened is shown with R acting at the friction angle from the normal in the direction to counteract the impending motion. Also shown is the frictional moment $M_B = 266$ lb-in. acting in the clockwise direction to oppose the motion. The angle between R and the screw axis is now $\phi - \alpha$, and we use Eq. 6/3a with the net moment equal to the applied moment M' minus M_B. Thus

2

$$M = Tr\tan(\phi - \alpha)$$

$$M' - 266 = 1600(0.5)\tan(11.31° - 3.64°)$$

$$M' = 374 \text{ lb-in.}$$

Thus, the force on the handle required to loosen the vise is

$$Q = M'/d = 374/8 = 46.8 \text{ lb} \qquad \textit{Ans.}$$

Helpful Hints

1. Be careful to calculate the helix angle correctly. Its tangent is the lead L (advancement per revolution) divided by the mean circumference $2\pi r$ and not by the diameter $2r$.

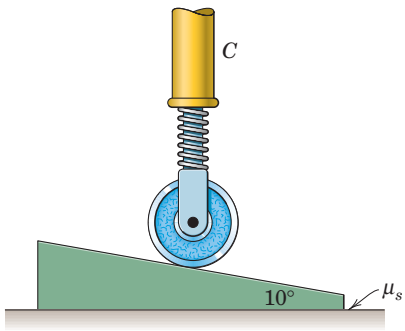
2. Note that R swings to the opposite side of the normal as the impending motion reverses direction.

PROBLEMS

(Unless otherwise instructed, neglect the weights of the wedges and screws in the problems which follow.)

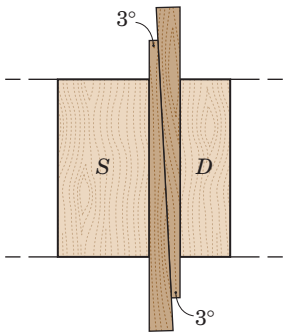
Introductory Problems

6/53 The 10° wedge is driven under the spring-loaded wheel whose supporting strut C is fixed. Determine the minimum coefficient of static friction μ_s for which the wedge will remain in place. Neglect all friction associated with the wheel.



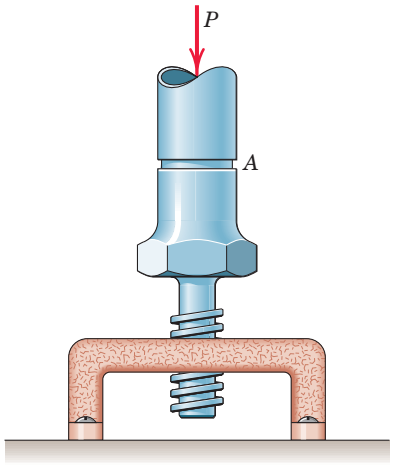
Problem 6/53

6/54 In wood-frame construction, two shims are frequently used to fill the gap between the framing S and the thinner window/door jamb D. The members S and D are shown in cross section in the figure. For the 3° shims shown, determine the minimum necessary coefficient of static friction necessary so that the shims will remain in place.



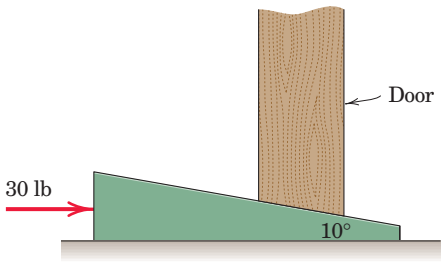
Problem 6/54

6/55 The device shown is used for coarse adjustment of the height of an experimental apparatus without a change in its horizontal position. Because of the slipjoint at A, turning the screw does not rotate the cylindrical leg above A. The mean diameter of the thread is $\frac{3}{8}$ in. and the coefficient of friction is 0.15. For a conservative design which neglects friction at the slipjoint, what should be the minimum number N of threads per inch to ensure that the single-threaded screw does not turn by itself under the weight of the apparatus?



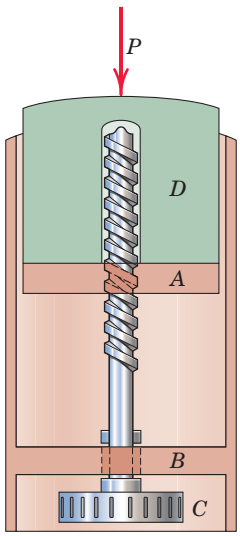
Problem 6/55

6/56 The 10° doorstop is inserted with a rightward horizontal force of 30 lb. If the coefficient of static friction for all surfaces is $\mu_s = 0.20$, determine the values N_U and N_L of the normal forces on the upper and lower faces of the doorstop. With the given information, can you determine the force P required to extract the doorstop?



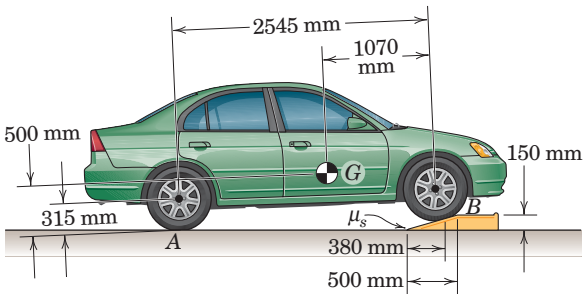
Problem 6/56

6/57 The elements of a deodorant dispenser are shown in the figure. To advance the deodorant D, the knob C is turned. The threaded shaft engages the threads in the movable deodorant base A; the shaft has no threads at the fixed support B. The 6-mm-diameter double square thread has a lead of 5 mm. Determine the coefficient of static friction μ_s for which the deodorant will not retract under the action of the force P. Neglect friction at B.



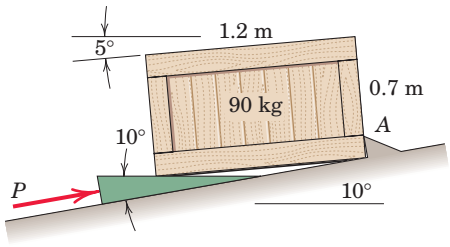
Problem 6/57

6/58 A 1600-kg rear-wheel-drive car is being driven up the ramp at a slow steady speed. Determine the minimum coefficient of static friction μ_s for which the portable ramp will not slip forward. Also determine the required friction force F_A at each rear drive wheel.



Problem 6/58

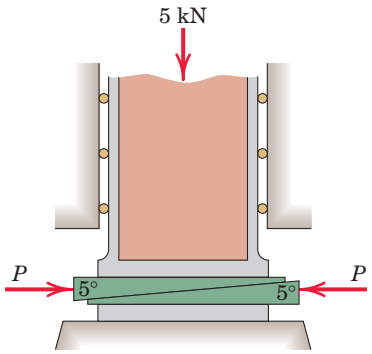
6/59 Determine the force P required to force the 10° wedge under the 90-kg uniform crate which rests against the small stop at A. The coefficient of friction for all surfaces is 0.40.



Problem 6/59

Representative Problems

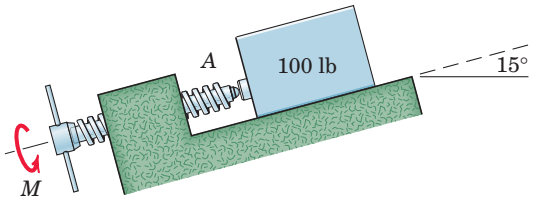
6/60 The two 5° wedges shown are used to adjust the position of the column under a vertical load of 5 kN. Determine the magnitude of the forces P required to raise the column if the coefficient of friction for all surfaces is 0.40.



Problem 6/60

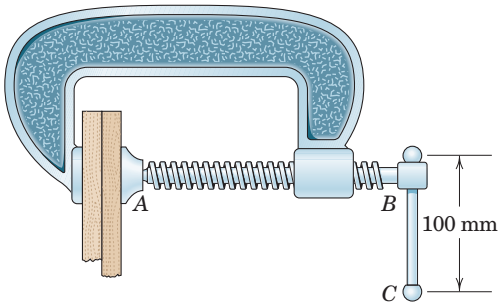
6/61 If the loaded column of Prob. 6/60 is to be lowered, calculate the horizontal forces P' required to withdraw the wedges.

6/62 Determine the torque M which must be applied to the handle of the screw to begin moving the 100-lb block up the 15° incline. The coefficient of static friction between the block and the incline is 0.50, and the single-thread screw has square threads with a mean diameter of 1 in. and advances 0.4 in. for each complete turn. The coefficient of static friction for the threads is also 0.50. Neglect friction at the small ball joint A.



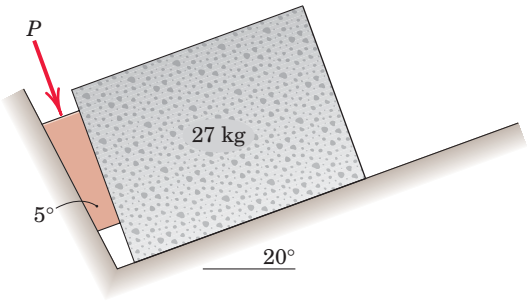
Problem 6/62

6/63 A compressive force of 600 N is to be applied to the two boards in the grip of the C-clamp. The threaded screw has a mean diameter of 10 mm and advances 2.5 mm per turn. The coefficient of static friction is 0.20. Determine the force F which must be applied normal to the handle at C in order to (a) tighten and (b) loosen the clamp. Neglect friction at point A.



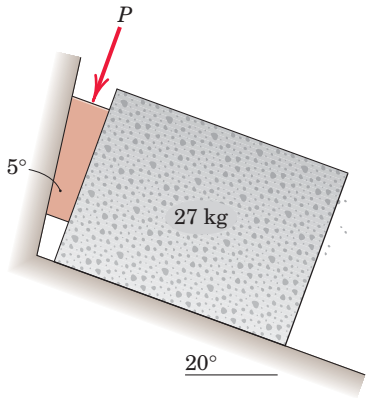
Problem 6/63

6/64 The coefficient of static friction for both wedge surfaces is 0.40 and that between the 27-kg concrete block and the 20° incline is 0.70. Determine the minimum value of the force P required to begin moving the block up the incline. Neglect the weight of the wedge.



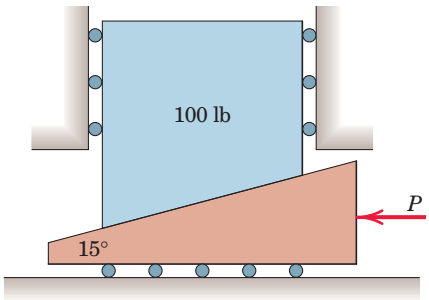
Problem 6/64

6/65 Repeat Prob. 6/64, only now the 27-kg concrete block begins to move down the 20° incline as shown. All other conditions remain as in Prob. 6/64.



Problem 6/65

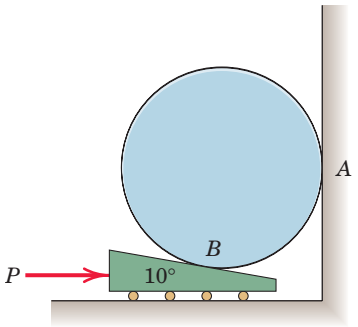
6/66 The coefficient of static friction μ_s between the 100-lb body and the 15° wedge is 0.20. Determine the magnitude of the force P required to begin raising the 100-lb body if (a) rollers of negligible friction are present under the wedge, as illustrated, and (b) the rollers are removed and the coefficient of static friction $\mu_s = 0.20$ applies at this surface as well.



Problem 6/66

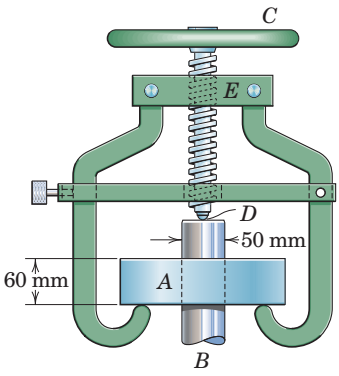
6/67 For both conditions (*a*) and (*b*) as stated in Prob. 6/66, determine the magnitude and direction of the force P' required to begin lowering the 100-lb body.

6/68 Calculate the horizontal force P on the light 10° wedge necessary to initiate movement of the 40-kg cylinder. The coefficient of static friction for both pairs of contacting surfaces is 0.25. Also determine the friction force F_B at point B. (*Caution:* Check carefully your assumption of where slipping occurs.)



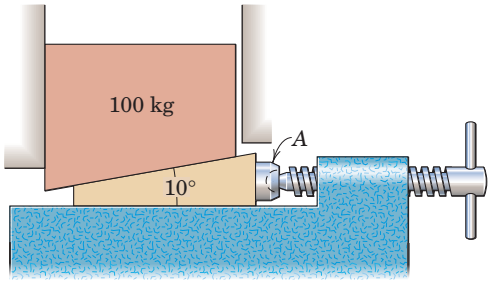
Problem 6/68

6/69 The collar A has a force fit on shaft B and is to be removed from the shaft by the wheel-puller mechanism shown. The screw has a single square thread with a mean diameter of 20 mm and a lead L of 6 mm. If a torque of 24 N·m is required to turn wheel C to slip the collar off the shaft, determine the average pressure p (compressive stress) between the collar and the shaft. The coefficient of friction for the screw at E is 0.25, and that for the shaft and collar is 0.30. Friction at the ball end D of the shaft is negligible.



Problem 6/69

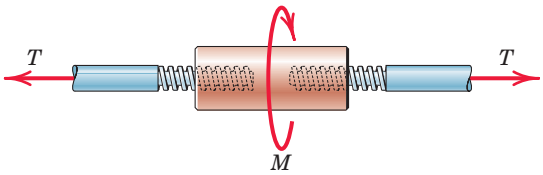
6/70 The vertical position of the 100-kg block is adjusted by the screw-activated wedge. Calculate the moment M which must be applied to the handle of the screw to raise the block. The single-thread screw has square threads with a mean diameter of 30 mm and advances 10 mm for each complete turn. The coefficient of friction for the screw threads is 0.25, and the coefficient of friction for all mating surfaces of the block and wedge is 0.40. Neglect friction at the ball joint A.



Problem 6/70

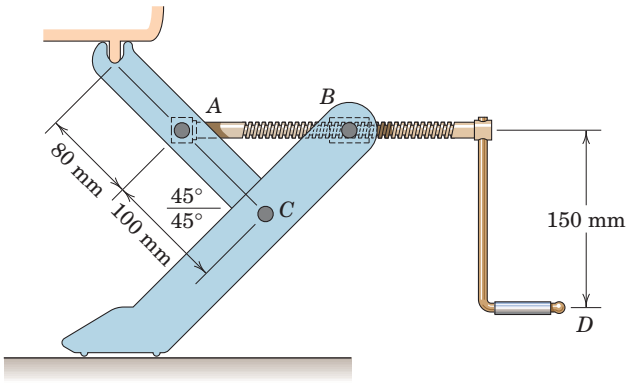
6/71 Calculate the moment M' which must be applied to the handle of the screw of Prob. 6/70 to withdraw the wedge and lower the 100-kg load.

6/72 The threaded collar is used to connect two shafts, both with right-hand threads on their ends. The shafts are under a tension $T = 8$ kN. If the threads have a mean diameter of 16 mm and a lead of 4 mm, calculate the torque M required to turn the collar in either direction with the shafts prevented from turning. The coefficient of friction is 0.24.



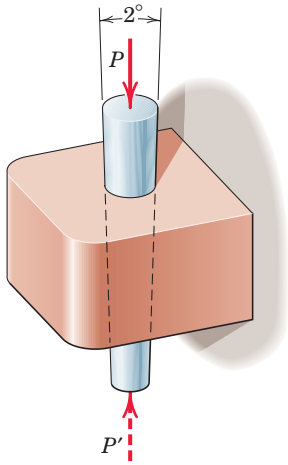
Problem 6/72

6/73 The jack shown is designed to lift small unit-body cars. The screw is threaded into the collar pivoted at B, and the shaft turns in a ball thrust bearing at A. The thread has a mean diameter of 10 mm and a lead (advancement per revolution) of 2 mm. The coefficient of friction for the threads is 0.20. Determine the force P normal to the handle at D required (a) to raise a mass of 500 kg from the position shown and (b) to lower the load from the same position. Neglect friction in the pivot and bearing at A.



Problem 6/73

▶6/74 The 2° tapered pin is forced into a mating tapered hole in the fixed block with a force $P = 400$ N. If the force required to remove the pin (with $P = 0$) is $P' = 300$ N, determine the coefficient of friction between the pin and the surface of the hole. (*Hint:* The pressure (stress) normal to the tapered pin surface remains unchanged until the pin actually moves. The distributed forces over the surface of the pin may be replaced by an equivalent single resultant force.)



Problem 6/74

6/6 Journal Bearings

A *journal bearing* is one which gives lateral support to a shaft in contrast to axial or thrust support. For dry bearings and for many partially lubricated bearings we may apply the principles of dry friction. These principles provide a satisfactory approximation for design purposes.

A dry or partially lubricated journal bearing with contact or near contact between the shaft and the bearing is shown in Fig. 6/8, where the clearance between the shaft and bearing is greatly exaggerated to clarify the action. As the shaft begins to turn in the direction shown, it will roll up the inner surface of the bearing until it slips. Here it will remain in a more or less fixed position during rotation. The torque M required to maintain rotation and the radial load L on the shaft will cause

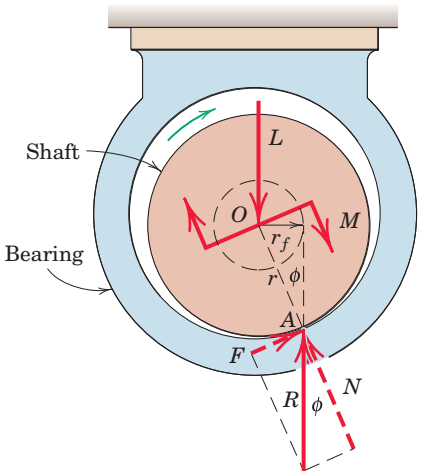


Figure 6/8

a reaction R at the contact point A. For vertical equilibrium R must equal L but will not be collinear with it. Thus, R will be tangent to a small circle of radius r_f called the *friction circle*. The angle between R and its normal component N is the friction angle ϕ. Equating the sum of the moments about A to zero gives

$$M = Lr_f = Lr \sin \phi \qquad \textbf{(6/4)}$$

For a small coefficient of friction, the angle ϕ is small, and the sine and tangent may be interchanged with only small error. Since $\mu = \tan \phi$, a good approximation to the torque is

$$M = \mu Lr \qquad \textbf{(6/4}\boldsymbol{a}\textbf{)}$$

This relation gives the amount of torque or moment which must be applied to the shaft to overcome friction for a dry or partially lubricated journal bearing.

6/7 Thrust Bearings; Disk Friction

Friction between circular surfaces under distributed normal pressure occurs in pivot bearings, clutch plates, and disk brakes. To examine these applications, we consider the two flat circular disks shown in Fig. 6/9. Their shafts are mounted in bearings (not shown) so that they can be brought into contact under the axial force P. The maximum torque which this clutch can transmit is equal to the torque M required to slip one disk against the other. If p is the normal pressure at any location between the plates, the frictional force acting on an elemental area is $\mu p\, dA$, where μ is the friction coefficient and dA is the area $r\, dr\, d\theta$ of the element. The moment of this elemental friction force about the shaft axis is $\mu pr\, dA$, and the total moment becomes

$$M = \int \mu pr\, dA$$

where we evaluate the integral over the area of the disk. To carry out this integration, we must know the variation of μ and p with r.

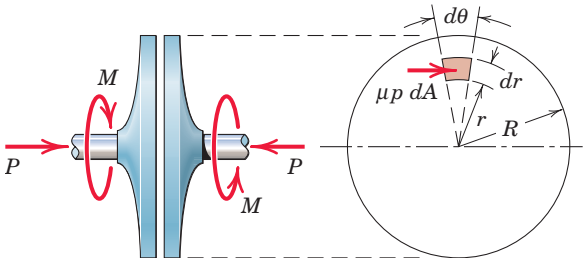


Figure 6/9

In the following examples we will assume that μ is constant. Furthermore, if the surfaces are new, flat, and well supported, it is reasonable to assume that the pressure p is uniform over the entire surface so that $\pi R^2 p = P$. Substituting the constant value of p in the expression for M gives

$$M = \frac{\mu P}{\pi R^2} \int_0^{2\pi} \int_0^R r^2\, dr\, d\theta = \tfrac{2}{3}\mu PR \qquad \textbf{(6/5)}$$

We may interpret this result as equivalent to the moment due to a friction force μP acting at a distance $\frac{2}{3}R$ from the shaft center.

If the friction disks are rings, as in the collar bearing shown in Fig. 6/10, the limits of integration are the inside and outside radii R_i and R_o, respectively, and the frictional torque becomes

$$M = \tfrac{2}{3}\mu P \frac{R_o^3 - R_i^3}{R_o^2 - R_i^2} \qquad \textbf{(6/5}\boldsymbol{a}\textbf{)}$$

After the initial wearing-in period is over, the surfaces retain their new relative shape and further wear is therefore constant over the surface. This wear depends on both the circumferential distance traveled

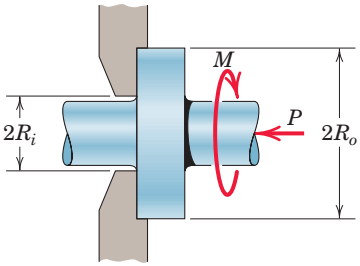


Figure 6/10

and the pressure p. Since the distance traveled is proportional to r, the expression $rp = K$ may be written, where K is a constant. The value of K is determined from the equilibrium condition for the axial forces, which gives

$$P = \int p\, dA = K \int_0^{2\pi} \int_0^R dr\, d\theta = 2\pi KR$$

With $pr = K = P/(2\pi R)$, we may write the expression for M as

$$M = \int \mu pr\, dA = \frac{\mu P}{2\pi R} \int_0^{2\pi} \int_0^R r\, dr\, d\theta$$

which becomes

$$M = \tfrac{1}{2}\mu PR \qquad \textbf{(6/6)}$$

The frictional moment for worn-in plates is, therefore, only $(\frac{1}{2})/(\frac{2}{3})$, or $\frac{3}{4}$ as much as for new surfaces. If the friction disks are rings of inside radius R_i and outside radius R_o, substitution of these limits gives for the frictional torque for worn-in surfaces

$$M = \tfrac{1}{2}\mu P(R_o + R_i) \qquad \textbf{(6/6}\boldsymbol{a}\textbf{)}$$

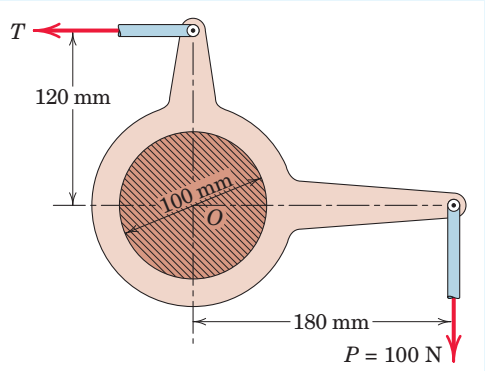
You should be prepared to deal with other disk-friction problems where the pressure p is some other function of r.

© Courtesy of General Motors/Zuma Press

A typical automotive disk brake

SAMPLE PROBLEM 6/8

The bell crank fits over a 100-mm-diameter shaft which is fixed and cannot rotate. The horizontal force T is applied to maintain equilibrium of the crank under the action of the vertical force $P = 100$ N. Determine the maximum and minimum values which T may have without causing the crank to rotate in either direction. The coefficient of static friction μ between the shaft and the bearing surface of the crank is 0.20.



Solution. Impending rotation occurs when the reaction R of the fixed shaft on the bell crank makes an angle $\phi = \tan^{-1}\mu$ with the normal to the bearing surface and is, therefore, tangent to the friction circle. Also, equilibrium requires that the three forces acting on the crank be concurrent at point C. These facts are shown in the free-body diagrams for the two cases of impending motion.

The following calculations are needed:

$$\text{Friction angle } \phi = \tan^{-1}\mu = \tan^{-1} 0.20 = 11.31°$$

$$\text{Radius of friction circle } r_f = r \sin\phi = 50 \sin 11.31° = 9.81 \text{ mm}$$

$$\text{Angle } \theta = \tan^{-1}\frac{120}{180} = 33.7°$$

$$\text{Angle } \beta = \sin^{-1}\frac{r_f}{\overline{OC}} = \sin^{-1}\frac{9.81}{\sqrt{(120)^2 + (180)^2}} = 2.60°$$

(a) Impending counterclockwise motion. The equilibrium triangle of forces is drawn and gives

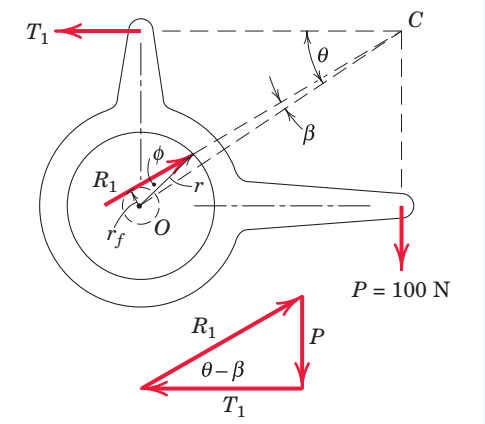
$$T_1 = P \cot(\theta - \beta) = 100 \cot(33.7° - 2.60°)$$

$$T_1 = T_{\max} = 165.8 \text{ N} \qquad \textit{Ans.}$$

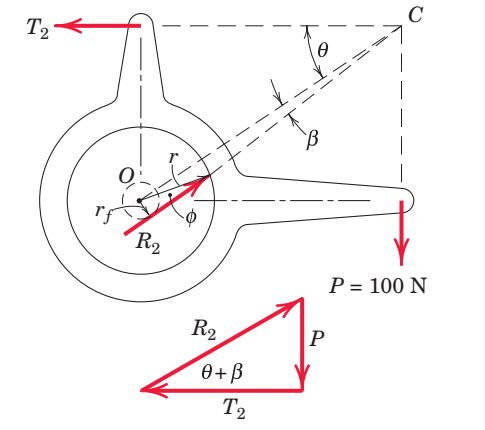
(b) Impending clockwise motion. The equilibrium triangle of forces for this case gives

$$T_2 = P \cot(\theta + \beta) = 100 \cot(33.7° + 2.60°)$$

$$T_2 = T_{\min} = 136.2 \text{ N} \qquad \textit{Ans.}$$



(*a*) Counterclockwise motion impends

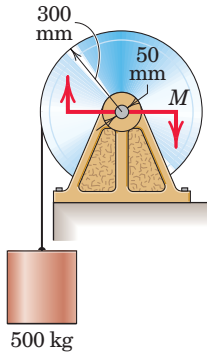


(*b*) Clockwise motion impends

PROBLEMS

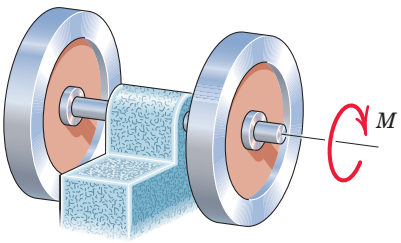
Introductory Problems

6/75 A torque M of 1510 N·m must be applied to the 50-mm-diameter shaft of the hoisting drum to raise the 500-kg load at constant speed. The drum and shaft together have a mass of 100 kg. Calculate the coefficient of friction μ for the bearing.



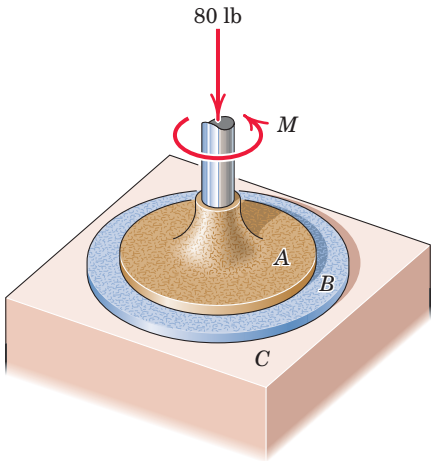
Problem 6/75

6/76 The two flywheels are mounted on a common shaft which is supported by a journal bearing between them. Each flywheel has a mass of 40 kg, and the diameter of the shaft is 40 mm. If a 3-N·m couple M on the shaft is required to maintain rotation of the flywheels and shaft at a constant low speed, compute (*a*) the coefficient of friction in the bearing and (*b*) the radius r_f of the friction circle.



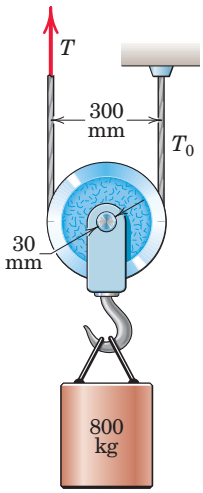
Problem 6/76

6/77 Circular disk A is placed on top of disk B and is subjected to a compressive force of 80 lb. The diameters of A and B are 9 in. and 12 in., respectively, and the pressure under each disk is constant over its surface. If the coefficient of friction between A and B is 0.40, determine the couple M which will cause A to slip on B. Also, what is the minimum coefficient of friction μ between B and the supporting surface C which will prevent B from rotating?



Problem 6/77

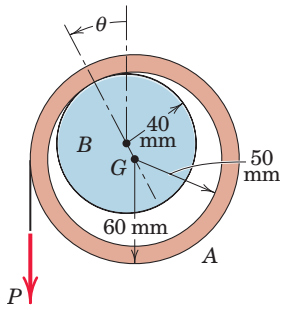
6/78 Determine the tension T in the cable to raise the 800-kg load if the coefficient of friction for the 30-mm bearing is 0.25. Also find the tension T_0 in the stationary section of the cable. The mass of the cable and pulley is small and may be neglected.



Problem 6/78

6/79 Calculate the tension T required to lower the 800-kg load described in Prob. 6/78. Also find T_0.

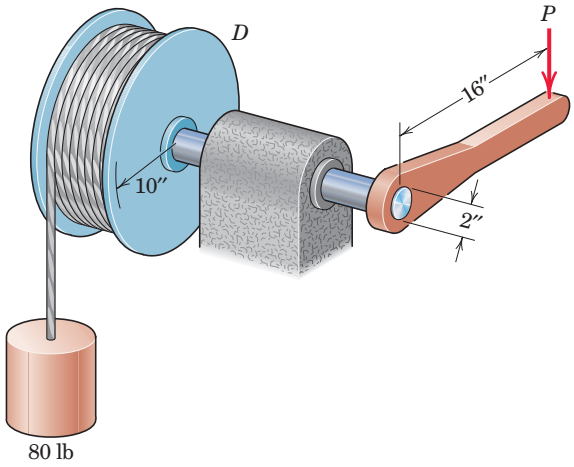
6/80 The 20-kg steel ring A with inside and outside radii of 50 mm and 60 mm, respectively, rests on a fixed horizontal shaft of 40-mm radius. If a downward force $P = 150$ N applied to the periphery of the ring is just sufficient to cause the ring to slip, calculate the coefficient of friction μ and the angle θ.



Problem 6/80

Representative Problems

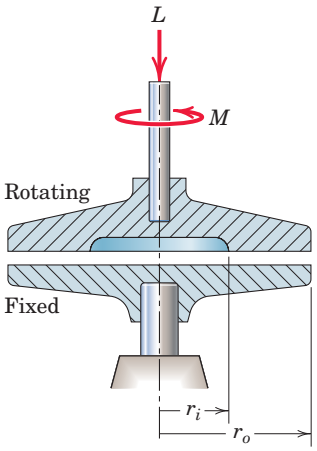
6/81 The weight of the drum D and its cable is 100 lb, and the coefficient of friction μ for the bearing is 0.20. Determine the force P required to raise the 80-lb cylinder if the bearing friction is (*a*) neglected and (*b*) included in the analysis. The weight of the shaft is negligible.



Problem 6/81

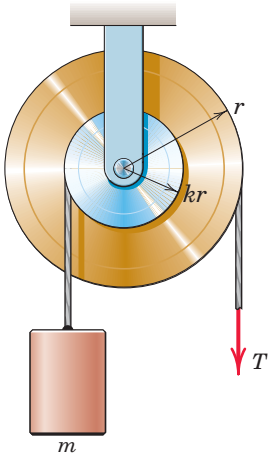
6/82 Determine the force P required to lower the 80-lb cylinder of Prob. 6/81. Compare your answer with the stated results of that problem. Is the no-friction value of P equal to the average of the forces required to raise and lower the cylinder?

6/83 The axial section of the two mating circular disks is shown. Derive the expression for the torque M required to turn the upper disk on the fixed lower one if the pressure p between the disks follows the relation $p = k/r^2$, where k is a constant to be determined. The coefficient of friction μ is constant over the entire surface.



Problem 6/83

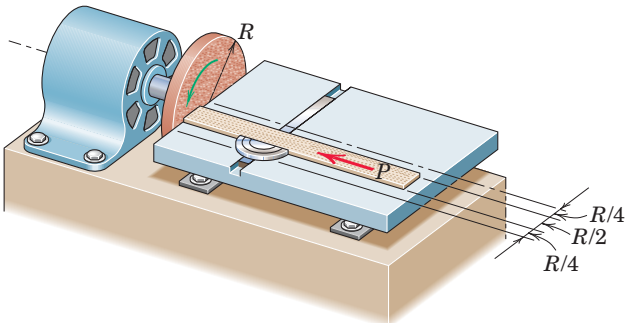
6/84 The two pulleys are fastened together and are used to hoist the cylinder of mass m. The fraction k can vary from near zero to one. Derive an expression for the tension T required to raise the cylinder at a steady speed if the coefficient of friction for the bearing of radius r_0 is μ, a small enough value to permit the substitution of μ for $\sin\phi$, where ϕ is the friction angle. The mass of the pulley unit is m_0. Evaluate your expression for T if $m = 50$ kg, $m_0 = 30$ kg, $r = 0.3$ m, $k = \frac{1}{2}$, $r_0 = 25$ mm, and $\mu = 0.15$.



Problem 6/84

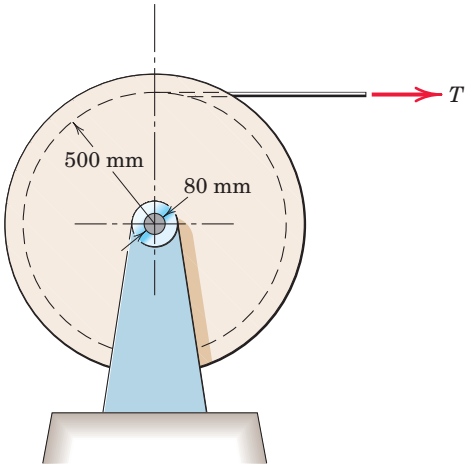
6/85 Repeat Prob. 6/84 for the case of lowering the cylinder of mass m at a steady speed.

6/86 An end of the thin board is being sanded by the disk sander under application of the force P. If the effective coefficient of kinetic friction is μ and if the pressure is essentially constant over the board end, determine the moment M which must be applied by the motor in order to rotate the disk at a constant angular speed. The board end is centered along the radius of the disk.



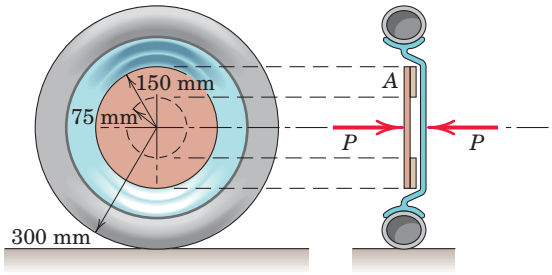
Problem 6/86

6/87 The telephone-cable reel has a mass of 250 kg and is mounted on an 80-mm-diameter shaft. If the coefficient of friction between the shaft and its bearing is 0.30, calculate the horizontal tension T required to turn the reel.



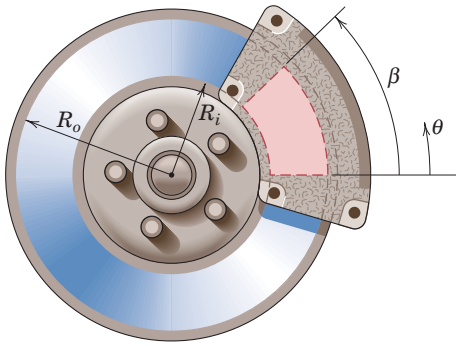
Problem 6/87

6/88 The front wheels of an experimental rear-drive vehicle have a radius of 300 mm and are designed with disk-type brakes consisting of a ring A with outside and inside radii of 150 mm and 75 mm, respectively. The ring, which does not turn with the wheel, is forced against the wheel disk with a force P. If the pressure between the ring and the wheel disk is uniform over the mating surfaces, compute the friction force F between each front tire and the horizontal road for an axial force $P = 1$ kN when the vehicle is powered at constant speed with the wheels turning. The coefficient of friction between the disk and ring is 0.35.



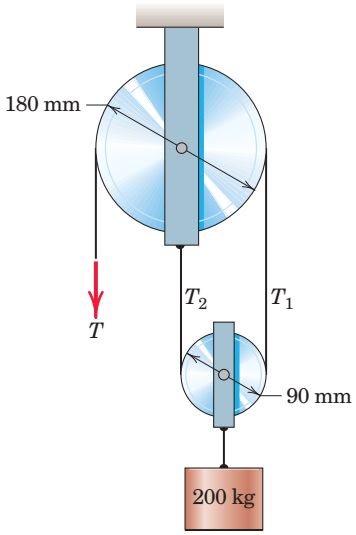
Problem 6/88

6/89 An automobile disk brake consists of a flat-faced rotor and a caliper which contains a disk pad on each side of the rotor. For equal forces P behind the two pads with the pressure p uniform over the pad, show that the moment applied to the hub is independent of the angular span β of the pads. Would pressure variation with θ change the moment?



Problem 6/89

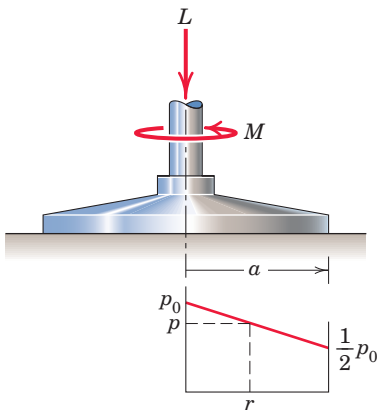
6/90 The pulley system shown is used to hoist the 200-kg block. The diameter of the bearing for the upper pulley is 20 mm, and that for the lower pulley is 12 mm. For a coefficient of friction $\mu = 0.25$ for both bearings, calculate the tensions T, T_1, and T_2 in the three cables if the block is being raised slowly.



Problem 6/90

6/91 Calculate the tensions T, T_1, and T_2 for Prob. 6/90 if the block is being lowered slowly.

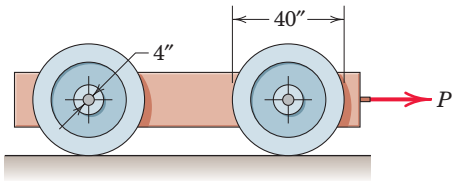
6/92 For the flat sanding disk of radius a, the pressure p developed between the disk and the sanded surface decreases linearly with r from a value p_0 at the center to $p_0/2$ at $r = a$. If the coefficient of friction is μ, derive the expression for the torque M required to turn the shaft under an axial force L.



Problem 6/92

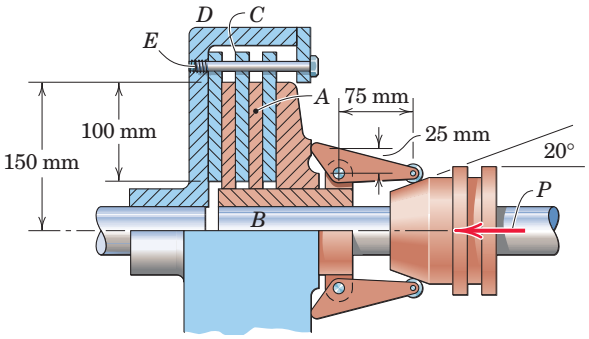
6/93 In addition to the radial pressure variation of Prob. 6/92, assume that the coefficient of friction μ decreases linearly with r from a value μ_0 at the center to $\mu_0/2$ at $r = a$. Derive an expression for the torque M required to turn the shaft under an axial load L and compare with the stated result for Prob. 6/92.

6/94 Each of the four wheels of the vehicle weighs 40 lb and is mounted on a 4-in.-diameter journal (shaft). The total weight of the vehicle is 960 lb, including wheels, and is distributed equally on all four wheels. If a force $P = 16$ lb is required to keep the vehicle rolling at a constant low speed on a horizontal surface, calculate the coefficient of friction which exists in the wheel bearings. (*Hint:* Draw a complete free-body diagram of one wheel.)



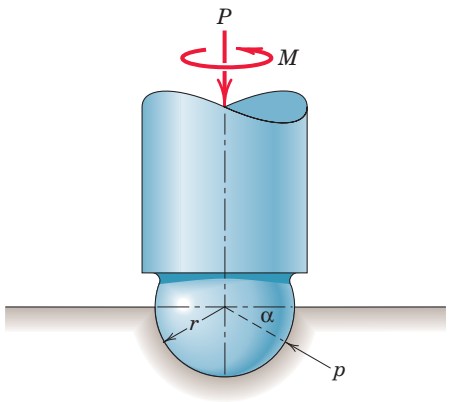
Problem 6/94

6/95 In the figure is shown a multiple-disk clutch designed for marine use. The driving disks A are splined to the driving shaft B so that they are free to slip along the shaft but must rotate with it. The disks C drive the housing D by means of the bolts E, along which they are free to slide. In the clutch shown there are five pairs of friction surfaces. Assume the pressure is uniformly distributed over the area of the disks and determine the maximum torque M which can be transmitted if the coefficient of friction is 0.15 and $P = 500$ N.



Problem 6/95

▶6/96 The hemispherical thrust bearing on the end of the shaft supports an axial force P. Derive the expression for the moment M required to rotate the shaft at constant speed if the pressure p is proportional to $\sin \alpha$ and the coefficient of friction is μ.



Problem 6/96

6/8 Flexible Belts

The impending slippage of flexible cables, belts, and ropes over sheaves and drums is important in the design of belt drives of all types, band brakes, and hoisting rigs.

Figure 6/11*a* shows a drum subjected to the two belt tensions T_1 and T_2, the torque M necessary to prevent rotation, and a bearing reaction R. With M in the direction shown, T_2 is greater than T_1. The free-body diagram of an element of the belt of length $r\,d\theta$ is shown in part *b* of the figure. We analyze the forces acting on this differential element by establishing the equilibrium of the element, in a manner similar to that used for other variable-force problems. The tension increases from T at the angle θ to $T + dT$ at the angle $\theta + d\theta$. The normal force is a differential dN, since it acts on a differential element of area. Likewise the friction force, which must act on the belt in a direction to oppose slipping, is a differential and is $\mu\,dN$ for impending motion.

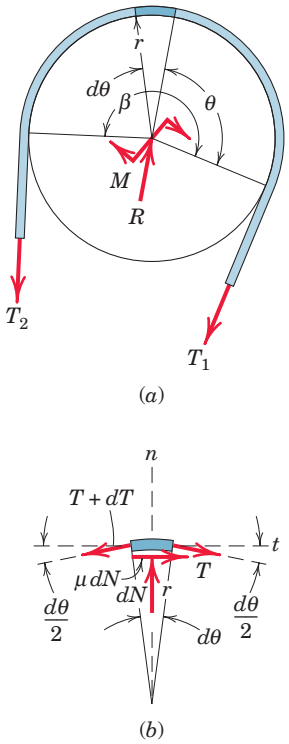


Figure 6/11

Equilibrium in the t-direction gives

$$T\cos\frac{d\theta}{2} + \mu\,dN = (T + dT)\cos\frac{d\theta}{2}$$

or

$$\mu\,dN = dT$$

since the cosine of a differential quantity is unity in the limit. Equilibrium in the n-direction requires that

$$dN = (T + dT)\sin\frac{d\theta}{2} + T\sin\frac{d\theta}{2}$$

or

$$dN = T\,d\theta$$

where we have used the facts that the sine of a differential angle in the limit equals the angle and that the product of two differentials must be neglected in the limit compared with the first-order differentials remaining.

Combining the two equilibrium relations gives

$$\frac{dT}{T} = \mu\,d\theta$$

Integrating between corresponding limits yields

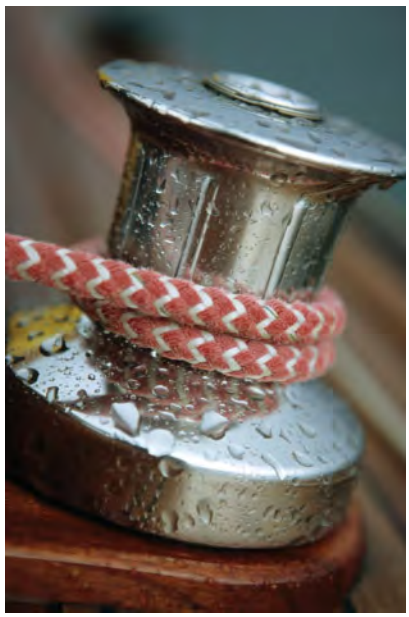
$$\int_{T_1}^{T_2}\frac{dT}{T} = \int_0^{\beta}\mu\,d\theta$$

or

$$\ln\frac{T_2}{T_1} = \mu\beta$$

where the $\ln(T_2/T_1)$ is a natural logarithm (base e). Solving for T_2 gives

$$T_2 = T_1 e^{\mu\beta} \qquad \textbf{(6/7)}$$



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Just one turn of a line around a fixed cylinder can produce a large change in tension.

Note that β is the total angle of belt contact and must be expressed in radians. If a rope were wrapped around a drum n times, the angle β would be $2\pi n$ radians. Equation 6/7 holds equally well for a noncircular section where the total angle of contact is β. This conclusion is evident from the fact that the radius r of the circular drum in Fig. 6/11 does not enter into the equations for the equilibrium of the differential element of the belt.

The relation expressed by Eq. 6/7 also applies to belt drives where both the belt and the pulley are rotating at constant speed. In this case the equation describes the ratio of belt tensions for slippage or impending slippage. When the speed of rotation becomes large, the belt tends to leave the rim, so Eq. 6/7 involves some error in this case.

6/9 Rolling Resistance

Deformation at the point of contact between a rolling wheel and its supporting surface introduces a resistance to rolling, which we mention only briefly. This resistance is not due to tangential friction forces and therefore is an entirely different phenomenon from that of dry friction.

To describe rolling resistance, we consider the wheel shown in Fig. 6/12 under the action of a load L on the axle and a force P applied at its center to produce rolling. The deformation of the wheel and supporting surfaces as shown is greatly exaggerated. The distribution of pressure p over the area of contact is similar to the distribution shown. The resultant R of this distribution acts at some point A and must pass through the wheel center for the wheel to be in equilibrium. We find the force P necessary to maintain rolling at constant speed by equating the moments of all forces about A to zero. This gives us

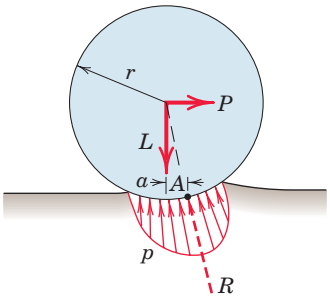


Figure 6/12

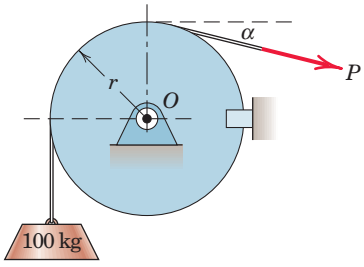
$$P = \frac{a}{r} L = \mu_r L$$

where the moment arm of P is taken to be r. The ratio $\mu_r = a/r$ is called the *coefficient of rolling resistance*. This coefficient is the ratio of resisting force to normal force and thus is analogous to the coefficient of static or kinetic friction. On the other hand, there is no slippage or impending slippage in the interpretation of μ_r.

Because the dimension a depends on many factors which are difficult to quantify, a comprehensive theory of rolling resistance is not available. The distance a is a function of the elastic and plastic properties of the mating materials, the radius of the wheel, the speed of travel, and the roughness of the surfaces. Some tests indicate that a varies only slightly with wheel radius, and thus a is often taken to be independent of the rolling radius. Unfortunately, the quantity a has also been called the coefficient of rolling friction in some references. However, a has the dimension of length and therefore is not a dimensionless coefficient in the usual sense.

SAMPLE PROBLEM 6/9

A flexible cable which supports the 100-kg load is passed over a fixed circular drum and subjected to a force P to maintain equilibrium. The coefficient of static friction μ between the cable and the fixed drum is 0.30. (a) For $\alpha = 0$, determine the maximum and minimum values which P may have in order not to raise or lower the load. (b) For $P = 500$ N, determine the minimum value which the angle α may have before the load begins to slip.



Solution. Impending slipping of the cable over the fixed drum is given by Eq. 6/7, which is $T_2/T_1 = e^{\mu\beta}$.

1 ***(a)*** With $\alpha = 0$ the angle of contact is $\beta = \pi/2$ rad. For impending upward motion of the load, $T_2 = P_{max}$, $T_1 = 981$ N, and we have

2 $$P_{max}/981 = e^{0.30(\pi/2)} \qquad P_{max} = 981(1.602) = 1572 \text{ N} \qquad \textit{Ans.}$$

For impending downward motion of the load, $T_2 = 981$ N and $T_1 = P_{min}$. Thus,

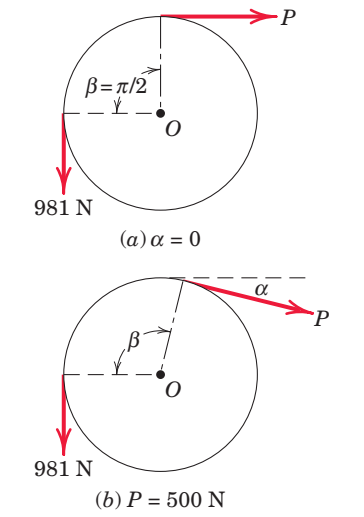
$$981/P_{min} = e^{0.30(\pi/2)} \qquad P_{min} = 981/1.602 = 612 \text{ N} \qquad \textit{Ans.}$$

(b) With $T_2 = 981$ N and $T_1 = P = 500$ N, Eq. 6/7 gives us

$$981/500 = e^{0.30\beta} \qquad 0.30\beta = \ln(981/500) = 0.674$$

$$\beta = 2.25 \text{ rad} \qquad \text{or} \qquad \beta = 2.25\left(\frac{360}{2\pi}\right) = 128.7°$$

3 $$\alpha = 128.7° - 90° = 38.7° \qquad \textit{Ans.}$$



Helpful Hints

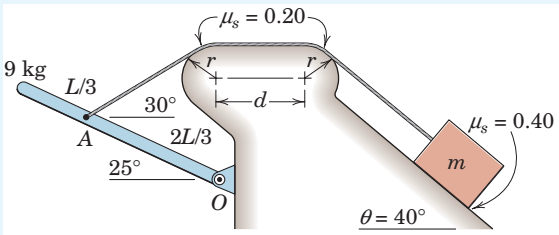
1 We are careful to note that β must be expressed in radians.

2 In our derivation of Eq. 6/7 be certain to note that $T_2 > T_1$.

3 As was noted in the derivation of Eq. 6/7, the radius of the drum does not enter into the calculations. It is only the angle of contact and the coefficient of friction which determine the limiting conditions for impending motion of the flexible cable over the curved surface.

SAMPLE PROBLEM 6/10

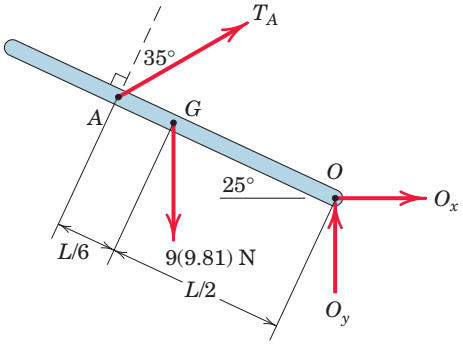
Determine the range of mass m over which the system is in static equilibrium. The coefficient of static friction between the cord and the upper curved surface is 0.20, while that between the block and the incline is 0.40. Neglect friction at the pivot O.



Solution. From the FBD of the uniform slender bar, we can determine the tension T_A in the cable at point A.

$$[\Sigma M_O = 0] \qquad -T_A\left(\frac{2L}{3}\cos 35°\right) + 9(9.81)\left(\frac{L}{2}\cos 25°\right) = 0$$

$$T_A = 73.3 \text{ N}$$



I. Motion of m impends up the incline.

The tension $T_A = 73.3$ N is the larger of the two tensions associated with the rough rounded surface. From Eq. 6/7 we have

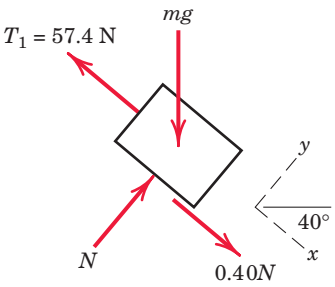
(1) $[T_2 = T_1 e^{\mu_s \beta}] \qquad 73.3 = T_1 e^{0.20[30° + 40°]\pi/180°} \qquad T_1 = 57.4 \text{ N}$

From the FBD of the block for Case I:

$$[\Sigma F_y = 0] \qquad N - mg\cos 40° = 0 \qquad N = 0.766mg$$

$$[\Sigma F_x = 0] \qquad -57.4 + mg\sin 40° + 0.40(0.766mg) = 0$$

$$mg = 60.5 \text{ N} \qquad m = 6.16 \text{ kg}$$



Case I

II. Motion of m impends down the incline.

The value $T_A = 73.3$ N is unchanged, but now this is the smaller of the two tensions in Eq. 6/7.

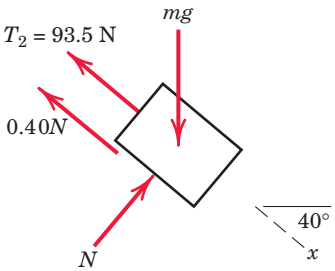
$$[T_2 = T_1 e^{\mu_s \beta}] \qquad T_2 = 73.3e^{0.20[30° + 40°]\pi/180°} \qquad T_2 = 93.5 \text{ N}$$

Considering the FBD of the block for Case II, we see that the normal force N is unchanged from Case I.

$$[\Sigma F_x = 0] \qquad -93.5 - 0.4(0.766mg) + mg\sin 40° = 0$$

$$mg = 278 \text{ N} \qquad m = 28.3 \text{ kg}$$

(2) So the requested range is $6.16 \le m \le 28.3$ kg. *Ans.*



Case II

Helpful Hints

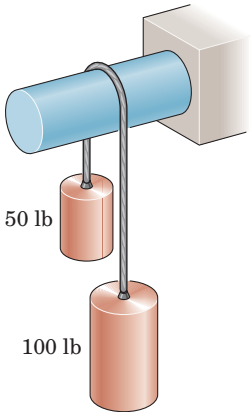
(1) Only the total angular contact enters Eq. 6/7 (as β). So our results are independent of the quantities r and d.

(2) Re-solve the entire problem if the ramp angle θ were changed to 20°, with all other given information remaining constant. Be alert for a surprising result!

PROBLEMS

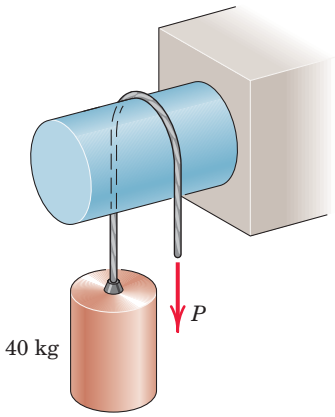
Introductory Problems

6/97 What is the minimum coefficient of friction μ between the rope and the fixed shaft which will prevent the unbalanced cylinders from moving?



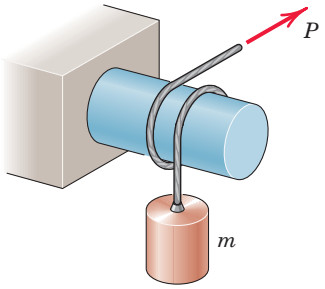
Problem 6/97

6/98 Determine the force P required to (a) raise and (b) lower the 40-kg cylinder at a slow steady speed. The coefficient or friction between the cord and its supporting surface is 0.30.



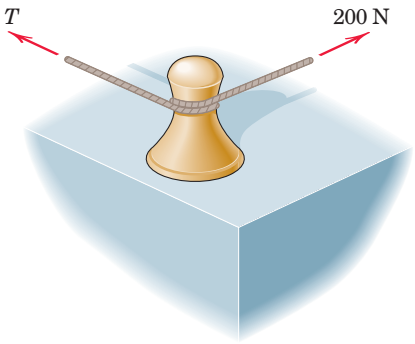
Problem 6/98

6/99 A force $P = mg/6$ is required to lower the cylinder at a constant slow speed with the cord making $1\frac{1}{4}$ turns around the fixed shaft. Calculate the coefficient of friction μ between the cord and the shaft.



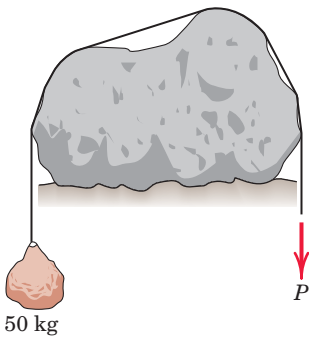
Problem 6/99

6/100 A dockworker adjusts a spring line (rope) which keeps a ship from drifting alongside a wharf. If he exerts a pull of 200 N on the rope, which has $1\frac{1}{4}$ turns around the mooring bit, what force T can he support? The coefficient of friction between the rope and the cast-steel mooring bit is 0.30.



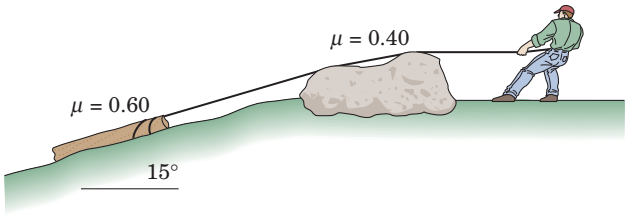
Problem 6/100

6/101 A 50-kg package is attached to a rope which passes over an irregularly shaped boulder with uniform surface texture. If a downward force $P = 70$ N is required to lower the package at a constant rate, (a) determine the coefficient of friction μ between the rope and the boulder. (b) What force P' would be required to raise the package at a constant rate?



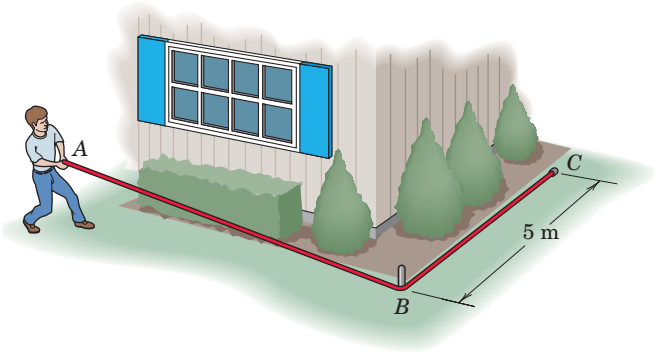
Problem 6/101

6/102 With what horizontal force P must the worker pull in order to move the 35-kg log up the 15° incline? The coefficient of friction between the log and the incline is 0.60 and that between the rope and the boulder is 0.40.



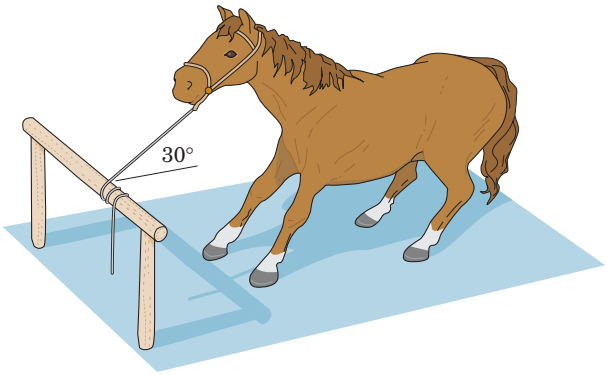
Problem 6/102

6/103 A garden hose with a mass of 1.2 kg/m is in full contact with the ground from B to C. What is the horizontal component P_x of the force which the gardener must exert in order to pull the hose around the small cylindrical guard at B? The coefficient of friction between the hose and the ground is 0.50, and that between the hose and the cylinder is 0.40. Assume that the hose does not touch the ground between A and B.



Problem 6/103

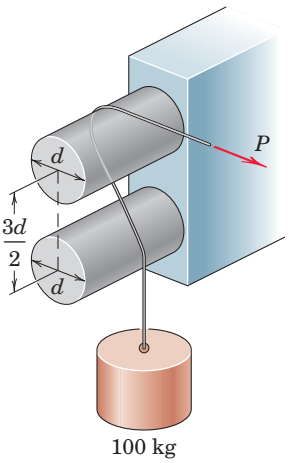
6/104 In western movies, cowboys are frequently observed hitching their horses by casually winding a few turns of the reins around a horizontal pole and letting the end hang free as shown—no knots! If the freely hanging length of rein weighs 2 oz and the number of turns is as shown, what tension T does the horse have to produce in the direction shown in order to gain freedom? The coefficient of friction between the reins and wooden pole is 0.70.



Problem 6/104

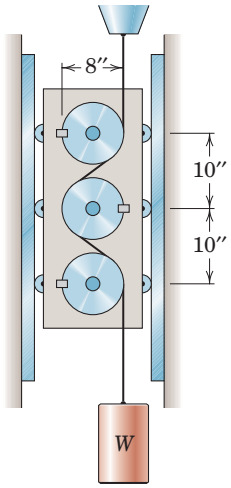
Representative Problems

6/105 Calculate the horizontal force P required to raise the 100-kg load. The coefficient of friction between the rope and the fixed bars is 0.40.



Problem 6/105

6/106 The vertical consists of a frame, three fixed 8-in.-diameter drums, and six small guide wheels with a combined weight of 50 lb. Determine the minimum weight W necessary to hold the trolley in equilibrium for any position in its vertical guides. The coefficient of friction between the light cable and the drums is 0.30.

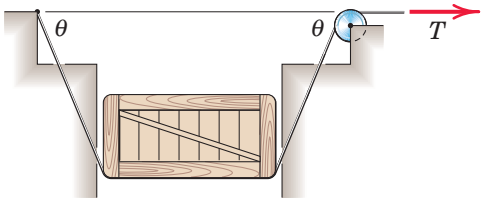


Problem 6/106

6/107 The 180-lb tree surgeon lowers himself with the rope over a horizontal limb of the tree. If the coefficient of friction between the rope and the limb is 0.60, compute the force which the man must exert on the rope to let himself down slowly.

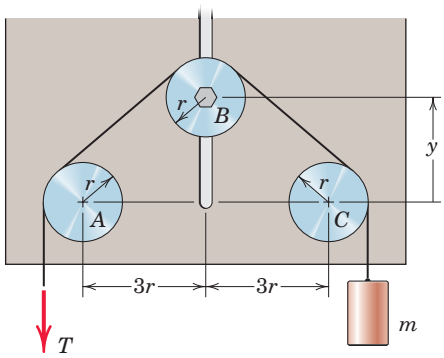
Problem 6/107

6/108 The uniform crate of mass m is supported by the tension in the flexible strap passing under it. For a given position θ, determine an expression for the tension T on the strap needed to begin (a) raising the crate and (b) lowering the crate if the coefficient of friction between the strap and the crate is μ. Neglect any friction between the crate and its vertical guides.



Problem 6/108

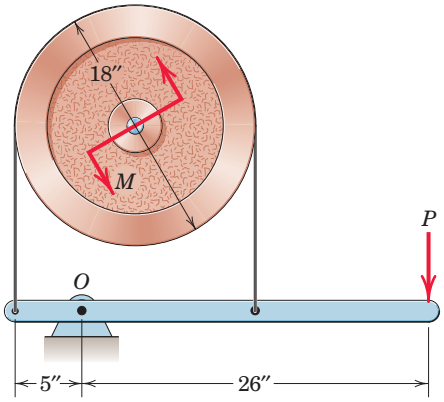
6/109 The positions of shafts A and C are fixed while that of shaft B can be varied via the vertical slot and the locking bolt. If the coefficient of static friction is μ at all interfaces, determine the dependence of T on the coordinate y, where T is the tension required to begin to lift the cylinder of mass m. All shafts are fixed against rotation.



Problem 6/109

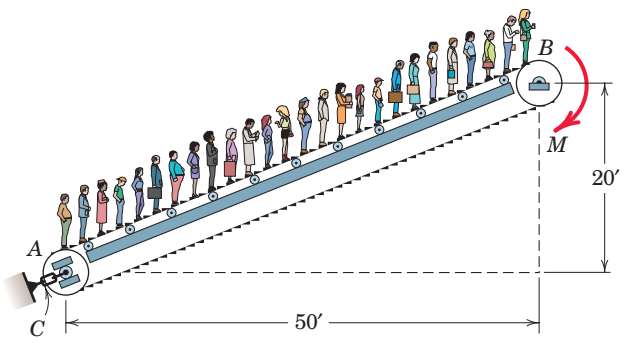
***6/110** Repeat Prob. 6/109, only now the coefficients of static friction are as follows: 0.60 at A and C and 0.20 at B. Plot T/mg as a function of y for $0 \le y \le 10r$, where r is the common radius of all three shafts. What are the limiting values of T/mg for $y = 0$ and for large values of y?

6/111 A counterclockwise moment $M = 1500$ lb-in. is applied to the flywheel. If the coefficient of friction between the band and the wheel is 0.20, compute the minimum force P necessary to prevent the wheel from rotating.



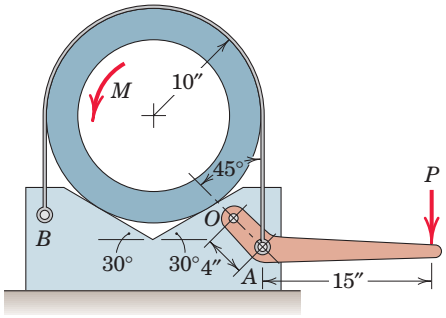
Problem 6/111

6/112 The endless belt of an escalator passes around idler drum A and is driven by a torque M applied to drum B. Belt tension is adjusted by a turnbuckle at C, which produces an initial tension of 1000 lb in each side of the belt when the escalator is unloaded. For the design of this system calculate the minimum coefficient of friction μ between drum B and the belt to prevent slipping if the escalator handles 30 people uniformly distributed along the belt and averaging 150 lb each. (*Note:* It can be shown that the increase in belt tension on the upper side of drum B and the decrease in belt tension at the lower drum A are each equal to half the component of the combined passenger weight along the incline.)



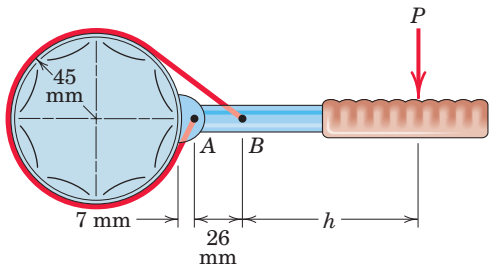
Problem 6/112

6/113 For the design of the band brake shown, find the couple M required to turn the pipe in the V-block against the action of the flexible band. A force $P = 25$ lb is applied to the lever, which is pivoted about O. The coefficient of friction between the band and the pipe is 0.30, and that between the pipe and the block is 0.40. The weights of the parts are negligible.



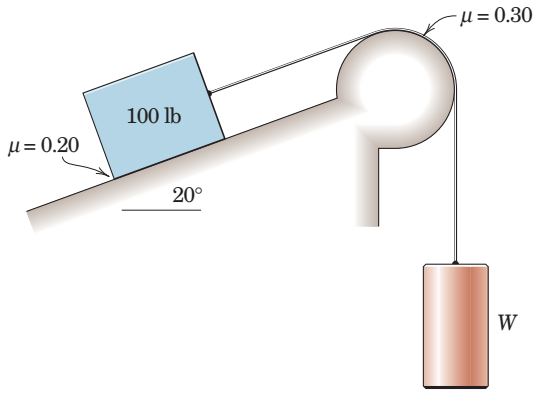
Problem 6/113

6/114 Shown in the figure is the design of a band-type oil-filter wrench. If the coefficient of friction between the band and the fixed filter is 0.25, determine the minimum value of h which ensures that the wrench will not slip on the filter, regardless of the magnitude of the force P. Neglect the mass of the wrench and assume that the effect of the small part A is equivalent to that of a band wrap which begins at the three-o'clock position and runs clockwise.



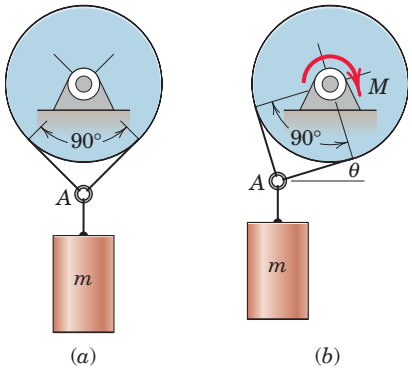
Problem 6/114

6/115 Determine the range of cylinder weights W for which the system is in equilibrium. The coefficient of friction between the 100-lb block and the incline is 0.20 and that between the cord and cylindrical support surface is 0.30.



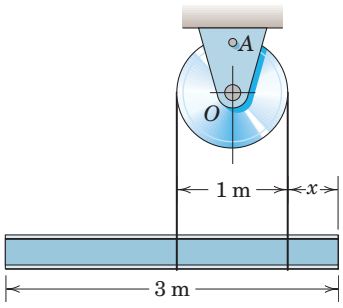
Problem 6/115

6/116 The cylinder of mass m is attached to the ring A, which is suspended by the cable which passes over the pulley, as shown in part a of the figure. A couple M applied to the pulley turns it until slipping of the cable on the pulley occurs at the position $\theta = 20°$, shown in part b of the figure. Calculate the coefficient of friction μ between the cable and the pulley.



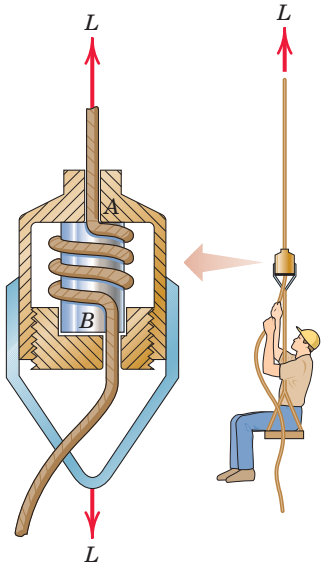
Problem 6/116

6/117 The uniform 3-m beam is suspended by the cable which passes over the large pulley. A locking pin at A prevents rotation of the pulley. If the coefficient of friction between the cable and the pulley is 0.25, determine the minimum value of x for which the cable will not slip on the pulley.



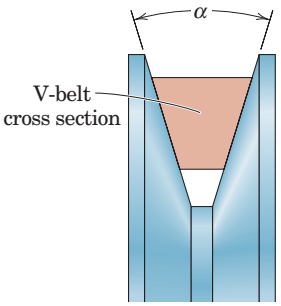
Problem 6/117

6/118 A device designed for lowering a person in a sling down a rope at a constant controlled rate is shown in the figure. The rope passes around a central shaft fixed to the frame and leads freely out of the lower collar. The number of turns is adjusted by turning the lower collar, which winds or unwinds the rope around the shaft. Entrance of the rope into the upper collar at A is equivalent to $\frac{1}{4}$ of a turn, and passage around the corner at B is also equivalent to $\frac{1}{4}$ of a turn. Friction of the rope through the straight portions of the collars averages 10 N for each collar. If three complete turns around the shaft, in addition to the corner turns, are required for a 75-kg man to lower himself at a constant rate without exerting a pull on the free end of the rope, calculate the coefficient of friction μ between the rope and the contact surfaces of the device. Neglect the small helix angle of the rope around the shafts.



Problem 6/118

6/119 Replace the flat belt and pulley of Fig. 6/11 by a V-belt and matching grooved pulley as indicated by the cross-sectional view accompanying this problem. Derive the relation among the belt tensions, the angle of contact, and the coefficient of friction for the V-belt when slipping impends. A V-belt design with $\alpha = 35°$ would be equivalent to increasing the coefficient of friction for a flat belt of the same material by what factor n?

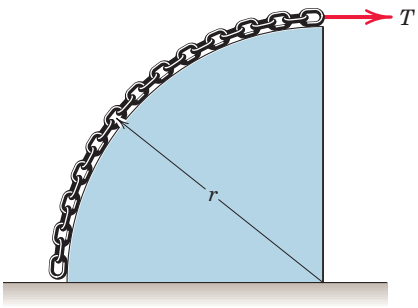


Problem 6/119

▶6/120 The chain has a mass ρ per unit length and a length $\pi r/2$. For a coefficient of friction μ between the chain and its quarter-circular guide, derive an expression for the tension T required to initiate motion upward along the guide in the vertical plane. (*Hint:* Obtain the equilibrium equations for a differential element of the chain and reduce them to the form $dT/d\theta + KT = f(\theta)$, which is a first-order, linear, nonhomogeneous equation whose solution takes the standard form

$$T = Ce^{-K\theta} + e^{-K\theta}\int e^{K\theta} f(\theta)\, d\theta$$

where C and K are constants.)



Problem 6/120

6/10 Chapter Review

In our study of friction we have concentrated on dry or Coulomb friction where a simple mechanical model of surface irregularities between the contacting bodies, Fig. 6/1, explains the phenomenon adequately for most engineering purposes. This model helps to visualize the three types of dry-friction problems which are encountered in practice. These problem types are:

1. Static friction of less than the maximum possible value and determined by the equations of equilibrium. (This usually requires a check to see that $F < \mu_s N$.)
2. Limiting static friction with impending motion ($F = \mu_s N$).
3. Kinetic friction where sliding motion occurs between contacting surfaces ($F = \mu_k N$).

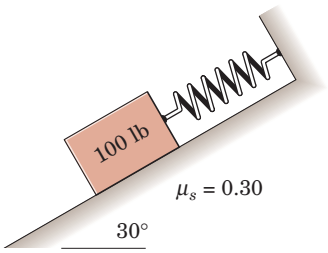
Keep in mind the following when solving dry-friction problems:

1. A coefficient of friction applies to a given pair of mating surfaces. It is meaningless to speak of a coefficient of friction for a single surface.
2. The coefficient of static friction μ_s for a given pair of surfaces is usually slightly greater than the kinetic coefficient μ_k.
3. The friction force which acts on a body is always in the direction to oppose the slipping of the body which takes place or the slipping which would take place in the absence of friction.
4. When friction forces are distributed over a surface or along a line, we select a representative element of the surface or line and evaluate the force and moment effects of the elemental friction force acting on the element. We then integrate these effects over the entire surface or line.
5. Friction coefficients vary considerably, depending on the exact condition of the mating surfaces. Computing coefficients of friction to three significant figures represents an accuracy which cannot easily be duplicated by experiment. When cited, such values are included for purposes of computational check only. For design computations in engineering practice, any handbook value for a coefficient of static or kinetic friction must be viewed as an approximation.

Other forms of friction mentioned in the introductory article of the chapter are important in engineering. Problems which involve fluid friction, for example, are among the most important of the friction problems encountered in engineering, and are studied in the subject of fluid mechanics.

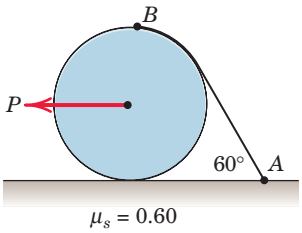
REVIEW PROBLEMS

6/121 A 100-lb block is placed on a 30° incline and released from rest. The coefficient of static friction between the block and the incline is 0.30. (*a*) Determine the maximum and minimum values of the initial tension T in the spring for which the block will not slip when released. (*b*) Calculate the friction force F on the block if $T = 40$ lb.



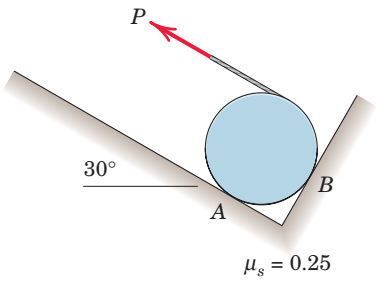
Problem 6/121

6/122 The circular cylinder weighs 50 lb and is held by a cord fixed to its periphery at B and to the ground at A. If the coefficient of static friction is 0.60, calculate the force P required to cause the cylinder to slip.



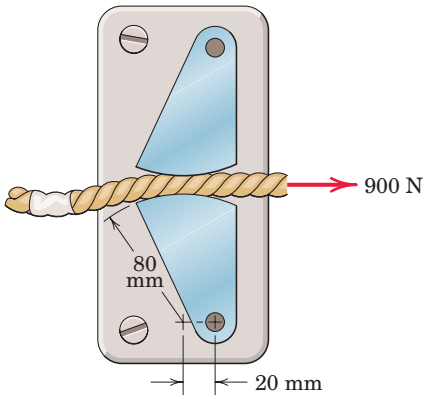
Problem 6/122

6/123 The homogeneous disk of mass m is resting on the right-angled supporting surfaces shown. The tension P in the cord is very gradually increased from zero. If the friction at both A and B is characterized by $\mu_s = 0.25$, what happens first—does the homogeneous disk slip in place or does it begin to roll up the incline? Determine the value of P at which this first movement occurs.



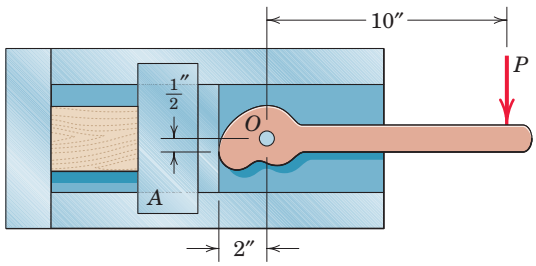
Problem 6/123

6/124 The figure shows a jam cleat designed for a sailboat where the large friction forces developed by the cams prevent the rope from slipping. Determine the force R supported by each cam bearing for the 900-N rope tension. The coefficient of static friction between the rope and the cams is 0.80.



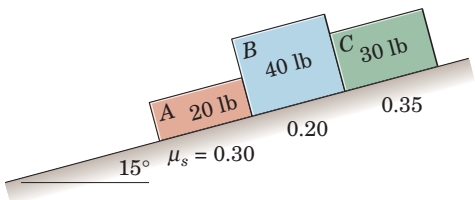
Problem 6/124

6/125 The design of the cam-lock vise provides a quick positive clamping action with a friction coefficient between the cam and the movable jaw A of 0.30. (*a*) As the cam and lever are rotating clockwise approaching the locking position shown with $P = 30$ lb, determine the clamping force C. (*b*) With P removed, find the friction force F in the locking position. (*c*) Determine the force P', opposite from P, required to release the clamp.



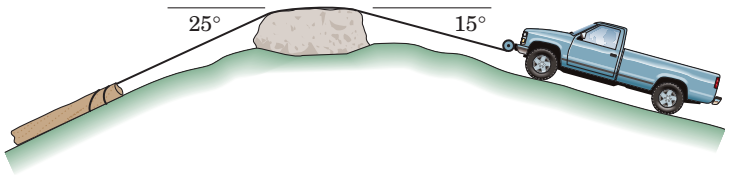
Problem 6/125

6/126 Three boxes are placed on the incline in contact with each other and released from rest. The coefficients of static friction under boxes A, B, and C are 0.30, 0.20, and 0.35, respectively. Describe what happens.



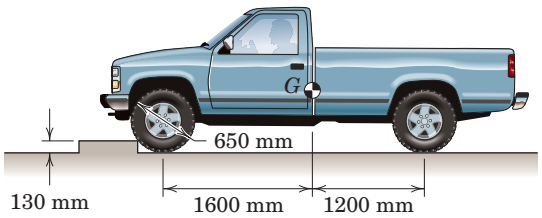
Problem 6/126

6/127 A 500-kg log is being steadily pulled up the incline by means of the cable attached to the winch on the truck. If the coefficient of kinetic friction is 0.80 between the log and the incline and 0.50 between the cable and rock, determine the tension T which must be developed by the winch.



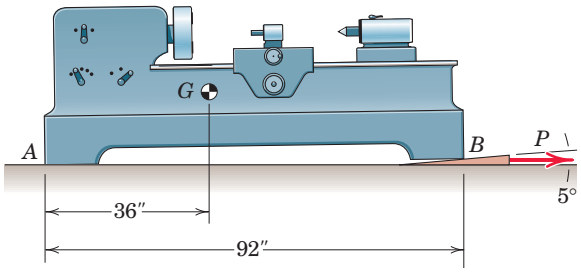
Problem 6/127

6/128 Calculate the torque M which the engine of the pickup truck must supply to the rear axle to roll the front wheels over the curbing from a rest position if the rear wheels do not slip. Determine the minimum effective coefficient of friction at the rear wheels to prevent slipping. The mass of the loaded truck with mass center at G is 1900 kg.



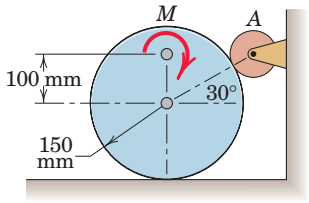
Problem 6/128

6/129 The 4600-lb lathe with mass center at G is positioned with the aid of the 5° steel wedge. Determine the horizontal force P required to remove the wedge if the coefficient of friction for all contacting surfaces is 0.30. Also show that no horizontal movement of the lathe takes places.



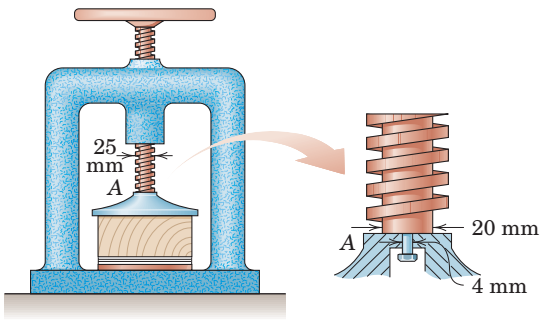
Problem 6/129

6/130 Under the action of the applied couple M the 25-kg cylinder bears against the roller A, which is free to turn. If the coefficients of static and kinetic friction between the cylinder and the horizontal surface are 0.50 and 0.40, respectively, determine the friction force F acting on the cylinder if (a) $M = 20$ N·m and (b) $M = 40$ N·m.



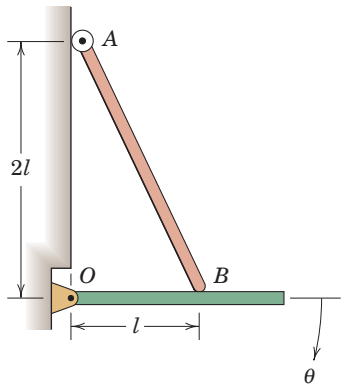
Problem 6/130

6/131 The screw of the small press has a mean diameter of 25 mm and has a single square thread with a lead of 8 mm. The flat thrust bearing at A is shown in the enlarged view and has surfaces which are well worn. If the coefficient of friction for both the threads and the bearing at A is 0.25, calculate the design torque M on the handwheel required (a) to produce a compressive force of 4 kN and (b) to loosen the press from the 4-kN compression.



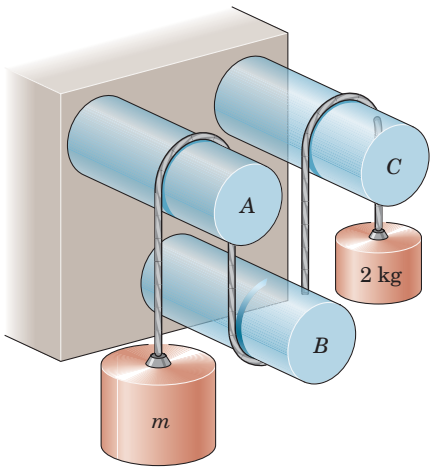
Problem 6/131

6/132 The small roller of the uniform slender rod rests against the vertical surface at A while the rounded end at B rests on the platform which is slowly pivoted downward beginning from the horizontal position shown. If the bar begins to slip when $\theta = 25°$, determine the coefficient of static friction μ_s between the bar and the platform. Neglect friction in the roller and the small thickness of the platform.



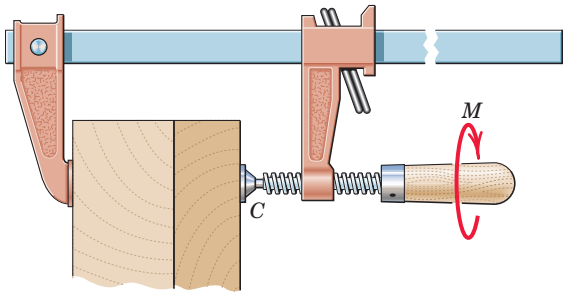
Problem 6/132

6/133 Determine the range of mass m over which the system is in equilibrium (a) if the coefficient of static friction is 0.20 at all three fixed shafts and (b) if the coefficient of static friction associated with shaft B is increased to 0.50.



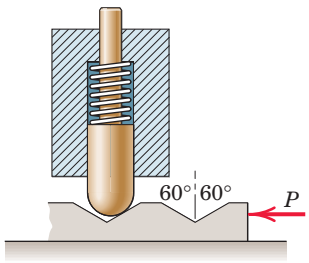
Problem 6/133

6/134 The bar clamp is being used to clamp two boards together while the glue between them cures. What torque M must be applied to the handle of the screw in order to produce an 80-lb compression between the boards? The $\frac{3}{8}$-in.-diameter single-thread screw has 12 square threads per inch, and the effective coefficient of friction is 0.20. Neglect any friction in the pivot contact at C. What torque M' is required to loosen the clamp?



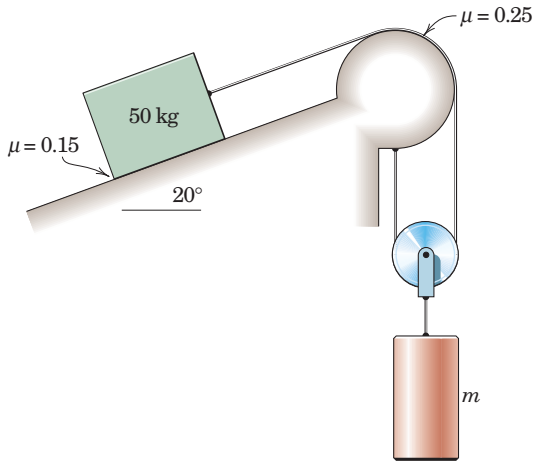
Problem 6/134

6/135 The detent mechanism consists of the spring-loaded plunger with a spherical end, which is designed to position the horizontal bar by engaging the spaced notches. If the spring exerts a force of 40 N on the plunger in the position shown and a force $P = 60$ N is required to move the detent bar against the plunger, calculate the coefficient of friction between the plunger and the detent. It is known from earlier tests that the coefficient of friction between the light bar and the horizontal surface is 0.30. Assume that the plunger is well lubricated and accurately fitted so that the friction between it and its guide is negligible.



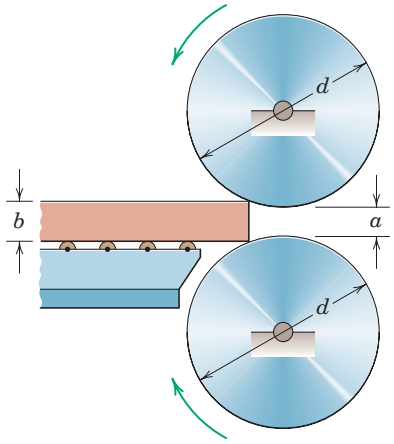
Problem 6/135

6/136 Determine the range of cylinder mass m for which the system is in equilibrium. The coefficient of friction between the 50-kg block and the incline is 0.15 and that between the cord and cylindrical support is 0.25.



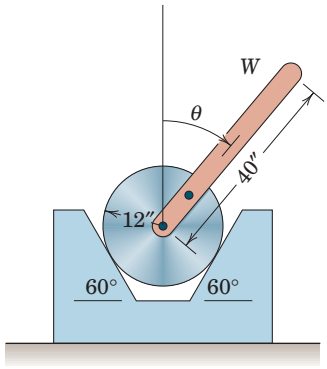
Problem 6/136

6/137 The elements of a rolling mill are shown here. In the design of the roller spacing, determine the maximum slab thickness b so that the slab will enter the rollers by means of friction alone if the coefficient of kinetic friction is μ_k. Assume that $(b - a)$ is small compared with d.



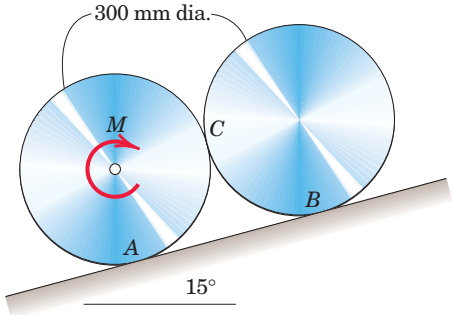
Problem 6/137

6/138 The cylinder weighs 80 lb and the attached uniform slender bar has an unknown weight W. The unit remains in static equilibrium for values of the angle θ ranging up to 45° but slips if θ exceeds 45°. If the coefficient of static friction is known to be 0.30, determine W.



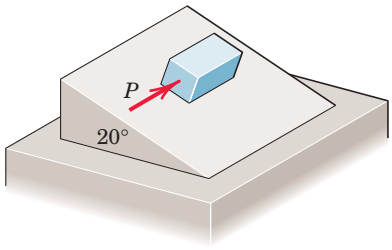
Problem 6/138

6/139 Calculate the couple M applied to the lower of the two 20-kg cylinders which will allow them to roll slowly *down* the incline. The coefficients of static and kinetic friction for all contacting surfaces are $\mu_s = 0.60$ and $\mu_k = 0.50$.



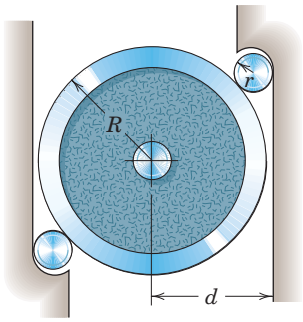
Problem 6/139

6/140 The 8-kg block is resting on the 20° inclined plane with a coefficient of static friction $\mu_s = 0.50$. Determine the minimum horizontal force P which will cause the block to slip.



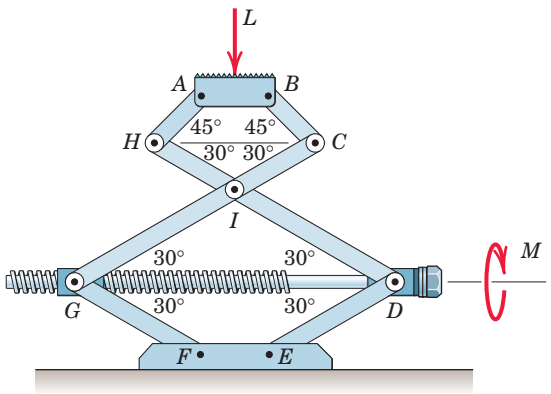
Problem 6/140

▶6/141 The device shown is designed to prevent clockwise rotation in the horizontal plane of the central wheel by means of frictional locking of the two small rollers. For given values of R and r and for a common coefficient of friction μ at all contact surfaces, determine the range of values of d for which the device will operate as described.



Problem 6/141

▶6/142 A scissors-type jack with a single square thread which engages the threaded collar G and turns in a ball thrust bearing at D is being designed. The thread is to have a mean diameter of 10 mm and a lead (advancement per revolution) of 3 mm. With a coefficient of friction of 0.20 for the greased threads, (*a*) calculate the torque M on the screw required to raise a load of 1000 kg from the position shown and (*b*) calculate the torque M required to lower the load from the same position. Assume that platform AB and line DG will remain horizontal under load. Neglect friction in the bearing at D.

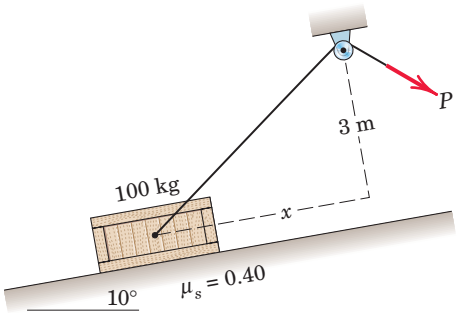


$\overline{GI} = \overline{DI} = 160$ mm, $\overline{HI} = \overline{CI} = 80$ mm

Problem 6/142

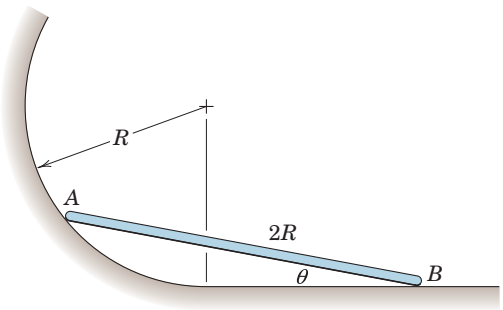
*Computer-Oriented Problems

***6/143** Plot the force P required to begin moving the 100-kg crate up the 10° incline starting from rest at various values of x from zero to 10 m. Determine the least possible value of P and the corresponding value of x.



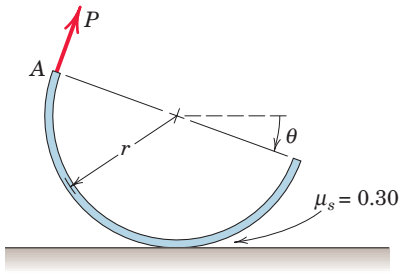
Problem 6/143

***6/144** The uniform slender bar of Prob. 6/41 is repeated here but now the ideal roller at B is removed. The coefficient of static friction at A is 0.70 and that at B is 0.50. Determine the maximum value of the angle θ for which equilibrium is possible.



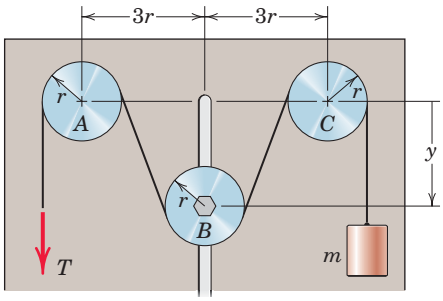
Problem 6/144

***6/145** The semicylindrical shell of mass m and radius r is rolled through an angle θ by the force P which remains tangent to its periphery at A as shown. If P is slowly increased, plot the tilt angle θ as a function of P up to the point of slipping. Determine the tilt angle θ_{max} and the corresponding value P_{max} for which slipping occurs. The coefficient of static friction is 0.30.



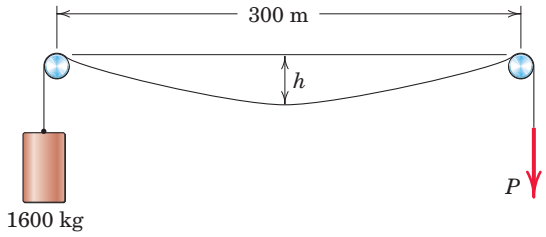
Problem 6/145

***6/146** The positions of shafts A and C are fixed while that of shaft B can be varied via the vertical slot and locking bolt. The coefficients of kinetic friction are 0.60 at A and C and 0.20 at B. Plot the quantity T/mg as a function of y for $0 \le y \le 10r$, where T is the tension required to lift the cylinder of mass m at a steady rate and r is the common radius of all three shafts. All shafts are fixed against rotation. What are the limiting values of T/mg for $y = 0$ and for large values of y?



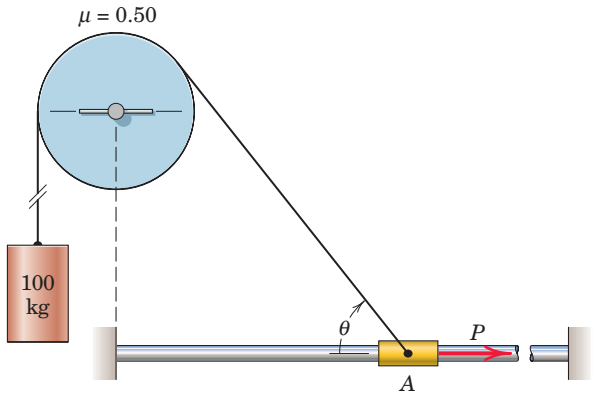
Problem 6/146

***6/147** A heavy cable with a mass of 12 kg per meter of length passes over the two fixed pipes 300 m apart on the same level. One end supports a 1600-kg cylinder. By experiment it is found that a downward force P of 60 kN is required to induce slipping of the cable over both pipes at a constant rate. Determine the coefficient of kinetic friction μ_k between the cable and the pipes, the maximum tension T in the cable between the pipes, and the sag h in the cable.



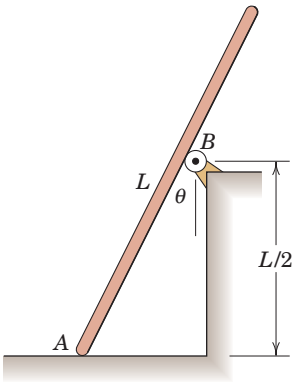
Problem 6/147

***6/148** The 100-kg load is elevated by the cable which slides over the fixed drum with a coefficient of friction of 0.50. The cable is secured to the slider A which is pulled slowly along its smooth horizontal guide bar by the force P. Plot P as a function of θ from $\theta = 90°$ to $\theta = 10°$ and determine its maximum value along with the corresponding angle θ. Check your plotted value of P_{max} analytically.



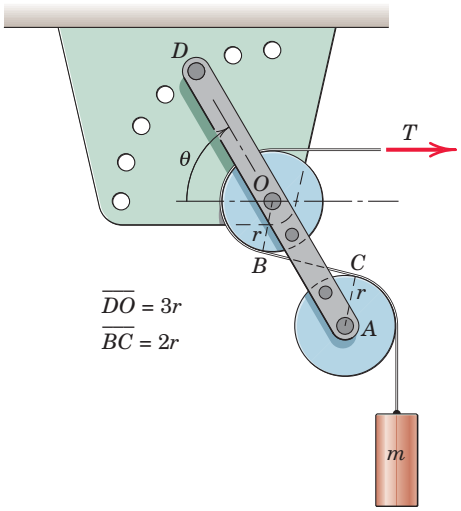
Problem 6/148

***6/149** The uniform slender pole rests against a small roller at B. End A will not slip on the horizontal surface if the coefficient of static friction μ_s is sufficiently large. (a) Determine the required minimum value of μ_s to prevent slipping for any value of θ from $\theta = 0$ to $\theta = 60°$ and plot μ_s versus θ. From these results find the range of θ for which the pole will be unstable if $\mu_s = 0.40$. (b) At what angle θ is the pole most unstable, and what is the least coefficient of static friction μ_s which would be required to prevent slipping for this angle?



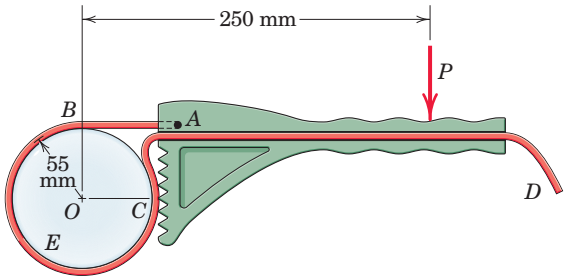
Problem 6/149

***6/150** The device is designed to permit an adjustment to the horizontal tension T in the cable passing around the two fixed wheels in order to lower the mass m. If the coefficient of friction between the cable and the wheel surfaces is 0.40, determine and plot the ratio T/mg as a function of θ in the range $0 < \theta < 90°$. Also find the value of the shear force V in the adjusting pin at D in terms of mg for $\theta = 60°$.



Problem 6/150

***6/151** The band wrench is useful for loosening and tightening such items as the whole-house water filter E shown. Assume that the teeth of the wrench do not slip on the band at point C and that the band is slack from C to its end D. Determine the minimum coefficient μ of static friction for which the band will not slip relative to the fixed filter.



Problem 6/151

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Problem Answers

(When a problem asks for both a general and a specific result, only the specific result might be listed below.)

Chapter 1

1/1 $\theta_x = 36.9°$, $\theta_y = 126.9°$, $\mathbf{n} = 0.8\mathbf{i} - 0.6\mathbf{j}$

1/2 $V = 16.51$ units, $\theta_x = 83.0°$

1/3 $V' = 14.67$ units, $\theta_x = 162.6°$

1/4 $\theta_x = 56.1°$, $\theta_y = 138.0°$, $\theta_z = 68.2°$

1/5 $m = 93.2$ slugs, $m = 1361$ kg

1/6 $W = 819$ N, $W = 184.1$ lb

1/7 $W = 578$ N, $m = 4.04$ slugs, $m = 58.9$ kg

1/8 $A + B = 8.40$, $A - B = 4.94$
$AB = 11.51$, $A/B = 3.86$

1/9 $F = 1.984(10^{20})$ N, $F = 4.46(10^{19})$ lb

1/10 $F = 1.358(10^{-9})$ N

1/11 $\theta = 5°$: $n_s = 0.1270\%$, $n_t = -0.254\%$
$\theta = 10°$: $n_s = 0.510\%$, $n_t = -1.017\%$
$\theta = 20°$: $n_s = 2.06\%$, $n_t = -4.09\%$

Chapter 2

2/1 $F_x = 460$ N, $F_y = -386$ N, $\mathbf{F} = 460\mathbf{i} - 386\mathbf{j}$ N

2/2 $\mathbf{F} = -346\mathbf{i} + 200\mathbf{j}$ lb
$F_x = -346$ lb, $F_y = 200$ lb
$\mathbf{F}_x = -346\mathbf{i}$ lb, $\mathbf{F}_y = 200\mathbf{j}$ lb

2/3 $\mathbf{F} = -6\mathbf{i} - 2.5\mathbf{j}$ kN

2/4 $F_x = 2650$ lb, $F_y = 1412$ lb

2/5 $\mathbf{F} = -1080\mathbf{i} - 1440\mathbf{j}$ N

2/6 $F_x = -F \sin \beta$, $F_y = -F \cos \beta$
$F_n = F \sin(\alpha+\beta)$, $F_t = F \cos(\alpha+\beta)$

2/7 $R = 3.80$ kN, $\theta = 338°$ (or $-21.6°$)

2/8 $F_n = -62.9$ N, $F = 97.9$ N

2/9 $\theta = 49.9°$, $R = 1077$ lb

2/10 (*a*) $F_n = 34.2$ N, $F_t = 94.0$ N
(*b*) $F_n = -17.36$ N, $F_t = 98.5$ N

2/11 $\mathbf{R} = 2.35\mathbf{i} - 3.45\mathbf{j}$ kips

2/12 $F_t = \dfrac{Fs}{r}$, $F_n = -\dfrac{F\sqrt{r^2 - s^2}}{r}$

2/13 $T = 5.83$ kN, $R = 9.25$ kN

2/14 $\mathbf{R} = 600\mathbf{i} + 346\mathbf{j}$ N, $R = 693$ N

2/15 $F_t = 1.286$ kN, $F_n = 1.532$ kN

2/16 $T_x = \dfrac{T(1 + \cos\theta)}{\sqrt{3 + 2\cos\theta - 2\sin\theta}}$

$T_y = \dfrac{T(\sin\theta - 1)}{\sqrt{3 + 2\cos\theta - 2\sin\theta}}$

2/17 $T_n = 66.7$ N, $T_t = 74.5$ N

2/18 $R = 50.2$ lb, $\theta = 84.3°$

2/19 $\mathbf{R} = 88.8\mathbf{i} + 245\mathbf{j}$ N

2/20 $R_a = 1170$ N, $R_b = 622$ N, $P_a = 693$ N

2/21 $F_a = 1093$ lb, $F_b = 980$ lb

$P_a = 400$ lb, $P_b = 207$ lb

2/22 $F_a = 1.935$ kN, $F_b = 2.39$ kN

$P_a = 3.63$ kN, $P_b = 3.76$ kN

2/23 (*a*) $R = 872$ N, $\theta = 36.6°$

(*b*) $\mathbf{R} = 520\mathbf{i} - 700\mathbf{j}$ N

2/24 $P = 537$ lb, $T = 800$ lb

2/25 $\theta = 51.3°$, $\beta = 18.19°$

2/26 $T_n = 333$ N, $T_t = -672$ N

2/27 $\theta = 51.3°$, $\beta = 18.19°$

2/28 BC: $P_t = -77.9$ N, $P_n = 45.0$ N

AB: $P_t = 63.6$ N, $P_n = 63.6$ N

►**2/29** $F_x = -101.2$ N, $F_y = 194.4$ N

►**2/30** $F_n = 50.4$ N, $F_t = -39.2$ N

2/31 $M_O = 2.68$ kN·m CCW, $\mathbf{M}_O = 2.68\mathbf{k}$ kN·m

$(x, y) = (-1.3, 0)$ and $(0, 0.78)$m

2/32 $M_B = 83.2$ lb-ft CW

2/33 $T = 36$ N

2/34 $M_O = \dfrac{Fbh}{\sqrt{b^2 + h^2}}$ CW

2/35 $M_O = 46.4$ N·m CW

2/36 $M_O = 43.4$ N·m CW

2/37 $M_O = 70.0$ lb-ft CW

2/38 $M_O = 5.64$ N·m CW

2/39 $M_O = mgb\left(1 - \dfrac{2}{\pi}\right)$ CW

2/40 $M_B = 48$ N·m CW

$M_A = 81.9$ N·m CW

2/41 $M_O = 0.268$ lb-in. CCW

2/42 $M_O = 191.0$ N·m CCW

2/43 $M_O = 102.8$ lb-ft CCW

2/44 $M_O = Tr\sin(\alpha + \theta)$ CW

$M_P = Tr[\cos\alpha + \sin(\alpha + \theta)]$ CW

2/45 $\theta = \tan^{-1}\left(\dfrac{h}{b}\right)$

2/46 $M_O = 130$ lb-in. CCW

2/47 $M_O = 0.1335 + 0.0282s$ N·mm

2/48 $M_O = 0.866$ N·m CCW

2/49 $M_O = 128.6$ lb-in. CW, $T = 64.3$ lb

2/50 $M_O = 23.5$ N·m CCW

2/51 $M_O = 5010$ lb-ft CCW

2/52 $\theta = 65.8°$, $M_O = 740$ lb-in. CW

2/53 $\theta = 57.5°$

2/54 $M_O = 80.2\mathbf{k}$ lb-ft

2/55 $M_O = 26.8$ N·m CCW

2/56 $F_{DE} = 141.6$ lb, $R = 220$ lb at 94.6°

2/57 $M_O = 0.902$ kN·m CW

2/58 $M_O = 41.5$ N·m CW

$\alpha = 33.2°$, $(M_O)_{\max} = 41.6$ N·m CW

2/59 $M = 1440$ lb-in. CW

2/60 $F = 12$ kN at 30°, $M_O = 24$ kN·m CW

$M_B = 76.0$ kN·m CW

2/61 $y_A = -0.4$ m

2/62 $F = 16.18$ N

2/63 $M = F(b\cos\theta + h\sin\theta)$ CCW

2/64 $F = 875$ lb

2/65 $R = 7$ lb at 60°, $M_O = 14.85$ lb-in. CCW

2/66 $R = 10$ kN down, $M_O = 0.75$ kN·m CCW

2/67 $P = 51.4$ kN

2/68 $d = \dfrac{M_A\sqrt{b^2 + h^2}}{Rb}$ (below A)

2/69 $F = 700$ lb

2/70 $F = 231$ N

2/71 $M = 2.60$ N·m CW

2/72 $R = 50$ N at 110°, $M_O = 17.29$ N·m CCW

2/73 $\mathbf{F} = 43.4\mathbf{i} + 246\mathbf{j}$ N, $M_O = 60.0$ N·m CW

2/74 $R = 250$ N at −45°, $M_C = 174.9$ N·m CW

2/75 $M = 21.7$ N·m CCW

2/76 $\mathbf{F} = F\left[\dfrac{\frac{b}{2}\mathbf{i} + h\mathbf{j}}{\sqrt{\frac{b^2}{4} + h^2}}\right]$, $M_B = \dfrac{3hbF}{4\sqrt{\frac{b^2}{4} + h^2}}$ CW

2/77 $y = -40.3$ mm

2/78 F at 67.5°, $M_O = 0.462FR$ CCW

2/79 $\theta = 56.9°$, $F = 1361$ lb

2/80 $R = 17.43$ kN at $\theta_x = 26.1°$

2/81 (*a*) $\mathbf{R} = -2F\mathbf{j}$, $\mathbf{M}_O = \mathbf{0}$

(*b*) $\mathbf{R} = \mathbf{0}$, $\mathbf{M}_O = Fd\mathbf{k}$

(*c*) $\mathbf{R} = -F\mathbf{i} + F\mathbf{j}$, $\mathbf{M}_O = \mathbf{0}$

2/82 (*a*) $\mathbf{R} = 2F\mathbf{i}$ along $y = -\dfrac{d}{2}$

(*b*) $\mathbf{R} = -2F\mathbf{i}$ along $y = \dfrac{3d}{4}$

(*c*) $\mathbf{R} = -F\mathbf{i} + \sqrt{3}F\mathbf{j}$ along $y = -\sqrt{3}x + \dfrac{d}{2}$

2/83 $d = 10.70$ m left of A

2/84 $R = 3$ kN down, $x = 4.33$ m

2/85 $R = 80$ N left, $y = 40$ mm

2/86 $y = -10$ in.

2/87 $\mathbf{R} = 1.644\mathbf{i} + 1.159\mathbf{j}$ kN, $M_A = 2.22$ kN·m CCW

2/88 $M = 148.0$ N·m CCW

2/89 $\mathbf{R} = 400\mathbf{i} - 3010\mathbf{j}$ lb

$M_A = 18{,}190$ lb-ft CW, $x = 6.05$ ft

2/90 $d = 4$ m below O

2/91 $\mathbf{R} = T(1.966\mathbf{i} + 0.259\mathbf{j})$, $M_O = 2.69T$ CW
$x = -10.39$ m
2/92 $x = 1.637$ m, $y = -0.997$ m
2/93 $\mathbf{R} = -15\mathbf{i} - 47.3\mathbf{j}$ kN, $x = 7.42$ m
$y = -23.4$ m
2/94 $\mathbf{R} = 346\mathbf{i} - 2200\mathbf{j}$ N
$M_A = 11\,000$ N·m CW, $x = 5$ m
2/95 $x = 7.83$ mm, $y = 14.55$ mm
2/96 $\mathbf{R} = 45\mathbf{i} - 15\mathbf{j}$ lb, $\mathbf{M}_A = -1650\mathbf{k}$ lb-in.
$y = -\frac{1}{3}x + \frac{110}{3}$ (in.)
2/97 $\mathbf{R} = 903\mathbf{i} + 175\mathbf{j}$ N, $x = 1.308$ m
$y = -0.253$ m
2/98 $\mathbf{R} = 185\mathbf{i} + 1133\mathbf{j}$ lb
$M_G = 460$ lb-ft CCW, $x = 4.06$ ft
2/99 $F_C = F_D = 6.42$ N, $F_B = 98.9$ N
2/100 $\mathbf{R} = 12.10\mathbf{i} - 18.71\mathbf{j}$ lb, $\mathbf{M}_O = -358\mathbf{k}$ lb-in.
$y = -1.547x + 29.6$ (in.)
2/101 $\mathbf{F} = 791\mathbf{i} + 403\mathbf{j} + 146.5\mathbf{k}$ lb, $F_y = 403$ lb
2/102 $\mathbf{F} = 2.05\mathbf{i} + 3.55\mathbf{j} + 2.87\mathbf{k}$ kN
$F_x = 2.05$ kN, $F_{OA} = 3.55$ kN
2/103 $\mathbf{T} = 2(-0.920\mathbf{i} + 0.383\mathbf{j} + 0.0767\mathbf{k})$ kN
2/104 $\theta_x = 155.7°$, $\theta_{xz} = 13.17°$
2/105 $\mathbf{T} = 0.876\mathbf{i} + 0.438\mathbf{j} - 2.19\mathbf{k}$ kN
$T_{AC} = 2.06$ kN
2/106 $\mathbf{F} = 92.7\mathbf{i} - 61.8\mathbf{j} - 61.8\mathbf{k}$ N
2/107 $\mathbf{T} = 0.321\mathbf{i} + 0.641\mathbf{j} - 0.962\mathbf{k}$ kN
2/108 $T_{GF} = 0.978$ kN
2/109 $\mathbf{F} = 500[0.781\mathbf{i} - 0.488\mathbf{j} + 0.390\mathbf{k}]$ lb
$F_x = 390$ lb, $F_y = -244$ lb, $F_z = 195.2$ lb
2/110 $F_{CD} = \sqrt{\frac{6}{5}}$ kN, $\theta = 56.8°$
2/111 $\mathbf{T} = -598\mathbf{i} + 411\mathbf{j} + 189.5\mathbf{k}$ N
2/112 $\mathbf{T} = 110.7\mathbf{i} - 775\mathbf{j} + 166.0\mathbf{k}$ lb
$\theta_x = 82.0°$, $\theta_y = 165.6°$, $\theta_z = 78.0°$
2/113 $\theta = 54.9°$
2/114 $T_{BC} = 251$ N
2/115 $T = 778$ lb, $\theta_{xy} = 8.88°$
2/116 $T_{CD} = 46.0$ N
2/117 $\mathbf{T} = 500(0.370\mathbf{i} + 0.410\mathbf{j} - 0.833\mathbf{k})$ N
$T_{DC} = 458$ N
►**2/118** $\mathbf{F} = \frac{F}{\sqrt{5 - 4\sin\phi}}[(2\sin\phi - 1)(\cos\theta\mathbf{i} + \sin\theta\mathbf{j}) + 2\cos\phi\mathbf{k}]$
►**2/119** $\mathbf{T} = 191.0\mathbf{i} + 28.7\mathbf{j} - 51.8\mathbf{k}$ lb, $T_{xz} = 197.9$ lb
►**2/120** $F_x = \frac{2acF}{\sqrt{a^2 + b^2}\sqrt{a^2 + b^2 + 4c^2}}$
$F_y = \frac{2bcF}{\sqrt{a^2 + b^2}\sqrt{a^2 + b^2 + 4c^2}}$
$F_z = F\sqrt{\frac{a^2 + b^2}{a^2 + b^2 + 4c^2}}$
2/121 $\mathbf{M}_1 = -cF_1\mathbf{j}$, $\mathbf{M}_2 = F_2(c\mathbf{j} - b\mathbf{k})$, $\mathbf{M}_3 = -bF_3\mathbf{k}$
2/122 $\mathbf{M}_O = -320\mathbf{i} + 160\mathbf{j}$ lb-ft, $M_O = 358$ lb-ft
2/123 $\mathbf{M}_O = F(-c\mathbf{i} + a\mathbf{k})$, $\mathbf{M}_A = Fa\mathbf{k}$
$\mathbf{M}_{OB} = -\frac{Fac}{a^2 + b^2}(a\mathbf{i} + b\mathbf{j})$
2/124 $\mathbf{M} = (-0.450\mathbf{i} + 0.300\mathbf{j})10^6$ lb-in.
2/125 $\mathbf{M} = (126\mathbf{i} - 36\mathbf{j} + 50.6\mathbf{k})10^3$ lb-in.
2/126 $M_O = 2.81$ kN·m
2/127 $\mathbf{M} = -75\mathbf{i} + 22.5\mathbf{j}$ N·m
2/128 $\mathbf{M}_O = \frac{\rho g}{2}(a^2\mathbf{j} - b^2\mathbf{i})$
2/129 $\mathbf{M}_z = 208\mathbf{k}$ lb-in.
2/130 $\mathbf{M}_O = -90.6\mathbf{i} - 690\mathbf{j} - 338\mathbf{k}$ lb-in.
$\mathbf{M}_{OA} = -690$ lb-in.
2/131 $\mathbf{M}_O = \rho a g(b + \frac{a}{2})[-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}]$
2/132 $\mathbf{M}_O = 480\mathbf{i} + 2400\mathbf{k}$ N·m
2/133 $\mathbf{M}_O = -98.7\mathbf{i} + 17.25\mathbf{j} + 29.9\mathbf{k}$ N·m
$M_x = -98.7$ N·m
2/134 $\mathbf{R} = -598\mathbf{i} + 411\mathbf{j} + 189.5\mathbf{k}$ N
$\mathbf{M}_O = -361\mathbf{i} - 718\mathbf{j} + 419\mathbf{k}$ N·m
2/135 $\mathbf{M} = -60\mathbf{i} + 48\mathbf{k}$ lb-in.
2/136 $\mathbf{M} = 3400\mathbf{i} - 51\,000\mathbf{j} - 51\,000\mathbf{k}$ N·m
2/137 $\mathbf{M}_O = -48.6\mathbf{j} - 9.49\mathbf{k}$ N·m, $d = 74.5$ mm
2/138 $\mathbf{R} = -38.3\mathbf{i} - 32.1\mathbf{j}$ N
$\mathbf{M}_O = 643\mathbf{i} - 766\mathbf{j} + 6250\mathbf{k}$ N·mm
2/139 $F = 228$ N
2/140 $\mathbf{F} = -1.716\mathbf{j} + 28.3\mathbf{k}$ kN
$\mathbf{M} = 5.15\mathbf{i} - 90\mathbf{k}$ kN·m
2/141 $M_x = 31.1$ N·m
$(M_x)_W = -31.1$ N·m, $M_{OB} = 0$
2/142 $\mathbf{M}_O = -2.89\mathbf{i} - 0.962\mathbf{k}$ kN·m
$\mathbf{M}_z = -0.962\mathbf{k}$ kN·m
2/143 $\mathbf{M}_A = -5.40\mathbf{i} + 4.68\mathbf{j}$ lb-in.
$\mathbf{M}_{AB} = -4.05\mathbf{i} - 2.34\mathbf{k}$ lb-in.
2/144 $\mathbf{M}_O = -260\mathbf{i} + 328\mathbf{j} + 88\mathbf{k}$ N·m
2/145 $\mathbf{M}_B = -1871\mathbf{i} + 1080\mathbf{j} - 3780\mathbf{k}$ lb-in.
$\mathbf{M}_C = -2840\mathbf{k}$ lb-in., $M_{BC} = 2540$ lb-in.
2/146 $M_z = 75.0$ N·m
$\mathbf{M}_O = -9.04\mathbf{i} - 414\mathbf{j} + 75.0\mathbf{k}$ N·m
2/147 $\mathbf{F} = \frac{F}{\sqrt{5}}(\cos\theta\mathbf{i} + \sin\theta\mathbf{j} - 2\mathbf{k})$
$\mathbf{M}_O = \frac{Fh}{\sqrt{5}}(\cos\theta\mathbf{j} - \sin\theta\mathbf{i})$
►**2/148** $\mathbf{M}_O = 136.4\mathbf{i} - 679\mathbf{k}$ lb-in.
$M' = 685$ lb-in.
2/149 $L = 1$ oz, $\theta = 30°$, $R = 5.82$ oz

2/150 $R = 600$ N down
$\mathbf{M}_O = -216\mathbf{i} + 216\mathbf{j}$ N·m, $\mathbf{R} \perp \mathbf{M}_O$
2/151 $\mathbf{R} = F\mathbf{k}$, $\mathbf{M}_O = Fb(4\mathbf{i} + \mathbf{j})$
$x = -b, y = 4b$
2/152 $x = 2.92$ m, $y = 6.33$ m
2/153 $\mathbf{R} = F(0.5\mathbf{j} - 0.1340\mathbf{k})$
$\mathbf{M}_O = Fb(1.866\mathbf{i} + 0.268\mathbf{j} + \mathbf{k})$, yes
2/154 $\mathbf{R} = 1000\mathbf{j} + 500\mathbf{k}$ N
$\mathbf{M}_O = 12\mathbf{i} - 300\mathbf{j} + 1200\mathbf{k}$ N·m
2/155 $\mathbf{R} = -1.2\mathbf{j}$ lb, $\mathbf{M}_G = -30\mathbf{i} + 72\mathbf{k}$ lb-in.
2/156 $\mathbf{R} = -8\mathbf{i}$ kN, $\mathbf{M}_G = 48\mathbf{j} + 820\mathbf{k}$ kN·m
2/157 $\mathbf{R} = -266\mathbf{j} + 1085\mathbf{k}$ N
$\mathbf{M}_O = -48.9\mathbf{j} - 114.5\mathbf{k}$ N·m, no
2/158 $\mathbf{R} = -20\mathbf{i} - 37.9\mathbf{j} + 12.65\mathbf{k}$ kN
$\mathbf{M} = 45.3\mathbf{j} + 40.9\mathbf{k}$ kN·m
2/159 $y = -4$ m, $z = 2.33$ m
2/160 $\mathbf{R} = 600\mathbf{j}$ lb, $\mathbf{M}_O = 1200\mathbf{i} + 4800\mathbf{k}$ lb-ft
$x = 8$ ft, $z = -2$ ft
2/161 $\mathbf{R} = 4\mathbf{i} + 3\mathbf{j} - 6\mathbf{k}$ kN, $R = 7.81$ kN
$\mathbf{M} = -0.4\mathbf{i} - 4\mathbf{j} + 4\mathbf{k}$ kN·m, $M = 5.67$ kN·m
2/162 $\mathbf{R} = 25\mathbf{i} - 60\mathbf{j} - 43.3\mathbf{k}$ lb
$\mathbf{M}_O = 1153\mathbf{i} - 830\mathbf{j} + 250\mathbf{k}$ lb-in.
2/163 $\mathbf{R} = -20\mathbf{j} - 40\mathbf{k}$ lb, $\mathbf{M}_O = -56\mathbf{i} - 320\mathbf{j}$ lb-in.
$x = -6.4$ in., $y = 1.4$ in.
2/164 $y = 0, z = \frac{7a}{2}, \mathbf{M} = -\frac{aT}{2}(\mathbf{i} + \mathbf{j})$
2/165 $\mathbf{R} = F(\mathbf{i} - 3\mathbf{k}), \mathbf{M} = \frac{3aF}{10}(\mathbf{i} - 3\mathbf{k})$
$y = \frac{a}{10}, z = 2a$
2/166 $x = 66.9$ mm, $z = 0, M = -99.6$ N·m
▶ **2/167** $x = a + c, y = \frac{b}{2}, \mathbf{M} = \frac{Fb}{2}(\mathbf{i} - \mathbf{k})$
▶ **2/168** $R = 44.7$ kN, $M = -26.9$ kN·m
$x = 0.221$ m, $y = -0.950$ m
2/169 $\mathbf{T}_B = -\mathbf{T}_A = -1.664\mathbf{i} + 1.109$ kN
2/170 $F = 20.7$ lb, $F_x = -5.36$ lb
2/171 $\mathbf{M}_1 = -cF_1\mathbf{i}, \mathbf{M}_2 = F_2(c\mathbf{i} - a\mathbf{k}), \mathbf{M}_3 = -aF_3\mathbf{k}$
2/172 $F = 300$ lb
2/173 $x = 266$ mm
2/174 $M_O = 189.6$ N·m CCW
2/175 $d = 0.733$ m
2/176 $R = 80$ lb down, $M_A = 1240$ lb-in. CW
$x = 15.5$ in.
2/177 $R = 4.75$ kN, $M_A = 21.1$ kN·m CCW
2/178 $\mathbf{M} = 864\mathbf{i} - 6720\mathbf{k}$ lb-in., $M = 6770$ lb-in.
2/179 (*a*) $\mathbf{T}_{AB} = -2.05\mathbf{i} - 1.432\mathbf{j} - 1.663\mathbf{k}$ kN
(*b*) $\mathbf{M}_O = 7.63\mathbf{i} - 10.90\mathbf{j}$ kN·m
$(M_O)_x = 7.63$ kN·m, $(M_O)_y = -10.90$ kN·m
$(M_O)_z = 0$
(*c*) $\mathbf{T}_{AO} = 2.69$ kN
2/180 $\mathbf{F} = F\left[\frac{h\mathbf{i} + r(\cos\theta - 1)\mathbf{j} + r\sin\theta\mathbf{k}}{\sqrt{h^2 + 2r^2(1 - \cos\theta)}}\right]$
2/181 $\mathbf{T} = 282\mathbf{i} + 112.7\mathbf{j} - 398\mathbf{k}$ N, $M_z = -45.0$ N·m
2/182 $\mathbf{M} = -20\mathbf{i} - 6.77\mathbf{j} - 37.2\mathbf{k}$ N·m
2/183 $R = 1093$ lb, $M = 9730$ lb-ft
***2/184** $\theta = 21.7°, T = 1.022$ kN
***2/185** $T = \dfrac{12.5\left(\theta + \frac{\pi}{4}\right)\sqrt{d^2 + 80d\cos\left(\theta + \frac{\pi}{4}\right) - 3200\sin\left(\theta + \frac{\pi}{4}\right) + 3200}}{d\sin\left(\theta + \frac{\pi}{4}\right) + 40\cos\left(\theta + \frac{\pi}{4}\right)}$
***2/186** $(M_O)_{\max} = 0.5FR$ at $\theta = 60°$
***2/187** $M = \dfrac{90\cos\theta[\sqrt{0.34 + 0.3\sin\theta} - 0.65]}{\sqrt{0.34 + 0.3\sin\theta}}$
***2/188** $M_B = 104\sin(\theta + 50°)$ (lb-in.)
$M_A = 154.8\sin\theta + 79.7\cos\theta$ (lb-in.)
***2/189** $\mathbf{M}_O = \dfrac{-76.8\cos\theta\mathbf{i} + 38.4\cos\theta\mathbf{j} - 38.4\sin\theta\mathbf{k}}{\sqrt{1.44 - 1.28\sin\theta}}$
***2/190** $\mathbf{M}_O = \dfrac{3000(\sin\theta + \cos\theta)}{\sqrt{11 + 6(\sin\theta - \cos\theta)}}\mathbf{k}$ lb-ft
$(M_O)_{\max} = 1414$ lb-ft at $\theta = 16.87°$
***2/191** $F_x = [\sqrt{x^2 + 0.0325} - 0.15]\left[\dfrac{-200x_A}{\sqrt{x^2 + 0.0325}}\right]$
$F_y = [\sqrt{x^2 + 0.0325} - 0.15]\left[\dfrac{20}{\sqrt{x^2 + 0.0325}}\right]$
$F_z = [\sqrt{x^2 + 0.0325} - 0.15]\left[\dfrac{30}{\sqrt{x^2 + 0.0325}}\right]$
***2/192** $\mathbf{M}_O = \dfrac{\sqrt{1.5 - 0.5\sin\theta} - 1}{\sqrt{1.5 - 0.5\sin\theta}}[-518\mathbf{i} + 259\cos\theta\mathbf{k}]$ lb-ft
$(M_x)_{\max} = 95.1$ lb-ft at $\theta = 0$ and $\theta = \pi$
$(M_z)_{\max} = 47.6$ lb-ft at $\theta = 0$ and $\theta = \pi$

Chapter 3

3/1 $F_A = 566$ N, $F_B = 283$ N
3/2 $N_A = 12$ lb down, $N_B = 24$ lb up
3/3 $N_A = 15.91$ lb, $N_B = 13.09$ lb
3/4 $A_y = 1864$ N, $B_y = 2840$ N
3/5 $O_x = 1500$ N, $O_y = 6100$ N
$M_O = 7560$ N·m CCW
3/6 $O_x = 0, O_y = 1736$ N, $M_O = 7460$ N·m CW
3/7 $W = 162$ lb
3/8 $T = 131.2$ lb
3/9 (*a*) $P = 5.59$ N, (*b*) $P = 5.83$ N
3/10 (*a*) $P = 6.00$ N, (*b*) $P = 6.25$ N
3/11 $m = 1509$ kg, $x = 1052$ mm

3/12 $P = 225$ N
3/13 $T_{AC} = 215$ N, $T_{BC} = 264$ N
3/14 $T = 81.2$ lb
3/15 $\theta = 18.43°$
3/16 $T = 160$ N
3/17 $T = 90$ N, $O = 94.9$ N
3/18 $N_A = 0.1091mg$, $O = 1.006mg$
3/19 $m_1 = 0.436m$
3/20 $F = 52.8$ N
3/21 $P = 40$ lb
3/22 $O = 64.6$ lb
3/23 $N_A = N_B = 12.42$ kN
3/24 $T = 38.3$ lb, $\overline{CD} = 5.23$ ft
3/25 (*a*) $F_C = 1705$ N, (*b*) $F_C = 1464$ N
3/26 $P = 111.1$ N
3/27 $D_x = L$, $D_y = 1.033L$, $A_y = 1.967L$
3/28 $F = 13.98$ N, $O = 48.8$ N
3/29 $W_L = 550$ lb
3/30 $T = 200$ lb, $A = 188.8$ lb
3/31 $A_x = 869$ N, $A_y = 2350$ N, $M_A = 8810$ N·m CW
3/32 $N_C = 0.433mg$, $O = 0.869mg$
3/33 $N_A = 0.892mg$, $O = 0.580mg$
3/34 $P = 166.7$ N, $T_2 = 1.917$ kN
3/35 $B = 133.3$ N, $C = 253$ N
3/36 $N_C = 18$ lb
3/37 $R = 4.38$ kN
3/38 $A = 1.082$ kN
3/39 $F = 1832$ N
3/40 $F_D = 710$ N, $O_x = 662$ N, $O_y = -185.6$ N
3/41 $F = 753$ N, $E = 644$ N
3/42 $F = 2.25W$, $O = 2.67W$
3/43 $P = 45.5$ N, $R = 691$ N
3/44 $C = 1276$ lb, $P = 209$ lb/in.^{2}
3/45 $M = 41.5$ lb-ft, $F = 35.3$ lb
3/46 (*a*) $n_A = -4.14\%$, $n_B = 4.98\%$
(*b*) $n_A = 3.15\%$, $n_B = -5.97\%$
3/47 $n_A = -32.6\%$, $n_B = 2.28\%$
3/48 $O = 4140$ N, $P = 2.58(10^6)$ Pa
3/49 $R_B = 1382$ N, $O = 2270$ N
3/50 $M = 4.94$ kN·m CCW
3/51 $F = 121.7$ lb
3/52 $P = 26.3$ N
3/53 $T = 0.1176kL + 0.366mg$
3/54 $O = 820$ lb
3/55 $C = \frac{mg}{2}(\sqrt{3} + \frac{2}{\pi})$, $F_A = 1.550mg$
3/56 $A_x = 346$ N, $A_y = 1100$ N, $F_y = 1100$ N
3/57 $F = 187.1$ lb
3/58 $A = 1266$ N, $B = 1514$ N
3/59 $M = 49.9 \sin\theta$ (N·mm) CW

▶**3/60** (*a*) $S = 0.669W$, $C = 0.770W$
(*b*) $S = 2.20W$, $C = 2.53W$
3/61 $\mathbf{R} = -173.9\mathbf{i} - 46.6\mathbf{k}$ N
$\mathbf{M} = -9.32\mathbf{i} - 21.7\mathbf{j} + 34.8\mathbf{k}$ N·m
3/62 $T_{AB} = T_{AC} = 47.8$ lb, $T_{AD} = 31.2$ lb
3/63 $T_A = T_B = 20.4$ lb, $T_C = 27.2$ lb
3/64 $T_{AB} = T_{AC} = 1698$ N, $D = 1884$ N
3/65 $T_A = T_B = 278$ N, $T_C = 262$ N
3/66 $O_x = 1.953$ lb, $O_y = -16.27$ lb, $O_z = 16.27$ lb
$M_x = 81.4$ lb-ft, $M_y = 9.76$ lb-ft
3/67 $T_1 = 4.90$ kN
3/68 $O = 300$ lb, $M = 8280$ lb-ft
3/69 $O = \rho g(a + b + c)$, $M = \frac{\rho g}{2}\sqrt{\frac{a^4}{4} + \frac{b^4}{9}}$
3/70 $m = 1509$ kg, $x = 1052$ mm, $y = -20.5$ mm
3/71 C: $N_A = 2350$ N, $N_B = 5490$ N, $N_C = 7850$ N
D: $N_A = 3140$ N, $N_B = 4710$ N, $N_D = 7850$ N
3/72 $O_x = 0$, $O_y = 0$, $O_z = \rho g(a + b + c)$
$M_x = \rho ga\left(\frac{a}{2} + b + c\right)$, $M_y = -\rho gb\left(\frac{b}{2} + c\right)$
$M_z = 0$
3/73 $O_x = 1962$ N, $O_y = 0$, $O_z = 6540$ N
$T_{AC} = 4810$ N, $T_{BD} = 2770$ N, $T_{BE} = 654$ N
3/74 $A_x = 48$ lb, $B_x = 12$ lb
3/75 $\theta = 9.49°$, $\overline{X} = 118.0$ mm
3/76 $O_x = 0$, $O_y = \rho gh(a + b + c)$, $O_z = 0$
$M_x = \rho ghb\left(\frac{b}{2} + c\right)$, $M_y = 0$
$M_z = \frac{\rho gh}{2}(ab + ac + c^2)$
3/77 $A = 275$ lb, $B = 158.5$ lb, $C = 317$ lb
3/78 $\Delta N_A = 66.1$ N, $\Delta N_B = 393$ N, $\Delta N_C = 522$ N
3/79 $T_A = T_B = 5.41$ kN, $T_C = 9.87$ kN
3/80 $T = 452$ lb, $T_0 = 267$ lb
3/81 $\theta = 30°$
3/82 $O_x = 869$ N, $O_y = 0$, $O_z = 2440$ N
$M_x = -524$ N·m, $M_y = 8700$ N·m
$M_z = 341$ N·m
3/83 $F_S = 3950$ N, $F_A = 437$ N, $F_B = 2450$ N
3/84 $R = mg/\sqrt{7}$
3/85 $P = 2.72$ lb, $F_A = 1.896$ lb, $F_B = 0.785$ lb
$N_A = 15.58$ lb, $N_B = 12.95$ lb
3/86 $k = 16.41$ lb/in.
3/87 $\Delta N_A = 1000$ N, $\Delta N_B = \Delta N_C = -500$ N
3/88 $B = 2.36$ kN
3/89 $C = 100$ lb, $B_{xy} = 45.5$ lb, $A_{xy} = 38.0$ lb
3/90 $P_{\min} = 18$ N, $B = 30.8$ N, $C = 29.7$ N
If $P = P_{\min}/2$: $D = 13.5$ N
3/91 $A = 31.4$ lb, $B = 72.1$ lb
3/92 $P = 0.206$ N, $A_y = 0.275$ N, $B_y = -0.0760$ N

3/93 $F_B = 70$ lb, $D_n = 101.1$ lb

▶**3/94** $M = 47.7$ N·m, $V = 274$ N, $P = 321$ N

▶**3/95** $M_x = -1.857$ N·m, $M_y = 1.411$ N·m
$M_z = -2.56$ N·m, $B_z = 14.72$ N

▶**3/96** $T = 277$ N, $B = 169.9$ N

▶**3/97** $P = 50$ N, $N_C = 58.1$ N, $N_A = 108.6$ N
$N_B = 32.4$ N

▶**3/98** $T = 471$ N, $O = 144.9$ N

3/99 $L = 1.676$ kN

3/100 $R = 566$ N

3/101 $A = 15$ lb, $B = 40$ lb, $T = 60.2$ lb

3/102 $P = 351$ N

3/103 $N_A = 159.9$ lb down, $N_B = 129.9$ lb up

3/104 $N_A = \sqrt{3}g\left(\frac{m}{2} - \frac{m_1}{3}\right)$

$(a)\ m_1 = 0.634m$, $(b)\ m_1 = \frac{3}{2}m$

3/105 $F_A = 486$ N down, $F_B = 686$ N up

3/106 $T_1 = 45.8$ kN, $T_2 = 26.7$ kN, $A = 44.2$ kN

3/107 $A' = 41.7$ lb

3/108 $D = 7.60$ kN

3/109 $\Delta F = 521$ N, $\Delta T = 344$ N

3/110 $\bar{x} = 199.2$ mm

3/111 $T_A = 147.2$ N, $T_B = 245$ N, $T_C = 196.2$ N

3/112 $T = 1053$ N

3/113 $F_A = 24.5$ lb

3/114 $P = \dfrac{mg}{3\sqrt{2}}$

3/115 $A = 183.9$ N, $B = 424$ N

▶**3/116** $O_x = -14\,900$ N, $O_y = 24\,400$ N, $O_z = 8890$ N
$M_x = -19\,390$ N·m, $M_y = -12\,120$ N·m
$M_z = 857$ N·m

***3/117** $T = mg\dfrac{\frac{\sqrt{3}}{2}\cos\theta - \frac{\sqrt{2}}{4}\cos(\theta + 15°)}{\cos\theta}$

***3/118** $T = \left[\dfrac{2750\cos\theta + 687\cos(\theta + 60°)}{4\sin(\theta + 60°)}\right] \times$
$[8 + 8\cos(\theta + 60°)]^{\frac{1}{2}}$
$\theta_{\max} = 79.1°$

***3/119** $(a)\ R = \frac{1}{3}[400x^2 - 400x + 1072]^{\frac{1}{2}}$
$(b)\ R_{\min} = 10.39$ kN at $x = \frac{1}{2}$ m
$(c)\ R_{\max} = 24.3$ kN at $x = 3.8$ m

***3/120** $(T_{AC})_{\max} = 600$ N at $x = 3.91$ m
$(T_{BC})_{\max} = 600$ N at $x = 6.09$ m

***3/121** $\alpha = 14.44°$, $\beta = 3.57°$, $\gamma = 18.16°$
$T_{AB} = 529$ lb, $T_{BC} = 513$ lb, $T_{CD} = 539$ lb

***3/122** $\overline{AB} = 17.01$ ft, $\overline{CD} = 8.99$ ft
$T_{AB} = 503$ lb, $T_{BC} = 493$ lb, $T_{CD} = 532$ lb

***3/123** $T_{45°} = 5.23$ N, $T_{90°} = 8.22$ N

***3/124** $T = 0$ at $\theta = 1.729°$

***3/125** $T = \dfrac{51.1\cos\theta - 38.3\sin\theta}{\cos\theta}\sqrt{425 - 384\sin\theta}$

***3/126** $T = 495$ N at $\theta = 15°$

Chapter 4

4/1 $AB = 2350$ N T, $AC = 981$ N T
$BC = 2550$ N C

4/2 $AB = 3400$ N T, $AC = 981$ N T
$BC = 1962$ N C

4/3 $AB = 1.115W$ T, $AC = 0.577W$ T
$BC = 0.816W$ C

4/4 $AB = 5$ kN T, $AC = 5\sqrt{5}$ kN T
$BC = 5\sqrt{2}$ kN C $AD = 0$, $CD = 15$ kN C

4/5 $AC = 849$ lb C, $AD = 849$ lb C, $DE = 849$ lb T

4/6 $AB = BE = CD = 1$ kN T
$AE = BD = 1.414$ kN C
$BC = 2$ kN T, $DE = 1$ kN C

4/7 $AB = DE = 96.0$ kN C, $AH = EF = 75$ kN T
$BC = CD = 75$ kN C
$BH = DF = CG = 60$ kN T, $CH = CF = 48.0$ kN C
$GH = FG = 112.5$ kN T

4/8 $AB = 14.42$ kN T, $AC = 2.07$ kN C, $AD = 0$
$BC = 6.45$ kN T, $BD = 12.89$ kN C

4/9 $AB = 12$ kN T, $AE = 3$ kN C, $BC = 5.20$ kN T
$BD = 6$ kN T, $BE = 5.20$ kN C
$CD = DE = 6$ kN C

4/10 $AB = 1000$ lb T, $AE = BC = DE = 707$ lb T
$DF = 1000$ lb C, $BE = CD = EF = 707$ lb C

4/11 $AB = BC = \dfrac{L}{2}$ T, $BD = 0$

4/12 $AB = 6.61$ kN T, $AE = 21.0$ kN T
$BE = CD = 12.62$ kN C, $BC = 2.00$ kN T
$CE = 12.62$ kN T, $DE = 7$ kN T

4/13 $AE = CD = 2000/\sqrt{3}$ lb C
$AB = BC = 1000/\sqrt{3}$ lb T
$BE = BD = 800/\sqrt{3}$ lb T, $DE = 1400/\sqrt{3}$ lb C

4/14 $AB = 9\sqrt{3}$ kN C, $AE = 5\sqrt{3}$ kN T
$BC = \frac{26}{3}\sqrt{3}$ kN C, $BD = 3\sqrt{3}$ kN C
$BE = \frac{7}{3}\sqrt{3}$ kN C
$CD = \frac{13}{3}\sqrt{3}$ kN T, $DE = \frac{11}{3}\sqrt{3}$ kN T

4/15 $EF = 3.87$ kips C, $DE = 4.61$ kips T
$DF = 4.37$ kips C, $CD = 2.73$ kips T
$FG = 7.28$ kips C

4/16 (a) BD slack, $AB = AD = 0$, $BC = L$ C
$AC = \frac{5L}{4}$ T, $CD = \frac{3L}{4}$ C
(b) BD slack, $AB = AD = BC = 0$
$CD = \frac{3L}{4}$ C, $AC = \frac{5L}{4}$ T

4/17 $AB = BC = 3.73$ kN C, $AD = 4.23$ kN T
$BD = 3.73$ kN T, $CD = 3.23$ kN T

4/18 $AB = BC = L$ T, $AF = EF = L$ C
$DE = CD = \frac{L}{2}$ T, $BF = DF = BD = 0$

4/19 $AB = 2.69$ kN C, $BH = 5.39$ kN C, $BG = 0$

4/20 $CF = 1538$ lb C, $CG = 4170$ lb T, $EF = 0$

4/21 $AB = DE = BC = CD = 3.35$ kN C
$AH = EF = 3$ kN T, $BH = DF = 1$ kN C
$CF = CH = 1.414$ kN T, $CG = 0$
$FG = GH = 2$ kN T

4/22 $AB = DE = 3.35$ kN C, $BC = CD = 2.24$ kN C
$AH = EF = GH = FG = 3$ kN T
$BH = DF = 0$, $CG = 1$ kN T
$BG = DG = 1.118$ kN C

4/23 $AB = AH = 130.7$ lb T, $BO = HO = 58.6$ lb C
$BC = GH = 207$ lb T, $CO = GO = 200$ lb C
$CD = FG = 315$ lb T, $DO = FO = 341$ lb C
$DE = EF = 392$ lb T, $EO = 400$ lb C

4/24 $FD = 24{,}500$ lb T

4/25 $AB = CD = 8.50$ kN T, $AF = 15.47$ kN C
$BC = 12.01$ kN T, $BF = 0.759$ kN C
$CE = 0$, $CF = 8.50$ kN C
$DE = EF = 9.81$ kN C

4/26 $GJ = 4.44$ tons C, $GI = 15.38$ tons T

▶**4/27** $CG = 0$

▶**4/28** $AB = 3.89$ kN C, $DB = 0$, $CD = 0.932$ kN C

▶**4/29** $EF = 75.1$ kN C, $KL = 40$ kN T, $GL = 20$ kN T

▶**4/30** $CM = 3.41L$ T

4/31 $CG = 14.14$ kips T

4/32 $CG = 0$, $GH = L$ T

4/33 $BE = 5.59$ kN T

4/34 $EF = 0.751L$ T

4/35 $AB = 7.2$ kN T, $BG = 3$ kN C, $GF = 7.8$ kN C

4/36 $BC = 4.32L$ T, $CF = 1.278L$ C, $EF = 2.53L$ C

4/37 $DG = L$ T

4/38 $BC = BE = \frac{2L}{\sqrt{3}}$ T, $BF = \frac{2L}{\sqrt{3}}$ C

4/39 $BG = 400\sqrt{5}$ lb C, $BF = 800$ lb T

4/40 $BC = CG = \frac{L}{3}$ T

4/41 $BC = 150$ lb T, $FG = 150$ lb C

4/42 $DG = \frac{L}{5}$ T

4/43 $BC = 3.00$ kN C, $CJ = 16.22$ kN C
$CI = 5.00$ kN T, $HI = 10.50$ kN T

4/44 $CG = \frac{2L}{\sqrt{3}}$ C, $GF = L\sqrt{3}$ C

4/45 $BH = 0.683L$ T, $CD = 1.932L$ C

4/46 $FN = GM = 84.8$ kN T, $MN = 20$ kN T

4/47 $AB = 3.78$ kN C

4/48 $GM = 0$

4/49 $BE = 0.787L$ T

4/50 $DJ = 0.45L$ T, $EJ = 0.360L$ T

4/51 $DE = 297$ kN C, $EI = 26.4$ kN T
$FI = 205$ kN T, $HI = 75.9$ kN T

4/52 $HP = 2.57L$ T

4/53 $BC = 100$ kN C, $CG = 0$, $EF = 45.7$ kN T
$FG = 25$ kN T

4/54 $CG = 0$

4/55 $JQ = 57.7$ kN C

▶**4/56** $DF = 768$ kN C, $EF = 364$ kN C
$A_x = 101.1$ kN right

▶**4/57** $DG = 0.569L$ C

▶**4/58** $FI = ER = 0$, $FJ = 7.81$ kN T
$EJ = 3.61$ kN C, $EK = 22.4$ kN C

4/59 $AB = -4.46$ kN, $AC = -1.521$ kN
$AD = 1.194$ kN

4/60 $BC = BD = CD = 55.6$ lb

4/61 $CD = 2.4L$

4/62 $B_x = P$

4/63 $AC = 0.283L$, $BD = 0$

4/64 $AC = 0.567L$, $BD = -1.014L$

4/65 $AF = \frac{\sqrt{13}}{3\sqrt{2}}P$, $D_x = \frac{P}{3\sqrt{2}}$

4/66 2

4/67 $AD = -0.625L$, $DG = -2.5L$

4/68 $AB = -\frac{4}{27}mg$, $AC = -\frac{5}{54}mg$

4/69 $BC = \frac{L\sqrt{2}}{4}$, $CD = 0$, $CE = -\frac{L\sqrt{3}}{2}$

4/70 $BC = CD = DE = EF = BF = 0.681L$

4/71 $BE = -2.36$ kN

▶**4/72** $FE = -\frac{P}{\sqrt{3}}$, $EG = \frac{P}{\sqrt{6}}$

4/73 $B = 4170$ lb

4/74 $M = 150$ lb-ft, $A_x = 86.6$ lb

4/75 $A_x = C_x = 0.293P$, $A_y = P$, $C_y = 0$

4/76 (a) $\mathbf{F} = 900(\cos 30° \mathbf{i} + \sin 30° \mathbf{j})$ N
(b) $\mathbf{F} = 779\mathbf{i}$ N

4/77 $F = 217$ N, $A = 297$ N

4/78 $C = 6470$ N

4/79 $EF = 100$ N T, $F = 300$ N

4/80 $F = 125.3P$

4/81 on AB: $B_x = 0.833$ kN left, $B_y = 0.833$ kN up
$A_x = 0.833$ kN right, $A_y = 4.16$ kN up
4/82 $D = 58.5$ N
4/83 $F = 25$ lb
4/84 $P = 217$ N
4/85 $N = 166.4$ N
4/86 $N = 13.19P$
4/87 $B = 68.8$ N
4/88 $R = 1111$ lb
4/89 $Q = P\dfrac{b}{a}$
4/90 $AB = 7400$ lb C, $EF = 0$
4/91 $P = 127.8$ N
4/92 $A = 315$ kN
4/93 $A = 999$ N, $F = 314$ N up
4/94 $T = \dfrac{24}{35}L$
4/95 $F = 45.4P$
4/96 $AB = 5310$ N C, $C = 4670$ N
4/97 $P = 2050$ N
4/98 $C = 1368$ N
4/99 $p = 267$ kPa
4/100 $AB = 1650$ lb T
4/101 $E_x = 1308$ N, $E_y = 122.6$ N
4/102 $M_A = 9.94$ N·m CCW
4/103 $AB = 142.8$ kN C
4/104 $CD = 127.8$ kN C
4/105 $P = 147.4$ N
4/106 $A = 34{,}000$ lb, $D = 17{,}100$ lb
4/107 $F = 1.063$ kN, $P = 1.260$ kN
4/108 $C = 1229$ lb, $A_n = B_n = 692$ lb
4/109 $C = 2090$ lb, $B_y = 1012$ lb
4/110 $A = 833$ N, $R = 966$ N
4/111 $C = 77.2$ lb
4/112 $T = \dfrac{\sqrt{2}\,PL}{x}$
4/113 $E = 820$ N
4/114 $A = 4550$ N, $B = 4410$ N, $C = D = 1898$ N
$E = F = 5920$ N
4/115 $A_y = 75$ N, $B_y = 150$ N, $D_y = 225$ N
$B_x = D_x = E_x = 173.2$ N, $E_y = 750$ N
$G_x = 0$, $G_y = 525$ N
4/116 $F_{AB} = 6710$ lb
4/117 $F_s = 45.2$ N
4/118 $A = 1.748$ kN
4/119 $C = 249$ N
4/120 $C = 235$ N
4/121 $AB = 9200$ N C
4/122 $HJ = 6220$ N C
4/123 $AB = 17{,}140$ lb C, $O = 16{,}590$ lb
4/124 $CE = 7440$ lb C
4/125 $F = 131.8$ kN, $H = 113.9$ kN
4/126 $M = 17.08$ N·m CCW
4/127 $P = 338$ lb, $E = 75.1$ lb
▶**4/128** $P = 26{,}900$ lb, $A = 14{,}600$ lb
4/129 $A_x = 0.833$ kN, $A_y = 5.25$ kN
$A_z = -12.50$ kN
4/130 $T = 1569$ lb, $EF = 429$ lb T
4/131 $BE = 7$ kN T
4/132 $F = 1478$ lb
4/133 $AB = DE = 13.52$ kips C
$AF = EF = 11.25$ kips T
$BC = CD = 9.01$ kips C
$BF = DF = 4.51$ kips C, $CF = 5$ kips T
4/134 $G = 181.8$ N
4/135 $CE = 655$ N C, $DE = 1057$ N T
$DF = 2250$ N C
4/136 $CH = 0$, $CD = 4.74$ kN C, $AH = 4.5$ kN T
4/137 $C = P\sqrt{2}$
4/138 $M = 1250$ lb-in. CCW
4/139 $BF = 24.3$ kN T
4/140 $BG = \dfrac{2L}{3\sqrt{3}}\,T$, $CG = \dfrac{4L}{3\sqrt{3}}\,T$
4/141 $AB = 2.26L$ T, $BI = L$ T, $CI = 0.458L$ T
4/142 $AB = 1.850L$ T, $BI = L$ T, $CI = 0.833L$ T
4/143 $AB = 59{,}900$ lb C, $p = 3420$ lb/in.^{2}
▶**4/144** $BE = 1.275$ kN T
▶**4/145** $R_x = 838$ lb, $R_y = 43.3$ lb
▶**4/146** $GJ = 70.8$ kN C, $FJ = 0$
▶**4/147** $AB = -\dfrac{\sqrt{2}L}{4}$, $DA = -\dfrac{\sqrt{2}L}{8}$
***4/148** $(F_A)_{\max} = 6$ kN at $\theta = -26.6°$
***4/149** $C = \dfrac{60}{\sin\theta}$ (lb)
***4/150** $R_{\max} = 1314$ lb at $\theta = 45°$
***4/151** $AB = -4\left(\cos\theta + \dfrac{\sqrt{3}}{3}\sin\theta\right)$
$AC = 4\left(-\cos\theta + \dfrac{\sqrt{3}}{3}\sin\theta\right)$
$BC = 2\sqrt{3}\left(\cos\theta + \dfrac{\sqrt{3}}{3}\sin\theta\right)$
***4/152** $(AC)_{\min} = 2.60$ kN T at $\alpha = 30°$
$(CD)_{\min} = 6$ kN C at $\alpha = 0$
***4/153** $(F_{DE})_{\max} = 3580$ N at $\theta = 0$
$(F_{DE})_{\min} = 0$ at $\theta_{\max} = 65.9°$
***4/154** (a) $BC = -260\,\dfrac{\sin(\theta - 22.6°)}{\cos\theta}$
(b) $BC = 0$ when $\theta = 22.6°$
(c) $0 \le \theta \le 77.7°$
***4/155** $(F_{BC})_{\max} = 2800$ N at $\theta = 5°$
***4/156** $\theta = 0$: $R = 75\,000$ N, $AB = 211\,000$ N
$C_x = -85\,400$ N
$R_{\min} = 49\,400$ N at $\theta = 23.2°$

Chapter 5

5/1 $(5.67, 3.67)$

5/2 $\bar{x} = 0, \bar{y} = 110.3 \text{ mm}$

5/3 $(0, -50.9, -180) \text{ mm}$

5/4 $(-50.9, 120, 69.1) \text{ mm}$

5/5 $\theta = 32.5°$

5/6 $\bar{y} = \dfrac{14R}{9\pi}$

5/7 $\bar{y} = \dfrac{2(h^3 - a^3)}{3(h^2 - a^2)}$

5/8 $\bar{x} = \dfrac{h(a + 2b)}{3(a + b)}, \bar{y} = \dfrac{(a^2 + ab + b^2)}{3(a + b)}$

5/9 $\bar{x} = 1.549, \bar{y} = 0.756$

5/10 $\bar{x} = \dfrac{3b}{10}, \bar{y} = \dfrac{3a}{4}$

5/11 $\bar{x} = \dfrac{2a}{\pi}, \bar{y} = \dfrac{\pi b}{8}$

5/12 $\bar{x} = \dfrac{7a}{6(\pi - 1)}, \bar{y} = \dfrac{a}{\pi - 1}$

5/13 $\bar{x} = \dfrac{3b}{5}, \bar{y} = \dfrac{3a}{8}$

5/14 $\bar{z} = \dfrac{3h}{4}$

5/15 $\bar{x} = 2.05, \bar{y} = 0.405$

5/16 $\bar{x} = \dfrac{a^2 + b^2 + ab}{3(a + b)}, \bar{y} = \dfrac{h(2a + b)}{3(a + b)}$

5/17 $\bar{x} = \dfrac{2a}{5}, \bar{y} = \dfrac{3b}{8}$

5/18 $\bar{x} = \dfrac{2a}{5}, \bar{y} = \dfrac{23b}{50}$

5/19 $\bar{x} = \dfrac{4}{9} \text{ ft}$

5/20 $\bar{x} = \bar{y} = \dfrac{2a}{3(\pi - 2)}$

5/21 $\bar{x} = \dfrac{17L}{28}$

5/22 $\bar{x} = \dfrac{12a}{25}, \bar{y} = \dfrac{3a}{7}$

5/23 $\bar{x} = \dfrac{a}{3\left(\dfrac{\pi}{2} - 1\right)}, \bar{y} = \dfrac{b}{3\left(\dfrac{\pi}{2} - 1\right)}$

5/24 $\bar{x} = 0.777a, \bar{y} = 0.223a$

5/25 $\bar{x} = \bar{y} = -\dfrac{8}{3\pi} \text{ in.}, \bar{z} = 5 \text{ in.}$

5/26 $\bar{y} = \dfrac{103b}{350}$

5/27 $\bar{y} = \dfrac{31b}{90}$

5/28 $\bar{x} = \dfrac{a}{\pi - 1}, \bar{y} = \dfrac{7a}{6(\pi - 1)}$

5/29 $\bar{y} = \dfrac{14\sqrt{2}}{9\pi} a$

5/30 $\bar{z} = \dfrac{5a}{8}$

5/31 $\bar{h} = 1.633 \text{ in.}$

5/32 $\bar{x} = \dfrac{4}{3\pi} (a + b)$

5/33 $\bar{h} = \dfrac{11h}{56}$

5/34 $h = \dfrac{R}{4}: \bar{x} = \dfrac{25R}{48}, h = 0: \bar{x} = \dfrac{3R}{8}$

5/35 $\bar{x} = \dfrac{5R}{8}$

5/36 $\bar{y} = \dfrac{3h}{8}$

5/37 $\bar{z} = 263 \text{ mm}$

5/38 $\bar{y} = 57.4 \text{ mm}$

5/39 $\bar{y} = \dfrac{h}{\pi}$

▶**5/40** $\bar{z} = \dfrac{r}{2}$

▶**5/41** $\bar{y} = \dfrac{\dfrac{2}{3} (a^2 - h^2)^{3/2}}{a^2\left(\dfrac{\pi}{2} - \sin^{-1} \dfrac{h}{a}\right) - h\sqrt{a^2 - h^2}}$

▶**5/42** $\bar{y} = 0.380a$

▶**5/43** $\bar{x} = \bar{y} = \left(\dfrac{4}{\pi} - \dfrac{3}{4}\right)a, \bar{z} = \dfrac{a}{4}$

▶**5/44** $\bar{x} = 1.542R$

▶**5/45** $\bar{x} = 1.583R$

▶**5/46** $\bar{x} = \bar{y} = 0.242a$

5/47 $\overline{X} = 132.1 \text{ mm}, \overline{Y} = 75.8 \text{ mm}$

5/48 $\overline{X} = 2.55 \text{ in.}, \overline{Y} = 1.362 \text{ in.}$

5/49 $\overline{X} = 233 \text{ mm}, \overline{Y} = 333 \text{ mm}$

5/50 $\overline{Y} = \dfrac{4h^3 - 2\sqrt{3}a^3}{6h^2 - \sqrt{3}\pi a^2}$

5/51 $\overline{X} = 244 \text{ mm}, \overline{Y} = 117.7 \text{ mm}$

5/52 $\overline{Y} = 9.93 \text{ in.}$

5/53 $\overline{X} = \overline{Y} = 0, \overline{Z} = 0.389r$

5/54 $\overline{X} = 3.50 \text{ in.}, \overline{Y} = 7.83 \text{ in.}$

5/55 $\overline{X} = 4.56 \text{ in.}, \overline{Y} = 3.14 \text{ in.}$

5/56 $\overline{Y} = 95.6 \text{ mm}$

5/57 $\overline{H} = 39.3 \text{ mm}$

5/58 $\overline{X} = \dfrac{3b}{10}, \overline{Y} = \dfrac{4b}{5}, \overline{Z} = \dfrac{3b}{10}$

5/59 $\overline{X} = \overline{Y} = \overline{Z} = \dfrac{13a}{28}$

5/60 $\overline{Y} = 5.35 \text{ in.}$

5/61 $\overline{X} = -8.26 \text{ mm}, \overline{Y} = -31.4 \text{ mm}, \overline{Z} = 10.33 \text{ mm}$

5/62 $\overline{X} = \dfrac{45R}{112}$

5/63 $\overline{X} = 0.395b$, $\overline{Y} = 0.25b$, $\overline{Z} = 0.345b$

5/64 $\overline{X} = 347$ mm, $\overline{Y} = -90.5$ mm

5/65 $\overline{X} = -33.3$ mm, $\overline{Y} = 20$ mm, $\overline{Z} = 16.36$ mm

5/66 $\overline{X} = 58.4$ mm, $\overline{Y} = \overline{Z} = 40$ mm

5/67 $\overline{X} = 1.595$ in., $\overline{Z} = 1.559$ in.

5/68 $l = (8 - \pi)r$

5/69 $\overline{X} = \dfrac{3a}{6 + \pi}$, $\overline{Y} = -\dfrac{2a}{6 + \pi}$, $\overline{Z} = \dfrac{\pi a}{6 + \pi}$

5/70 $\overline{Z} = 0.642R$

5/71 $\overline{Z} = 0.671R$

5/72 $\overline{X} = 0.475a$, $\overline{Y} = 0.424a$, $\overline{Z} = -0.1008a$

5/73 $\overline{X} = 56.4$ mm, $\overline{Y} = 95.0$ mm, $\overline{Z} = 56.8$ mm

5/74 $\overline{X} = 2.48$ in., $\overline{Y} = 2.71$ in., $\overline{Z} = -0.882$ in.

5/75 $\overline{Y} = 28.8$ mm

▶ **5/76** $\overline{X} = -0.509L$, $\overline{Y} = 0.0443R$, $\overline{Z} = -0.01834R$

5/77 $A = 10\,300$ mm^2, $V = 24\,700$ mm^3

5/78 $V = 0.345(10^6)$ mm^3

5/79 $V = 2.83(10^5)$ mm^3

5/80 $A = 2\sqrt{2}\pi a^2$

5/81 $V = \dfrac{2\pi\sqrt{2}a^3}{3}$

5/82 8.82 gal

5/83 $m = 0.228$ kg

5/84 $A = 177\,100$ mm^2

5/85 $m = 0.293$ kg

5/86 $A = 29.3$ in.^{2}, $V = 6.56$ in.^{3}

5/87 $A = 1074$ in.^{2}

5/88 $A = \pi r^2(\pi - 1)$

5/89 $m = 0.639$ kg

5/90 $A = \pi a^2(\pi - 2)$

5/91 $m = 84.5$ kg

5/92 $\bar{r} = 77.8$ mm

5/93 $A = 157.9$ in.^{2}

5/94 $V = 361\,000$ mm^3

5/95 $A = 105\,800$ mm^2, $V = 1.775(10^6)$ mm^3

5/96 $A = 89.3$ in.^{2}, $V = 33.1$ in.^{3}

5/97 $V = 4bc\left(a + \dfrac{b\pi}{8}\right)$

5/98 $A = 4.62$ m^2

5/99 $W = 608$ kN

5/100 $m = 1.126(10^6)$ Mg

5/101 $A_x = 0$, $A_y = 625$ N, $B_y = 1375$ N

5/102 $R_A = 13.33$ lb up, $R_B = 207$ lb up

5/103 $R_A = 6$ kN up, $M_A = 3$ kN·m CW

5/104 $R_A = 360$ lb up, $M_A = 2880$ lb-ft CCW

5/105 $A_x = 750$ N, $A_y = 3.07$ kN, $B_y = 1.224$ kN

5/106 $R_A = 2230$ N up, $R_B = 2170$ N up

5/107 $R_A = R_B = \dfrac{w_0L}{4}$ up

5/108 $A_x = 0$, $A_y = \dfrac{2w_0L}{9}$, $B_y = \dfrac{5w_0L}{18}$

5/109 $M_O = 6600$ lb-ft CCW

5/110 $R_A = 5.5$ kips down, $M_A = 71.5$ kip-ft CW

5/111 $R_A = 24.1$ kN up, $R_B = 19.87$ kN up

5/112 $P = 6.96$ kN

5/113 $R_A = \dfrac{2w_0l}{\pi}$ up, $M_A = \dfrac{w_0l^2}{\pi}$ CCW

5/114 $R_A = \dfrac{2w_0l}{3}$ up, $M_A = \dfrac{w_0l^2}{3}$ CCW

5/115 $R_A = 14.29$ kN down, $R_B = 14.29$ kN up

5/116 $R_A = 3.8$ kN up, $M_A = 10.20$ kN·m CCW

5/117 $A = 2.5$ kN, $B = 6.61$ kN

5/118 $R_A = 667$ lb up, $R_B = 1167$ lb up

5/119 $A_y = 34.7$ kN, $M_A = 76$ kN·m CCW

5/120 $R_A = 1900$ lb up, $R_B = 1600$ lb up

5/121 $A_x = 0$, $A_y = \dfrac{2w_0l}{3}$ up, $M_A = \dfrac{2w_0l^2}{5}$ CW

5/122 $B_x = 4$ kN, $B_y = 1.111$ kN up, $A_y = 5.56$ kN up

5/123 $R_A = R_B = 7$ kN up

▶ **5/124** $C_A = V_A = pr$, $M_A = pr^2$ CCW

5/125 $x = \dfrac{l}{2}$: $V = \dfrac{P}{3}$, $M = \dfrac{Pl}{6}$

5/126 $M = \dfrac{-Pl}{2}$

5/127 $M_B = -1488$ lb-ft

5/128 $d = 2.67$ m

5/129 $V = -C/l$

5/130 $V = \dfrac{3M_0}{2l}$, $M = \dfrac{M_0}{4}$

5/131 $M_C = -1667$ lb-ft

5/132 $M_{\max} = \dfrac{5Pl}{16}$ at $x = \dfrac{3l}{4}$

5/133 $M_{\max} = \dfrac{5Pl}{6}$

5/134 $M = -1$ kN·m

5/135 $M_{\max} = \dfrac{wl^2}{8}$

5/136 $M_{\max} = 3000$ lb-ft

5/137 $M = 4800$ lb-ft

5/138 $V = 2.4$ kN, $M = -4.8$ kN·m

5/139 $w = -2200 + 120x^2$, $M = 6400$ lb-ft

5/140 $M_A = -\dfrac{w_0l^2}{3}$

5/141 $M_{\max} = \dfrac{w_0l^2}{12}$

5/142 $M_{\max} = \dfrac{w_0L^2}{24}$

5/143 $x = \dfrac{a + l}{2}$, $(M_A)_{\max} = \dfrac{L}{4l}(l - a)^2$

5/144 $M = -Fh$

5/145 $M_B = -0.40$ kN·m, $x = 0.2$ m

5/146 $V = 1467$ lb, $M = -200$ lb-ft

5/147 $V = -600$ N, $M = 4800$ N·m
$M_{max} = 5620$ N·m at $x = 4.25$ m

5/148 $V = -1400$ N, $M = 0$
$M_{max} = 2800$ N·m at $x = 3.25$ m

5/149 $T = 500$ N, $V = -62.5$ N, $M = 64.4$ N·m

5/150 $V = -9$ kN, $M = -15$ kN·m
$R_A = 28$ kN up, $R_B = 16$ kN down

5/151 $V = 2000 - 100x - x^3$

$$M = -12{,}500 + 2000x - 50x^2 - \frac{1}{4}x^4$$

5/152 $V = 1.6$ kN, $M = -2.33$ kN·m

5/153 $V = w_0\left(\frac{l}{3} - x + \frac{4x^3}{3l^2}\right)$

$$M = w_0\left(-\frac{l^2}{16} + \frac{xl}{3} - \frac{x^2}{2} + \frac{x^4}{3l^2}\right)$$

►**5/154** $x = 0.207L$, $M_{max} = 0.0214wL^2$

5/155 $y = 0.750$ in.

5/156 $T_0 = 50.0(10^6)$ lb, $C = 44.7(10^6)$ lb

5/157 $T_0 = 18$ lb at C, $T_{max} = 51.3$ lb at A and B

***5/158** $T_A = 58\,100$ N, $T_B = 58\,800$ N

5/159 $m = 270$ kg, $\overline{AC} = 79.1$ m

***5/160** $m = 282$ kg, $\overline{AC} = 79.6$ m

5/161 $T_0 = 33.7$ kN, $T_A = 37.4$ kN, $T_B = 40.8$ kN

5/162 $T_0 = \frac{w_0 l^2}{2h}\left(1 + \frac{w_1 - w_0}{3w_0}\right)$

5/163 $y = (123.5x^2 - 0.01372x^4)10^{-4}$ ft

►**5/165** $T_C = 236$ lb, $L = 23.0$ ft

►**5/166** $L = 46.2$ m

►**5/167** Catenary: $T_0 = 408$ lb; Parabolic: $T_0 = 400$ lb

5/168 $l = 49.7$ ft

***5/169** $L = 33.0$ ft

***5/170** $T = 1665$ N, $s = 32.1$ m

***5/171** $h = 3.36$ m, $T_h = 3.36$ N, $T_v = 0.756$ N

***5/172** $T_h = 8.53$ N, $T_v = 2.06$ N

5/173 $s = 251$ m, $H = 24.5$ m

***5/174** $T = 3770$ N, $h = 33.8$ m

***5/175** $h = 2.69$ m, $\theta = 55.9°$

***5/176** $T_A = 736$ N, $T_B = 2310$ N

***5/177** $T_0 = 1210$ N

***5/178** $T_0 = 6630$ N, $T = 6690$ N

***5/179** $L = 13.06$ m, $T_B = 229$ N, $\theta_A = 12.64°$

***5/180** $\theta_A = 16.98°$, $\theta_B = 28.0°$, $T_A = 355$ N
$T_B = 384$ N

***5/181** (*a*) $L_P = 11.32$ m, (*b*) $L_C = 11.40$ m

***5/182** $\delta = 0.724$ m

***5/183** $n = 29.0\%$

►***5/184** $x_B = 125.0$ m, $y_B = -33.6$ m

5/185 0.872 lb, 1 lb

5/186 $V = 5.71$ m^3

5/187 $r = 0.8$, $r = 0.577$

5/188 $m = 90.2$ kg

5/189 $h = 1.988$ m

5/190 $C = 95.5$ kN

5/191 $\sigma = 462$ MPa

5/192 $T = 26.7$ N

5/193 $\theta = \sin^{-1}\left(\frac{h}{L}\sqrt{\frac{\rho}{\rho'}}\right)$

5/194 $\sigma = 10.74$ kPa, $P = 1.687$ kN

5/195 $P = 12.57$ kN

5/196 $C = 2.04$ kN

5/197 $T = 8960$ lb

5/198 $h = 2.75$ in.

5/199 $C = 666$ lb

5/200 $T = 121.6$ lb, $h = 4.42$ ft

5/201 $\rho_s = \rho_l\left(\frac{h}{2r}\right)^2\left(3 - \frac{h}{r}\right)$

5/202 $h = a\sqrt{3}$

5/203 $M = 11.22(10^4)$ lb-ft

5/204 $d = 5.77$ in.

5/205 $R = 562$ lb, $H = 2.25$ ft

5/206 $T = 609$ lb

5/207 $Q = \frac{1}{2}\pi r p_0$

5/208 A requires 12,470 lb/ft less than B

5/209 $R = 156.0(10^6)$ lb

5/210 $m = 6.57$ kg

5/211 $\sigma = 262$ kPa, $\Delta T = 1192$ N

5/212 $\theta = 48.2°$

5/213 $d = 0.300$ m

►**5/214** $m = 4.24$ Mg

5/216 $\overline{GM} = 0.530$ m

5/217 $\overline{X} = 93.3$ mm, $\overline{Y} = 66.7$ mm

5/218 $\bar{x} = \frac{23b}{25}, \bar{y} = \frac{2b}{5}$

5/219 $\overline{X} = 0.1263r$, $\overline{Y} = 0.1036r$

5/220 $\overline{X} = \frac{s(2 + 2\sin\theta + \sin\theta\cos\theta)}{2(1 + \sin\theta)}$

$\overline{Y} = \frac{s}{2}(1 + \sin\theta)$

5/221 $m = 1.183$ kg

5/222 $m = 11.55$ kg

5/223 $V = 280\,000$ mm^3, $A = 30\,900$ mm^2

5/224 $\overline{Y} = 0.461b$, $\overline{Z} = 0.876b$

5/225 $\overline{X} = 1.845$ mm, $\overline{Y} = 0$
$\overline{Z} = 6.02$ mm, $m = 0.0693$ kg

5/226 $\overline{X} = 24.4$ mm, $\overline{Y} = 60$ mm
$\overline{Z} = 29.7$ mm, $m = 0.932$ kg

5/227 $\bar{x} = \frac{2h}{3}, \bar{z} = \frac{4r}{3\pi}$

5/228 $S = 147.9$ ft

5/229 $h = r\sqrt{3}$

5/230 $\bar{y} = \frac{15a}{14\pi}$

5/231 $\bar{h} = \dfrac{11H}{28}$

5/232 $\bar{h} = \dfrac{4H}{9}$

5/233 $\bar{h} = \dfrac{5H}{16}$

5/234 $P = 348$ kN

5/235 $M_{\max} = 500$ lb-ft at $x = 6$ ft

5/236 $M_{\max} = 186.4$ N·m at $x = 0.879$ m, 5.12 m

5/237 $M = -11.12$ N·m, $V = 55.6$ N

5/238 $V = 2.8$ kN, $M = 6.53$ kN·m, $T = 1.867$ kN·m

5/239 $P = \dfrac{\rho g a b}{6}\left(h + \dfrac{a}{2}\right)$

5/240 $A = 1.440(10^6)$ lb, $M = 7.78(10^8)$ lb-ft

▶**5/241** $V = 0.931(10^6)$ lb, $M = 2.21(10^8)$ lb-ft

***5/242** $\theta = 46.8°$

***5/243** $\bar{y} = 3.74$

***5/244** $V_{\max} = 1900$ lb at $x = 0$
$M_{\max} = 9080$ lb-ft at $x = 9.63$ ft

***5/245** $\bar{X}_{\max} = 322$ mm at $x = 322$ mm

***5/246** $\rho = 8.63$ kg/m

***5/247** sag $= 3.98$ m at 393 m right of A
$T_A = 175\,800$ N, $T_B = 169\,900$ N

***5/248** $d = 197.7$ m, horizontal thruster
$T_h = 10$ N, $T_v = 1.984$ N

Chapter 6

6/1 $F = 85$ lb left

6/2 $F = 379$ N left

6/3 $\theta = 4.57°$

6/4 (*a*) $F = 66.0$ N
(*b*) $P = 516$ N
(*c*) $F = 148.5$ N

6/5 $P_{\text{slip}} = 0.448mg$, $P_{\text{tip}} = 0.536mg$

6/6 (*a*) $F = 46.4$ lb up
(*b*) $F = 40$ lb up

6/7 $\mu_s = 0.0959$, $F = 0.0883mg$, $P = 0.1766mg$

6/8 $\mu_s = 0.1763$

6/9 $F_A = F_B = 126.6$ N, $x_{\max} = 86.2$ mm

6/10 $\theta = 16.75°$, $P = 12$ N

6/11 (*a*) $\theta = 26.6°$, (*b*) $\theta = 18.43°$, (*c*) $\theta = 8.13°$

6/12 $M = 19.81$ N·m

6/13 $\mu_{\min} = 0.630$, $\theta = 32.2°$

6/14 $d \le 0.2L$

6/15 $\mu_s = 0.25$: $\theta = 63.4°$
$\mu_s = 0.50$: $\theta = 45°$

6/16 $\mu_s = 0.25$: $\theta = 61.8°$
$\mu_s = 0.50$: $\theta = 40.9°$

6/17 $\mu_2 < \dfrac{m_1}{m_1 + m_2}\mu_1$, $\mu_2 < 0.533$

6/18 Slips first: $a > \mu b$; tips first: $a < \mu b$

6/19 (*a*) $\theta = 8.09°$, (*b*) $\theta = 26.2°$

6/20 (*a*) and (*b*) $P = 19.2$ lb

6/21 $M = 76.3$ N·m

6/22 $\mu = 0.268$

6/23 (*a*) $P = 14.7$ lb, (*b*) $P = 36.8$ lb

6/24 $M = 2.94$ N·m

6/25 $3.05 \le W \le 31.7$ lb

6/26 (*a*) 445 N, (*b*) 321 N

6/27 $s = 2.55$ m

6/28 $\mu_s = 0.365$

6/29 $\mu_s = 0.287$

6/30 $\mu_s = 0.1575$

6/31 $h = 156.6$ mm, $N = 1333$ N

6/32 $F = 400$ lb

6/33 $\theta = 20.7°$

6/34 $20.5 \le W \le 112.9$ lb

6/35 $\theta = \tan^{-1}\left(\mu\dfrac{a+b}{a}\right)$

6/36 $\theta = \sin^{-1}\dfrac{3\pi\mu_s}{4 - 3\pi\mu_s}$, $\mu_s = \dfrac{2}{3\pi}$

6/37 (*a*) $P = 24.5$ N, (*b*) $P = 109.1$ N

6/38 $\mu_s = 0.268$

6/39 $x = \dfrac{a - b\mu_s}{2\mu_s}$

6/40 $2.37 \le \dfrac{l}{d} \le 8.14$

6/41 $\theta = 6.29°$

6/42 $P = mg\sqrt{\mu_s^2\cos^2\theta - \sin^2\theta}$
$\beta = \tan^{-1}\left(\dfrac{mg\sin\theta}{P}\right)$

6/43 $P = \dfrac{M}{rl}\left(\dfrac{b}{\mu_s} - e\right)$

6/44 (*a*) $\theta = -8.02°$, (*b*) $\theta = 8.85°$

6/45 $P = 22.1$ kN

6/46 $R = 8.81$ kN

6/47 $\mu_s = 0.1264$

6/48 $\mu_s = 0.761$

6/49 $\mu_s = 0.0824$, 40.2 N

6/50 $(\mu_s)_{\min} = 0.25$ at A, $(\mu_s)_{\min} = 0.389$ at B
$R = 1291$ lb

6/51 (*a*) $\theta_{\min} = 61.2°$, $(\mu_s)_B = 1.667$
(*b*) $\theta_{\min} = 45°$, $(\mu_s)_B = 0.667$
(*c*) $\theta_{\min} = 10.30°$, $(\mu_s)_B = 0.0952$

6/52 $k = 20.8(10^3)$ N/m
6/53 $\mu_s = 0.1763$
6/54 $\mu_s = 0.0262$
6/55 $N = 5.66$ threads per inch
6/56 $N_U = 53.5$ lb, $N_L = 50.8$ lb
6/57 $\mu_s = 0.265$
6/58 $\mu_s = 0.3$, $F_A = 1294$ N
6/59 $P = 449$ N
6/60 $P = 4.53$ kN
6/61 $P' = 3.51$ kN
6/62 $M = 24.8$ lb-in.
6/63 (a) $F = 8.52$ N, (b) $F = 3.56$ N
6/64 $P = 333$ N
6/65 $P = 105.1$ N
6/66 (a) $P = 49.4$ lb, (b) $P = 69.4$ lb
6/67 (a) $P' = 6.45$ lb left, (b) $P' = 13.55$ lb right
6/68 $F_A = F_B = 24.6$ N, $P = 98.6$ N
6/69 $p = 2400$ kPa
6/70 $M = 7.30$ N·m
6/71 $M' = 3.02$ N·m
6/72 $M = 30.9$ N·m
6/73 (a) $P = 78.6$ N, (b) $P = 39.6$ N
►**6/74** $\mu = 0.1222$
6/75 $\mu = 0.271$
6/76 (a) $\mu = 0.1947$, (b) $r_f = 3.82$ mm
6/77 $M = 96$ lb-in., $\mu = 0.3$
6/78 $T = 4020$ N, $T_0 = 3830$ N
6/79 $T = 3830$ N, $T_0 = 4020$ N
6/80 $\mu = 0.609$, $\phi = 31.3°$
6/81 (a) $P = 50$ lb, (b) $P = 52.9$ lb
6/82 $P = 47.2$ lb, No
6/83 $M = \mu L \dfrac{r_o - r_i}{\ln(r_o/r_i)}$
6/84 $T = 258$ N
6/85 $T = 233$ N
6/86 $M = \dfrac{1}{2}\mu PR$
6/87 $T = 56.4$ N
6/88 $F = 136.1$ N
6/89 $M = \dfrac{4\mu P}{3}\dfrac{R_o^3 - R_i^3}{R_o^2 - R_i^2}$
6/90 $T = 1069$ N, $T_1 = 1013$ N, $T_2 = 949$ N
6/91 $T = 899$ N, $T_1 = 949$ N, $T_2 = 1013$ N
6/92 $M = \dfrac{5}{8}\mu La$
6/93 $M = \dfrac{2}{5}\mu_0 La$
6/94 $\mu = 0.204$
6/95 $M = 335$ N·m
►**6/96** $M = \mu Pr$
6/97 $\mu = 0.221$
6/98 (a) $P = 1007$ N, (b) $P = 152.9$ N
6/99 $\mu = 0.228$
6/100 $T = 2.11$ kN
6/101 (a) $\mu = 0.620$, (b) $P' = 3.44$ kN
6/102 $P = 320$ N
6/103 $P_x = 55.2$ N
6/104 $T = 1720$ lb
6/105 $P = 3.30$ kN
6/106 $W = 24.5$ lb
6/107 $P = 23.7$ lb
6/108 (a) $T_a = \dfrac{mg}{\sin\theta(1 + e^{-2\mu\theta})}$
(b) $T_b = \dfrac{mg}{\sin\theta(1 + e^{2\mu\theta})}$
6/109 $T = mge^{\pi\mu}$
*__6/110__ $\theta = 0$: $\dfrac{T}{mg} = 6.59$, $\theta \to \dfrac{\pi}{2}$: $\dfrac{T}{mg} \to e^{\pi\mu_B}$
6/111 $P = 142.0$ lb
6/112 $\mu = 0.768$
6/113 $M = 1834$ lb-in.
6/114 $h = 27.8$ mm
6/115 $8.66 \le W \le 94.3$ lb
6/116 $\mu = 0.214$
6/117 $x = 0.813$ m
6/118 $\mu = 0.1948$
6/119 $T_2 = T_1 e^{\mu\beta/\sin(\alpha/2)}$, $n = 3.33$
►**6/120** $T = \dfrac{\rho g r}{1 + \mu^2}\left[2\mu e^{\mu\pi/2} + 1 - \mu^2\right]$
6/121 (a) $T_{\max} = 76.0$ lb, $T_{\min} = 24.0$ lb
(b) $F = 10$ lb
6/122 $P = 93.7$ lb
6/123 Rotates first at $P = 0.232mg$
6/124 $R = 1855$ N
6/125 $C = 273$ lb, $F = 68.2$ lb, $P' = 2.73$ lb
6/126 A and B slip, C does not slip
6/127 $T = 7980$ N
6/128 $M = 3.00$ kN·m, $\mu_{\min} = 0.787$
6/129 $P = 913$ lb
6/130 (a) $F = 133.3$ N, (b) $F = 127.6$ N
6/131 (a) $M = 24.1$ N·m, (b) $M = 13.22$ N·m
6/132 $\mu_s = 0.767$
6/133 (a) $0.304 \le m \le 13.17$ kg
(b) $0.1183 \le m \le 33.8$ kg
6/134 $M = 4.12$ lb-in., $M' = 1.912$ lb-in.
6/135 $\mu = 0.368$
6/136 $12.44 \le m \le 78.0$ kg
6/137 $b = a + \dfrac{\mu_k^2 d}{2}$
6/138 $W = 70.1$ lb
6/139 $M = 10.16$ N·m

6/140 $P = 25.3$ N

▶ **6/141** $d_{\min} = \dfrac{2r + (1 - \mu^2)R}{1 + \mu^2}$

$d_{\max} = R + 2r$

▶ **6/142** (a) $M = 35.7$ N·m, (b) $M' = 12.15$ N·m

***6/143** $P_{\min} = 517$ N at $x = 7.5$ m

***6/144** $\theta = 21.5°$

***6/145** $\theta_{\max} = 59.9°$, $P_{\max} = 0.295mg$

***6/146** $y = 0$: $\dfrac{T}{mg} = 21.2$; y large: $\dfrac{T}{mg} \rightarrow 81.3$

***6/147** $\mu_k = 0.298$, $T = 30.7$ kN, $h = 58.1$ m

***6/148** $P_{\max} = 2430$ N, $\theta = 26.6°$

***6/149** $\mu_s = \dfrac{\sin\theta}{1 + \tan\theta(\tan\theta - \sin\theta)}$

(a) $24.0° < \theta < 57.2°$

(b) $\theta = 41.8°$, $\mu_s = 0.554$

***6/150** $V = 0.867mg$

***6/151** $\mu = 0.420$

Chapter 7

7/1 $\theta = 2\tan^{-1}\left(\dfrac{4P}{mg}\right)$

7/2 $M = mgl\sin\dfrac{\theta}{2}$

7/3 $R = P\dfrac{b}{r}$

7/4 $P = 2\rho gb$

7/5 $\theta = \cos^{-1}\left(\dfrac{M}{mgb}\right)$

7/6 $\theta = \tan^{-1}\left(\dfrac{2mg}{3P}\right)$

7/7 $M = mg(b\cos\theta - a\sin\theta)$

7/8 $Q = P\dfrac{b}{a}$

7/9 $P = 4kl\,(\tan\theta - \sin\theta)$

7/10 $R = 11.67P$

7/11 $P = \dfrac{3}{2}mg$

7/12 $F = 0.8R\cos\theta$

7/13 $M = \dfrac{3}{2}mgl\sin\dfrac{\theta}{2}$

7/14 $M = (C - mg)\dfrac{r}{n}$

7/15 $e = 0.983$

7/16 $C = P\sqrt{\left(\dfrac{2b}{h}\right)^2 - 1}$

7/17 $\theta = \tan^{-1}\left(\dfrac{mg}{P}\right)$, no

7/18 $F = \dfrac{mgl}{4h}\sin 2\theta$

7/19 $e = 0.597$

7/20 $F = mg\dfrac{R}{r}\sin\theta$

7/21 $P = mg\dfrac{\cos\theta}{\cos\frac{\theta}{2}}$

7/22 $M = 2mgb\sqrt{1 - \left(\dfrac{h}{2b}\right)^2}$

7/23 $C = \dfrac{Pe(d + c)}{2bc}$

7/24 $p = \dfrac{2mg}{A}$

7/25 $P = 309$ lb

7/26 $P = W$, no, $Q = \dfrac{W}{2}\left(1 + \dfrac{2e}{\sqrt{l^2 - h^2}}\right)$

7/27 $P = \dfrac{2Feb}{c(b - a)}$

7/28 $C = 2mg\sqrt{1 + \left(\dfrac{b}{L}\right)^2 - 2\dfrac{b}{L}\cos\theta\cot\theta}$

7/29 $F = \dfrac{2\pi M}{L\sqrt{\tan\theta + \dfrac{a}{b}}}$

7/30 $N = 1.6P$

7/31 $\theta_1 = \cos^{-1}\left(\dfrac{2M}{3mgl}\right)$, $\theta_2 = \cos^{-1}\left(\dfrac{2M}{mgl}\right)$

7/32 $M = 7.88$ N·m

7/33 $P = 3.5$ kN

▶ **7/34** $Q = 13.18$ kN

7/35 $x = 0$: unstable; $x = \dfrac{1}{2}$: stable; $x = -\dfrac{1}{2}$: stable

7/36 $\theta = \cos^{-1}\left(\dfrac{mg}{2kb}\right)$, $k_{\min} = \dfrac{mg}{b\sqrt{3}}$

7/37 $k_{\min} = \dfrac{mg}{4L}$

7/38 $l < \dfrac{2R}{mg}$

7/39 (a) unstable, (b) stable

7/40 $x = \sqrt{\left(\dfrac{F}{k}\right)^2 - h^2} \quad \left(k < \dfrac{F}{h}\right)$

7/41 $\theta = \sin^{-1}\left(\dfrac{M}{kb^2}\right)$

7/42 $M > \dfrac{m}{2}$

7/43 Stable

7/44 $\theta = 0$ and 180°: unstable
$\theta = 120°$ and 240°: stable

7/45 $\theta = 2\sin^{-1}\left[\dfrac{P + \dfrac{mg}{2}}{2kb}\right]$

7/46 $K_{\min} = \dfrac{1}{2}mgl$

7/47 $P = \dfrac{4kb^2}{a}\sin\theta\,(1 - \cos\theta)$

7/48 $k = \dfrac{mg}{2b}$

7/49 $h < \dfrac{2kb^2}{mg}$

7/50 $\theta = \sin^{-1}\dfrac{M}{kb^2}, M < kb^2$

7/51 $k < \dfrac{L}{2l}$

7/52 $k = \dfrac{mgb}{a^2}$

7/53 $h = \dfrac{mgr^2}{K}$

7/54 $\theta = 0$: $k > \dfrac{2mg}{L}$ for stability
$\cos\theta = \dfrac{2mg}{kL}$: unstable

7/55 $k_{\max} = 1.125$ lb/in.

7/56 $\rho < 6r$

7/57 $h < b\sqrt{2}$

▶**7/58** $M = \dfrac{(2m_1 + m_2)pg}{4\pi}\cot\theta$

▶**7/59** $k = \dfrac{mg(r + a)}{8a^2}$

▶**7/60** $h = 265$ mm

7/61 $P = 1047$ N

7/62 $\theta = 28.1°$

7/63 $h_{\max} = 0.363r$

7/64 $h < 2r$

7/65 $F = 960{,}000$ lb

7/66 A: a, b, d; B: c, e, f

7/67 $h < r$

7/68 $\mu_s = 0.1443$

7/69 $\theta = -6.82°$: stable; $\theta = 207°$: unstable

7/70 $x = 2.05$ m: stable

7/71 $b_{\min} = \dfrac{2R}{3}$

▶**7/72** $k_{\min} = \dfrac{mg}{2b}\left(1 + \dfrac{b^2}{l^2}\right)$

▶**7/73** $\theta = 0$: stable if $k < \dfrac{mg}{a}$
$\theta = \cos^{-1}\left[\dfrac{1}{2}\left(1 + \dfrac{mg}{ka}\right)\right]$: stable if $k > \dfrac{mg}{a}$

***7/74** $x = 130.3$ mm

***7/75** $\theta = 23.0°$

***7/76** $V = 56.3x^2 - 231x$

***7/77** $\theta = 78.0°$: stable; $\theta = 260°$: unstable

***7/78** $\theta = 24.8°$: unstable

***7/79** $\theta = 71.7°$

***7/80** $\dfrac{R}{P} = 0.943\dfrac{\sin\left(\dfrac{\pi}{4} - \theta\right)}{\sin\theta}$

Appendix A

A/1 $A \cong 1600$ mm^2

A/2 $I_x = \dfrac{bh^3}{3}, I_y = \dfrac{hb^3}{9}\, I_z = \dfrac{bh}{3}\left(h^2 + \dfrac{b^2}{3}\right)$

A/3 $I_y = \dfrac{hb^3}{4}$

A/4 $I_y = 21.5$ in.^{4}

A/5 $A = 4800$ mm^2

A/6 $I_x = \dfrac{bh^3}{4}, I_y = \dfrac{hb^3}{48}$
$I_z = \dfrac{bh}{4}\left(h^2 + \dfrac{b^2}{12}\right)$

A/7 $I_A = \dfrac{3}{4}\pi r^4, I_B = r^4\left(\dfrac{3\pi}{4} - \dfrac{4}{3}\right)$

A/8 $I_x = 4.38(10^6)$ mm^4

A/9 $k_B = a/\sqrt{3}, k_O = a$

A/10 $I_x = I_y = \dfrac{1}{2}\pi r^3 t, I_C = \pi r^3 t\left(1 - \dfrac{4}{\pi^2}\right)$

A/11 $I_x = I_y = \dfrac{1}{3}Ab^2, I_O = \dfrac{2}{3}Ab^2$

A/12 $I_x = 0.1963a^4, I_y = 1.648a^4, k_O = 1.533a$

A/13 $k_A = 14.43$ mm

A/14 $I_x = h^3\left(\dfrac{a}{4} + \dfrac{b}{12}\right), I_y = \dfrac{h}{12}(a^3 + a^2b + ab^2 + b^3)$
$I_O = \dfrac{h}{12}[h^2(3a + b) + a^3 + a^2b + ab^2 + b^3]$

A/15 $I_x = \dfrac{4ab^3}{9\pi}$

A/16 $k_x = 0.754,\ k_y = 1.673,\ k_z = 1.835$

A/17 $I_{x'} = I_x = \dfrac{a^4}{24}$

A/18 $I_x = h^3\left(\dfrac{b_2}{4} + \dfrac{b_1}{12}\right)$

$I_y = \dfrac{h}{48}(b_1{}^3 + b_1{}^2 b_2 + b_1 b_2{}^2 + b_2{}^3)$

A/19 $I_y = \dfrac{\pi a^3 b}{4},\ k_O = \dfrac{1}{2}\sqrt{a^2 + b^2}$

A/20 $k_M = \dfrac{a}{\sqrt{6}}$

A/21 $I_x = \dfrac{a^4}{28},\ I_y = \dfrac{a^4}{20}$

A/22 $n = -0.518\%$

A/23 $I_{x'} = 0.315a^4$

A/24 $I_y = 73.1(10^8)\ \text{mm}^4,\ I_{y'} = 39.0(10^8)\ \text{mm}^4$

A/25 $I_x = 51.2\ \text{in.}^4$

A/26 $I_x = \dfrac{16ab^3}{105}$

A/27 $I_x = \dfrac{5}{8}\pi r^4,\ k_O = 1.830r$

A/28 $I_x = 140.8\ \text{in.}^4$

A/29 $k_y = 53.1\ \text{mm}$

A/30 $I_x = \dfrac{a^4}{4}\left(\alpha + \dfrac{1}{2}\sin 2\alpha\right),\ I_y = \dfrac{a^4}{4}\left(\alpha - \dfrac{1}{2}\sin 2\alpha\right)$

A/31 $I_{x'} = I_{y'} = I_x = I_y$

A/32 $I_x = 25.6\ \text{in.}^4,\ I_y = 30.5\ \text{in.}^4,\ I_O = 56.1\ \text{in.}^4$

A/33 $k_x = k_y = \dfrac{\sqrt{5}}{4}a,\ k_z = \dfrac{\sqrt{10}}{4}a$

▶ **A/34** $I_x = 0.1988r^4$

A/35 $I_x = 0.0833bh^3,\ I_x = 0.0781bh^3$

A/36 $I_y = 0.785R^4,\ I_y = 0.702R^4$

A/37 $n = 3.68\%$

A/38 $n_A = 50\%,\ n_{I_y} = 22.2\%$

A/39 $k_x = k_y = \dfrac{\sqrt{5}}{4}a,\ k_z = \dfrac{\sqrt{10}}{4}a$

A/40 $\bar{I}_x = 649\ \text{in.}^4$

A/41 $I_x = 130.8(10^6)\ \text{mm}^4$

A/42 $I_x = 4.94a^4,\ I_y = 3.37a^4$

A/43 $I_x = 0.01825a^4,\ I_y = 0.1370a^4$

A/44 $\bar{I}_x = 22.6(10^6)\ \text{mm}^4,\ \bar{I}_y = 9.81(10^6)\ \text{mm}^4$

A/45 $I_x = \dfrac{58}{3}a^4$

A/46 $n = 0.1953 + 2.34y^2,\ n = 9.57\%$

A/47 $k_A = 78.9\ \text{mm}$

A/48 $k_O = 7.92\ \text{in.}$

A/49 $I_x = \dfrac{hb}{9}\left(\dfrac{7}{4}h^2 + \dfrac{2}{9}b^2 + hb\right),\ n = 176.0\%$

A/50 $I_x = \dfrac{5\sqrt{3}}{16}a^4$

A/51 $I_x = 254\ \text{in.}^4,\ I_{x_{\text{exact}}} = 245\ \text{in.}^4$

A/52 $k_O = 0.778a$

A/53 $k_C = 261\ \text{mm}$

A/54 $I_x = h^3\left(\dfrac{b_2}{4} + \dfrac{b_1}{12}\right)$

$I_y = \dfrac{h}{48}(b_1{}^3 + b_1{}^2 b_2 + b_1 b_2{}^2 + b_2{}^3)$

A/55 $h = 1.900\ \text{in.}$

A/56 $I_x = 0.222h^4,\ I_y = 0.0281h^4,\ I_O = 0.251h^4$

A/57 $I_x = 0.319a^4$

A/58 $I_{x_0} = 6.24(10^6)\ \text{mm}^4$

A/59 $I_O = 86.5(10^6)\ \text{mm}^4$

▶ **A/60** $b = 161.1\ \text{mm}$

A/61 (a) and (c): $I_{xy} = 360(10^4)\ \text{mm}^4$

(b) and (d): $I_{xy} = -360(10^4)\ \text{mm}^4$

A/62 $I_x = 2.44(10^8)\ \text{mm}^4,\ I_y = 9.80(10^8)\ \text{mm}^4$

$I_{xy} = -14.14(10^6)\ \text{mm}^4$

A/63 $I_{xy} = 110.2\ \text{in.}^4$

A/64 (a) and (c): $I_{xy} = 9.60(10^6)\ \text{mm}^4$

(b): $I_{xy} = -4.71(10^6)\ \text{mm}^4$

(d): $I_{xy} = -2.98(10^6)\ \text{mm}^4$

A/65 $I_{xy} = \dfrac{1}{6}bL^3 \sin 2\alpha$

A/66 $I_{xy_1} = I_{xy_3} = \dfrac{bL^3}{6}\sin 2\alpha$

$I_{xy_2} = I_{xy_4} = \dfrac{-bL^3}{6}\sin 2\alpha$

A/67 $I_{xy} = 18.40(10^6)\ \text{mm}^4$

A/68 $I_{xy} = \dfrac{1}{2}r^3 b,\ I_{x_0y_0} = \left(\dfrac{1}{2} - \dfrac{2}{\pi}\right)r^3 b$

A/69 $I_{xy} = \dfrac{b^2h^2}{24},\ I_{x_0y_0} = -\dfrac{b^2h^2}{72}$

A/70 $I_{xy} = \dfrac{b^2h^2}{8}$

A/71 $\bar{I}_{xy} = -0.01647r^4$

A/72 $I_{xy} = -1968\ \text{in.}^4$

A/73 $I_{xy} = \dfrac{2}{3}r^4$

A/74 $A = 1.316(10^4)\ \text{mm}^2$

A/75 $I_{xy} = \dfrac{h^2}{24}(3a^2 + 2ab + b^2)$

A/76 $I_{xy} = \dfrac{1}{8}a^2b^2$

A/77 $I_{xy} = \dfrac{r^4}{16}(1 - \cos 2\theta)$

A/78 $I_{xy} = \dfrac{5}{16}s^4$

A/79 $I_{xy} = \dfrac{1}{16}a^2b^2$

A/80 $I_{x'} = 0.0277b^4$, $I_{y'} = 0.1527b^4$
$I_{x'y'} = 0.0361b^4$

A/81 $I_{\max} = 3.79a^4$, $I_{\min} = 0.373a^4$, $\alpha = 111.5°$

A/82 $I_{x'} = \frac{r^4}{16}(\pi - \sqrt{3})$, $I_{y'} = \frac{r^4}{16}(\pi + \sqrt{3})$, $I_{x'y'} = \frac{r^4}{16}$

A/83 $I_x = 0.446b^4$, $I_y = 0.280b^4$

A/85 $I_{\min} = 0.505a^4$, $I_{\max} = 6.16a^4$, $\alpha = 112.5°$

A/86 $I_{\min} = 0.252b^4$, $I_{\max} = 3.08b^4$, $\alpha = -22.5°$

A/87 $I_{\max} = 183.6 \text{ in.}^4$, $\alpha = -16.85°$

***A/88** $(I_{x'})_{\min} = 2.09(10^8) \text{ mm}^4$ at $\theta = 22.5°$

***A/89** $I_{\min} = 3.03(10^6) \text{ mm}^4$ at $\theta = 64.1°$

***A/90** $(I_{x'})_{\min} = 0.0432a^4$ at $\theta = 21.8°$
$(I_{x'})_{\max} = 1.070a^4$ at $\theta = 111.8°$
$(I_{y'})_{\min} = 0.0432a^4$ at $\theta = 111.8°$
$(I_{y'})_{\max} = 1.070a^4$ at $\theta = 21.8°$
$(I_{x'y'})_{\min} = -0.514a^4$ at $\theta = 66.8°$
$(I_{x'y'})_{\max} = 0.514a^4$ at $\theta = 156.8°$

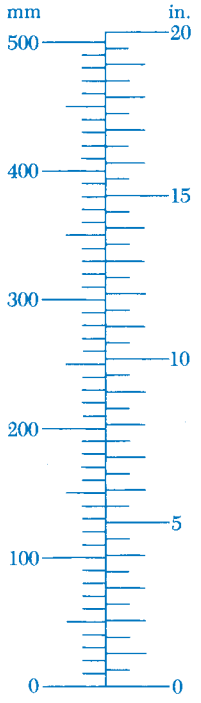
***A/91** $I_{x'y'} = -203 \sin 2\theta - 192 \cos 2\theta$, $\theta = 68.3°$

***A/92** $I_{\max} = 1.820(10^6) \text{ mm}^4$, $\alpha = 30.1°$

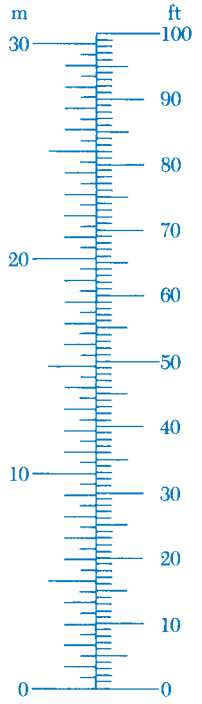
***A/93** $I_{\min} = 0.0547b^4$ at $\theta = 41.1°$
$I_{\max} = 0.286b^4$ at $\theta = 131.1°$

***A/94** $I_{\min} = 20.8(10^6) \text{ mm}^4$ at $\theta = 9.26°$
$I_{\max} = 147.2(10^6) \text{ mm}^4$ at $\theta = 99.3°$

Conversion Charts Between SI and U.S. Customary Units



Length



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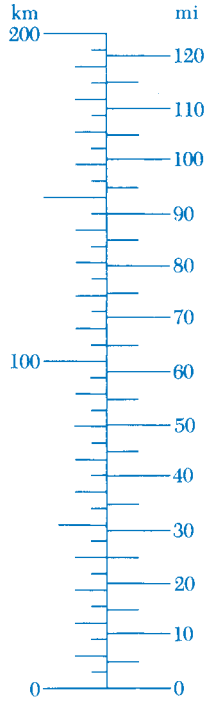

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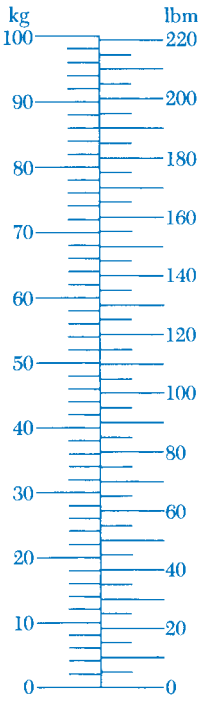
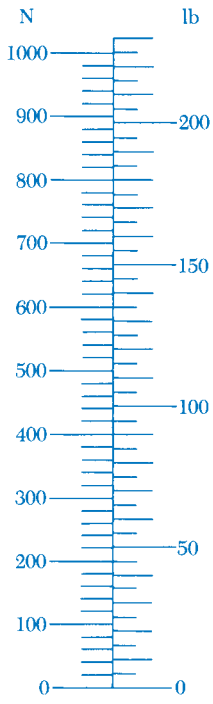
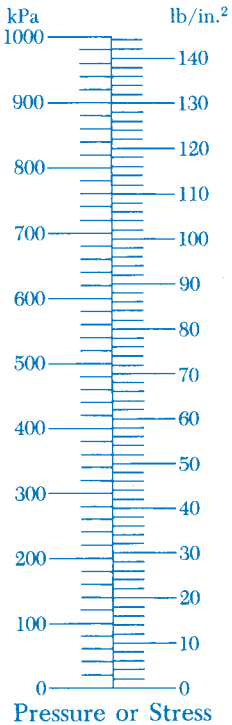


Mass

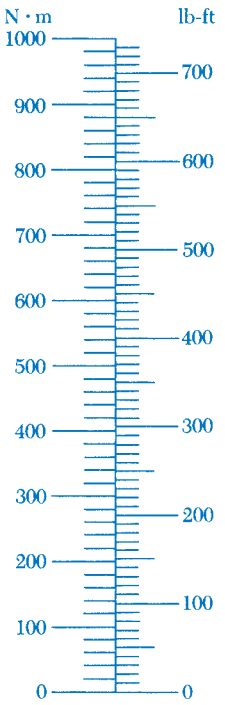


Force

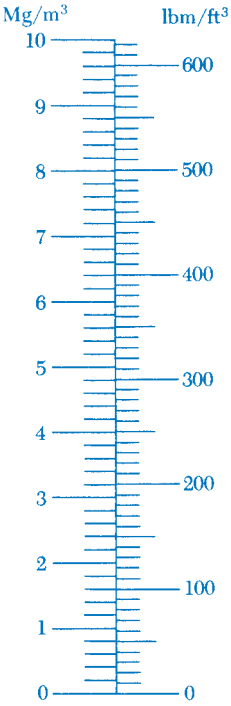


Pressure or Stress

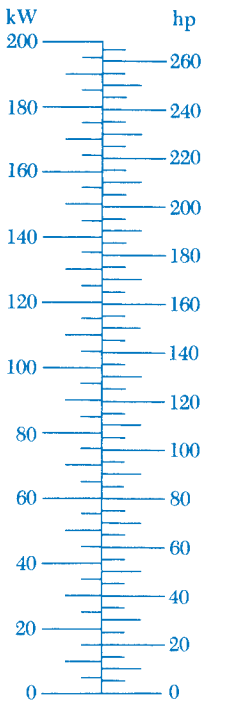
Conversion Charts Between SI and U.S. Customary Units (cont.)



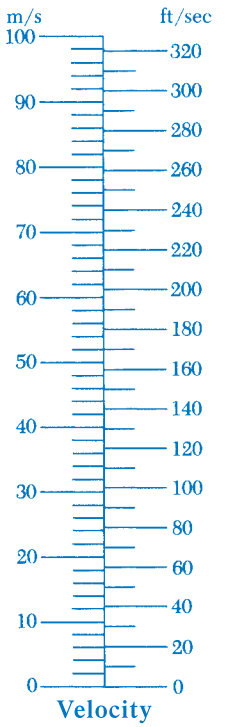
Moment or Torque



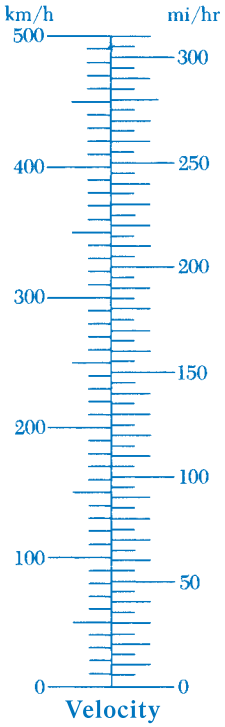
Density



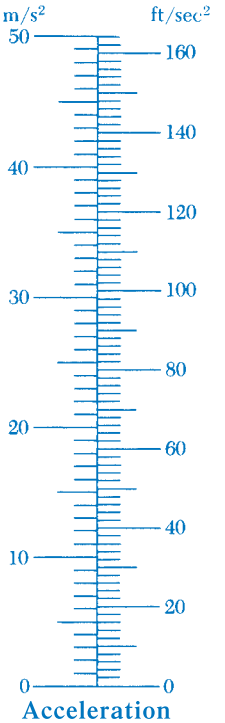
Power



Velocity



Velocity



Acceleration