Math 401 **Preparation for Calculus II** ***selected solutions***

"Directions on original document."

1. Solve for 'x' in each of the following functions.

a) $y = \sqrt[3]{x}$

$x = y^3$

b) $y = x^2 - 3$

$x = \pm\sqrt{y+3}$

Two possible solutions…

c) $y = 4x$

$x = \frac{1}{4}y$

d) $y = e^x$

$x = \ln y$

e) $y = \cos x$ if $0 \le x \le \pi$

$x = \arccos y$

f) A fun little challenge. ☺ $y = \dfrac{e^x - e^{-x}}{2}$

Solution: $y = \dfrac{e^x - e^{-x}}{2}$ Multiply by 2.

$2y = e^x - e^{-x}$ Multiply by e^x

$2ye^x = e^{2x} - 1$ Set equal to zero.

$e^{2x} - 2ye^x - 1 = 0$ This is in quadratic form. Complete the Square.

$e^{2x} - 2ye^x + y^2 = 1 + y^2$ Factor the left side.

$\left(e^x - y\right)^2 = 1 + y^2$ Square root.

$e^x - y = \sqrt{1+y^2}$ add 'y'

$e^x = y + \sqrt{1+y^2}$ take the natural logarithm of both sides

$x = \ln\left(y + \sqrt{1+y^2}\right)$ ☺

2. Functions related questions.

a) If $f(x)=\dfrac{e^{x}-e^{-x}}{e^{x}+e^{-x}}$, then $f(2x)=\dfrac{e^{2x}-e^{-2x}}{e^{2x}+e^{-2x}}$

b) If $f(x)=1+x+\dfrac{x^2}{2!}+\dfrac{x^3}{3!}+\dfrac{x^4}{4!}+\dots$, then $f(-x)=1+(-x)+\dfrac{(-x)^2}{2!}+\dfrac{(-x)^3}{3!}+\dfrac{(-x)^4}{4!}+\dots$

$$=1-x+\frac{x^2}{2!}-\frac{x^3}{3!}+\frac{x^4}{4!}-\dots$$

3. Evaluate each limit. If a Limit is undefined, indicate which case it is: $\{\infty,-\infty$, or <u>DNE</u> (Does not exist $\}$.

a) $\displaystyle \lim_{x\to 2}\frac{x^2-4}{x^3-8}= \qquad \lim_{x\to 2}\frac{(x+2)(x-2)}{(x-2)(x^2+2x+4)}=\frac{4}{12}=\frac{1}{3}$

b) $\displaystyle \lim_{h\to 0}\frac{(5+h)^2-5^2}{h}= \qquad =\lim_{h\to 0}\frac{10h+h^2}{h}=\lim_{h\to 0}(10+h)=10$

c) $\displaystyle \lim_{x\to\infty}\frac{x+5}{x^2+2x-15}= \quad 0$

d) $\displaystyle \lim_{x\to\infty}\frac{7x^3+2}{4x^2-5x^3-1}= \quad -\frac{7}{5}$

e) $\displaystyle \lim_{x\to-\infty}e^{-x}= \quad \infty$

f) $\displaystyle \lim_{x\to\infty}e^{-x}= \quad 0$

g) $\displaystyle \lim_{x\to\infty}(\sin x)= \quad$ DNE

h) $\displaystyle \lim_{x\to\frac{\pi}{2}^+}(\tan x)= \quad -\infty$

i) $\displaystyle \lim_{x\to\infty}\frac{3x^2+x\sin x}{x^2}= \qquad \lim_{x\to\infty}\left(3+\frac{\sin x}{x}\right)=3+0=3$

j) $\displaystyle \lim_{x\to 0}\frac{\sin x}{x}=1$

4. Use the limit definition of the derivative to evaluate: $\dfrac{d\left(5x^2-3x+2\right)}{dx}$

You must show your work and use the proper notation throughout.

$$\frac{d\left(f(x)\right)}{dx}=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)= \quad \cdots \quad =10x-3$$

You should be able to set up and evaluate this. (Section 3.1)

5. Find the derivative of each function using known formulas.

a) $y=x^4$ $\qquad \dfrac{dy}{dx}=4x^3$

b) $y=\sqrt[3]{u}$ $\qquad \dfrac{dy}{du}=\frac{1}{3}u^{-2/3}=\dfrac{1}{3u^{2/3}}$

c) $y=\dfrac{1}{x}=x^{-1}$ $\qquad y'=-x^{-2}=\dfrac{-1}{x^2}$

d) $\dfrac{d\left(\sec\theta\right)}{dx}=$ $\qquad \sec\theta\tan\theta$

e) $y=e^x$ $\qquad \dfrac{dy}{dx}=e^x$

f) $y=\arcsin t$ $\qquad \frac{dy}{dt}=\dfrac{1}{\sqrt{1-t^2}}$

6. Answer each of the following.

a) Given the function $f(x) = x^2 - \ln x + 1$, write an equation of the line tangent to the function at $x = 3$.

Note: We need a y-coordinate and the slope (derivative).

$f(3) = 10 - \ln 3$ $\quad f'(x) = 2x - \frac{1}{x}$ $\quad f'(3) = 6 - \frac{1}{3} = \frac{17}{3}$

Point-Slope equation of the tangent line: $\left(y - (10 - \ln 3)\right) = \frac{17}{3}(x - 3)$

b) Given the function $g(\theta) = \tan\theta$, write an equation of the normal line to the function at $\theta = \frac{\pi}{6}$.

The normal line is perpendicular to the tangent line.

The process is similar to the previous problem: $\left(y - \frac{1}{\sqrt{3}}\right) = -\frac{3}{4}\left(x - \frac{\pi}{6}\right)$

7. Find the derivative (with respect to x) of each function.

a) (using the power rule) $y = \frac{x^3}{\sqrt{x}} = x^{5/2}$ $\quad y' = \frac{5}{2}x^{3/2}$

b) (using the quotient rule) $y = \frac{x^3}{\sqrt{x}}$

$$y' = \frac{\sqrt{x}\cdot 3x^2 - x^3\cdot\frac{1}{2}x^{-1/2}}{\left(\sqrt{x}\right)^2} = \frac{\left(\sqrt{x}\cdot 3x^2 - x^3\cdot\frac{1}{2}x^{-1/2}\right)2\sqrt{x}}{x\cdot 2\sqrt{x}} = \frac{\left(6x^3 - x^3\right)}{x\cdot 2\sqrt{x}} = \frac{5x^3}{2x\cdot\sqrt{x}}$$

c) Verify that the results of part (a) and part (b) are the same.

Part(b) $\rightarrow \frac{5x^3}{2x\cdot\sqrt{x}} = \frac{5x^3}{2x^{3/2}} = \frac{5}{2}x^{3/2}$ = Part (a)

d) $y = x\cot x$ (product) $\frac{d(x\cot x)}{dx} = 1\cdot\cot x - x\csc^2 x = \cot x - x\csc^2 x$

e) $y = \frac{\ln x}{x}$ $\quad y' = \frac{x\left(\frac{1}{x}\right) - (1)\ln x}{x^2} = \frac{1 - \ln x}{x^2}$

8. Find the derivative of each function with respect to x.

a) $y = (4x+3)^9$

$$\frac{dy}{dx} = 9(4x+3)^8 \cdot 4 = \quad 36(4x+3)^8$$

b) $f(x) = e^{3x^2+7}$

$$\frac{dy}{dx} = 6x \cdot e^{3x^2+7}$$

d) $y = \dfrac{1}{\sqrt{3x^2+7}}$

$$\frac{dy}{dx} = -\frac{1}{2}(3x^2+7)^{-3/2} \cdot 6x = \quad \frac{-3x}{(3x^2+7)^{3/2}}$$

d) $y = \ln(\sec x + \tan x)$

$$\frac{dy}{dx} = \frac{1}{\sec x + \tan x} \cdot (\sec x \tan x + \sec^2 x) = \quad \sec x$$

e) $f(x) = \dfrac{e^x - e^{-x}}{2} = \dfrac{1}{2}(e^x - e^{-x})$

$$f'(x) = \frac{1}{2}(e^x + e^{-x})$$

f) $f(x) = \dfrac{2}{e^{5x} + e^{-5x}} \qquad = 2(e^{5x} + e^{-5x})^{-1}$ (We can now use the power rule with the chain rule.)

$$f'(x) = -2(e^{5x} + e^{-5x})^{-2}(5e^{5x} - 5e^{5x}) \qquad = -2\frac{(5e^{5x} - 5e^{5x})}{(e^{5x} + e^{-5x})^2}$$

9. Solve for $\dfrac{dy}{dx}$ by using implicit differentiation.

a) $x^2 + 5x^3y - 4y^5 = 7$

$$\frac{d(x^2 + 5x^3y - 4y^5)}{dx} = \frac{d(7)}{dx}$$

$$2x + 15x^2y + 5x^3 \cdot \tfrac{dy}{dx} - 20y^4 \cdot \tfrac{dy}{dx} = 0$$

$$5x^3 \cdot \tfrac{dy}{dx} - 20y^4 \cdot \tfrac{dy}{dx} = -2x - 15x^2y$$

$$\frac{dy}{dx} = \frac{-2x - 15x^2y}{5x^3 - 20y^4}$$

b) $x = \sin y$

$$\frac{d(x)}{dx} = \frac{d(\sin y)}{dx}$$

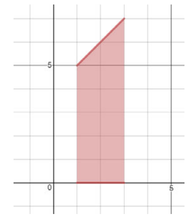
$$1 = \cos y \cdot \tfrac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\cos y}$$ (This is a solution.)

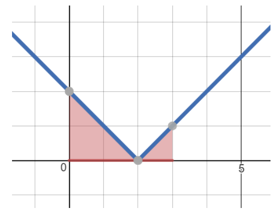
$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \quad \frac{1}{\sqrt{1-x^2}}$$

10. Sketch the region indicated by each definite integral. Use geometry to evaluate. (No Calculus involved.)

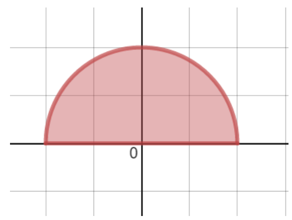
a) $\int_1^3 (x+4)\,dx \quad =12$ Trapezoidal region or 'rectangle + triangle.'



b) $\int_0^3 |x-2|\,dx \quad =2\tfrac{1}{2}$ Two triangles.



c) $\int_{-2}^{2} \sqrt{4-x^2}\,dx \quad =\tfrac{1}{2}\pi\cdot 2^2 \quad =2\pi$ Semicircle.



11. Evaluate each indefinite integral. These are all known anti-derivative formulas.
Here are some examples.

a) $\int x\,dx \quad =\dfrac{x^2}{2}+C$

i) $\int \sqrt[5]{x}\,dx = \dfrac{5}{6}x^{6/5}+C$

k) $\int \frac{1}{x^4}\,dx = \int x^{-4}\,dx \;=\dfrac{x^{-3}}{-3}+C = -\dfrac{1}{3x^3}+C$
(Write in power rule form first.)

l) $\int \dfrac{1}{x\sqrt{x^2-9}}\,dx \quad =\dfrac{1}{3}\operatorname{arcsec}\dfrac{|x|}{3}+C$

12. Evaluate each definite integral. Use the Fundamental Theorem of Calculus.

a) $\int_{-1}^{2} x^2\,dx \qquad =\left.\dfrac{x^3}{3}\right|_{-1}^{2} = \dfrac{8}{3}-\dfrac{-1}{3} = \;3$

b) $\int_0^{3\pi/4} (4+\sin x)\,dx \qquad = 4x-\cos x\Big|_0^{3\pi/4} \qquad = \left(4\left(3\pi/4\right)-\cos 3\pi/4\right)-\left(0-\cos 0\right) \qquad = 3\pi+\dfrac{\sqrt{2}}{2}+1$

13. Evaluate each indefinite integral. (Show the u-substitution used for each problem.)

a) $\int (6x+5)^{10}dx \quad = \int u^{10}\,\frac{du}{6} = \frac{1}{6}\cdot\frac{u^{11}}{11}+C \quad = \quad \frac{1}{66}(6x+5)^{11}+C$

Let $u = 6x+5$ so $du = 6dx$

b) $\int \frac{\cos x}{\sqrt[3]{4+\sin x}}dx \quad = \int u^{-1/3}du \quad = \frac{3u^{2/3}}{2}+C \quad = \frac{3(4+\sin x)^{2/3}}{2}+C$

Let $u = 4+\sin x$ so $du = \cos x dx$

c) $\int \frac{t}{3t^2+5}dt \quad = \frac{1}{6}\int \frac{1}{u}du \quad = \frac{1}{6}\ln|u|+C \quad = \frac{1}{6}\ln|3t^2+5|+C$

Let $u = 3t^2+5$ so $du = 6tdt$

d) $\int \tan\theta\sec^2\theta d\theta \quad = \int udu \quad = \frac{u^2}{2}+C \quad = \frac{(\tan\theta)^2}{2}+C$ or $= \frac{\tan^2\theta}{2}+C$

Let $u = \tan\theta$ so $du = \sec^2\theta d\theta$

e) $\int \frac{\ln y}{y}dy \quad = \int udu \quad = \frac{u^2}{2}+C \quad = \frac{(\ln y)^2}{2}+C$ (There is no other rewrite for this.)

Let $u = \ln y$ so $du = \frac{1}{y}dy$

f) $\int \frac{-1}{e^x}dx \quad = \int -1\cdot e^{-x}dx \quad = \int e^u du \quad = e^u + C \quad = e^{-x}+C$

Let $u = -x$ so $du = -dx$

14. Expand each sum. (The first 4 terms must be shown. The last term must be shown if it exists.) *Do Not Simplify*.

Example: $$\sum_{k=1}^{10} k^2+3 = \left(1^2+3\right)+\left(2^2+3\right)+\left(3^2+3\right)+\left(4^2+3\right)+\cdots+\left(10^2+3\right)$$

a) $$\sum_{k=1}^{74} k^3+1 = \left(1^3+1\right)+\left(2^3+1\right)+\left(3^3+1\right)+\left(4^3+1\right)+\cdots+\left(74^3+1\right)$$

b) $$\sum_{k=1}^{n} k^2+3 = \left(1^2+3\right)+\left(2^2+3\right)+\left(3^2+3\right)+\left(4^2+3\right)+\cdots+\left(n^2+3\right)$$

c) $$\sum_{k=1}^{\infty} \frac{k}{k+4} = \left(\frac{1}{1+4}\right)+\left(\frac{2}{2+4}\right)+\left(\frac{3}{3+4}\right)+\left(\frac{4}{4+4}\right)+\cdots$$

d) $$\sum_{k=0}^{\infty} \left(\frac{2}{3}\right)^k = (1)+\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\cdots$$

e) $$\sum_{k=1}^{\infty} \frac{1}{\sqrt{k}} = (1)+\left(\frac{1}{\sqrt{2}}\right)+\left(\frac{1}{\sqrt{3}}\right)+\left(\frac{1}{\sqrt{4}}\right)+\cdots$$