The formal definition of a limit:

Let f be a function defined on an open interval containing c (except possibly at c) and let L be a real number (infinity does not technically count). The statement

$$\lim_{x \to c} f(x) = L$$

Means that for all values of x that are approaching c, f evaluated at those values, $f(x)$, will create values which are approaching the number L.

So as $x \to c\ then\ f(x) \to L$

Formally this means that:

for every, $\forall$, $\epsilon > 0$, there exists, $\exists$, $\delta > 0$, such that, s.t.,

if it is true that $0 < |x - c| < \delta$ then it must be true that $0 < |f(x) - L| < \epsilon$

or symbolically written as

$\forall\ \epsilon > 0, \exists,\ \delta > 0, s.t.$ if $0 < |x - c| < \delta$ then $0 < |f(x) - L| < \epsilon$

Breaking down the key notation in the definition:

Let's recall what this means: $0 < |x - c| < \delta$

It is saying that all the x-values you want to try, their distance from c is less than the number δ (which is not zero).

In this definition it is assumed that we will only pick those x-values whose distance is less than δ.

Now what does $0 < |f(x) - L| < \epsilon$ mean?

It is saying that for all the x-values we do try, the value of the function evaluated at those x-values will have a distance from the limit L which is less than the number ϵ (which is also not zero).

Breaking down the logic in the definition:

$\forall\ \epsilon > 0, \exists,\ \delta > 0, s.t.$ if $0 < |x - c| < \delta$ then $0 < |f(x) - L| < \epsilon$

Is a conditional or "if then" logical statement much like the statement

"if it is raining outside then I will bring an umbrella"

In order for this statement to be true both the hypothesis "the if part" AND the conclusion "the then part" will both be true.

So if $\lim_{x \to c} f(x) = L$ is correct, then for every ϵ imaginable, I am charged with finding a corresponding δ such that whenever $0 < |x - c| < \delta$ is true then $0 < |f(x) - L| < \epsilon$ must also be true.

If I cannot find that δ, then I cannot prove that the limit actually exists.

The approach we take to doing a formal ϵ, δ proof:

Lets assume that

$0 < |x - c| < \delta$ is true, meaning we are only picking x-values "within delta away from c".

Lets investigate to see if plugging in those x-values will put us "within ϵ away from our limit L

So lets see if $0 < |f(x) - L|$ is in fact $< \epsilon$.

Start with the expression

$|f(x) - L|$

can you make it look like something times $|x - c|$?

Does $|f(x) - L| = \cdots = k \cdot |x - c|$? Where k is a constant real number.

If so, great, then it looks like we can prove this one by using the our assumption that $|x - c| < \delta$

By using this assumption we have

$|f(x) - L| = \cdots = k \cdot |x - c| < k \cdot \delta$

So it appears that $|f(x) - L|$ IS already less than some constant non-zero number!! Great!

Note: If we cannot then we are in some trouble and either the limit does not exist or we will have to learn some new techniques to prove the limit exist. You can see this in an example in this video https://www.youtube.com/watch?v=gLpQgWWXgMM

If we set that non-zero number equal to ϵ then without assuming the conclusion being true we can prove that

if $0 < |x - c| < \delta$ then $0 < |f(x) - L| = \cdots = k \cdot |x - c| < k \cdot \delta = \epsilon$

So as long as we do pick x values that are δ away from c, we are guaranteed that the distance that the function is from L will be less than ϵ.

Example proof:

Lets prove that

$$\lim_{x \to 3}(2x - 5) = 1$$

I.E. $\forall\, \epsilon > 0, \exists,\ \delta > 0, s.t.$ if $0 < |x - 3| < \delta$ then we need to show that $0 < |(2x - 5) - 1| < \epsilon$

Lets assume that

$0 < |x - 3| < \delta$ is true, meaning we are only picking x-values "within delta away from c".

So lets see if $0 < |f(x) - L|$ is in fact $< \epsilon$.

$|(2x - 5) - 1| = |2x - 5 - 1| = |2x - 6| = 2 \cdot |x - 3| < 2 \cdot \delta = \epsilon \to \delta = \frac{\epsilon}{2}$

Now we have just found the δ that will guarantee us that the function gets close to the number 1.

Lets plug it in.

Assuming that $|x - 3| < \delta$ is true, and letting $\delta = \frac{\epsilon}{2}$

We will have that

Math 400 2.2 Formal Def of a limit Handout

$\forall\, \epsilon > 0, \exists,\ \delta = \frac{\epsilon}{2}, s.t.$ if $0 < |x-3| < \delta$ is true then

$$|(2x-5)-1| = |2x-5-1| = |2x-6| = 2 \cdot |x-3| < 2 \cdot \delta = 2 \cdot \frac{\epsilon}{2} = \epsilon \qquad \text{QED.}$$