# Separation of Variables

A differential equation, DE or diff eq., that can be written in the form

$$M\left(x\right)+N\left(y\right)\frac{dy}{dx}$$

Where $M(x)$ is a continuous function of $x$ alone and $N\left(y\right)$ is a continuous function of $y$ alone is called **separable**.

 The method of finding the solution by means of collecting all $x$ terms with $dx$ and all $y$ terms with $dy$, then solving the DE is called **separation of variables**.

## Example: Find the general solution of the differential equation $4yy^{'}-3e^{x}=0$

## Solution:

L4e: 6.3.12



## Example: Find the particular solution of the differential equation $y\sqrt{1-x^{2}}y^{'}-x\sqrt{1-y^{2}}=0$ with I.C. $y\left(0\right)=1$

## Solution:

L4e: 6.3.18



## Example: Find all functions, $f$, such that all tangents to the graph of $f$ pass through the origin.

Hint: the instantaneous slope will equal the average slope, and this average slope always contains the origin.

## Solution:

L4e: 6.3.26



# Looking Forward: Differential Equations (if time or interest allows)

In math we call functions that behave as though they have a multiplicative scaling factor, a **homogeneous Function.** If all the arguments can be substituted in such a way that they all are multiplied by the same scaling factor, then the value of function can be thought of as being multiplied by that scaling factor.

Def: if a function of one variable, $f(x)$, is **homogeneous** of degree $n$, then it has the following multiplicative scaling property:

$$f\left(tx\right)=t^{n}f(x)$$

 We call $n$ the **degree of homogeneity**.

# Example: Show that the area function of a circle is a homogeneous function of $r$, with degree of homogeneity, 2.

## Solution:

For a Circle, the area function is given by $A(r)=πr^{2}$ it is homogeneous because if you double the radius, the area will be a scalar multiple of the function, and the scalar has degree of homogeneity of 2.

$A(2r)=π\left(2r\right)^{2}=2^{2}πr^{2}$.

So from below we know that if the first circle has area, A, then the second circle has area 4A. If you triple the radius of the first, then its area will be $3^{2}$A or 9 times larger.



This provides a nice intuition for what the definition of homogeneous means.

# Example: We are very familiar with homogeneous functions, consider $f\left(x\right)=x^{2},f\left(x\right)=x^{3}, etc.$ Show that they both satisfy the definition of a single variable homogeneous function and find the degree of homogeneity.

## Solution:

Since $f\left(tx\right)=t^{2}∙x^{2}$, it satisfies the definition of a homogeneous function with degree 2.

Likewise, $f\left(tx\right)=t^{3}∙x^{3}$ satisfies the definition of a homogeneous function with degree 3.

But this is not limited only to functions of one variable.

Def: if a function of two variables, $f(x,y)$, is **homogeneous** of degree $n$, then it has the following multiplicative scaling property:

$$f\left(tx,ty\right)=t^{n}f\left(x,y\right)$$

 Again, we call $n$ the **degree of homogeneity**.

# Example: Ellipsoids, $f\left(x,y\right)=x^{2}+y^{2}$, and multivariate polynomials (where each term has the same degree), are among some of many homogeneous functions. Prove that ellipsoids and the degree 3 polynomial $f\left(x,y\right)=x^{2}y+x^{3}-3y^{2}x$ are homogeneous, but the degree polynomial $x^{3}+x^{2}+1=f(x)$ is not.

## Solution:

Since $f\left(tx,ty\right)=t^{2}∙x^{2}+t^{2}y^{2}=t^{2}(x^{2}+y^{2})$ its satisfies the definition of a homogeneous function with degree 2, because each answer is a multiplicative scaling factor, $t^{2}$, of the original.

Likewise, if $f\left(x,y\right)=x^{2}y+x^{3}-3y^{2}x$ , then, $f\left(tx,ty\right)=t^{3}∙x^{2}y+t^{3}∙x^{3}-3t^{3}∙y^{2}x=t^{3}\left(x^{2}y+x^{3}-3y^{2}x\right)=t^{3} f\left(tx,ty\right)$ satisfies the definition of a homogeneous function with degree 3.

However

$x^{3}+x^{2}+1=f(x)$ is not homogeneous because, $f\left(tx\right)=t^{3}x^{3}+t^{2}x^{2}+1\ne t^{n}f(x)$

# Definition of Homogeneous Differential Equation:

A **homogeneous differential equation** is an equation that can be expressed of the form

$$M\left(x,y\right)dx+N\left(x,y\right)dy=0$$

Where $M\left(x,y\right) \& N\left(x,y\right)$ are homogeneous functions of the same degree.

# Theorem 6.2: Change of variables for Homogeneous Differential Equations

If $M\left(x,y\right)dx+N\left(x,y\right)=0$ is homogeneous, then it can be transformed into a differential equation whose variables are separable by using the substitution

$$y=vx$$

Where $v$ is a differentiable function of $x$.

# ­Example: Solve the homogeneous diff eq. $y^{'}=\frac{x^{3}+y^{3}}{xy^{2}}$

## Solution:

L4e: 6.3.36

